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Summary of The Fokker-Planck Equation Approach to Asset*Price Fluctuations

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Abstract

This paper investigates the possibility that adaptive expectation behavior causes large-scale fluctuations in asset prices. Although traditional financial theories assume the rationality of traders, empirical studies show that traders in the real market form expectations by using an adaptive scheme. The adaptive behavior of traders introduces path dependency into the asset price dynamics, which causes fluctuations in the asset market.

This paper employs the Fokker-Planck equation approach to investigate dynamical behavior of the model. The model is proved to have (at least) two stable situations, and transition from one situation to the other induced by stochastic shocks generates large-scale fluctuations in asset prices.

1 The Model

This section formulates the model of asset price dynamics as simultaneous SDEs of the asset price and expectations. Suppose a market consists of one asset and n homogeneous traders. Let S_t be the asset price at period $t \in \mathcal{T}$, where time set $\mathcal{T}$ is assumed to be continuous and infinite both in the future and past. At the initial period $t = 0$ the traders are given infinite past data of the asset price, $\{S_\tau\}_{\tau<0}$.

At period $t(> 0)$ trader $j \in \{1, 2, \cdots, n\}$ is assumed to hold an expectation $S^e_{j,t+\Delta t}$, which denotes an asset price at period $t + \Delta t$ expected by trader j. Trader j's expectation price $S^e_{j,t+\Delta t}$ is decomposed into two parts: a common element shared by all the traders, A_t, and a specific element peculiar only to trader j, $\epsilon_{j,t}$. The

*In this paper the word "asset" denotes anything traded in order to receive capital gain, and includes *i.e.* stocks, foreign currencies and rare metals. Dividends from assets are ignored in this paper.

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common expectation A_t is linearly dependent on all past data of the price up to period t, and takes the following form:

$$A_t = \int_{-\infty}^{t} K(t,\tau) S_\tau \, d\tau \ , \tag{1}$$

where $K(t,\tau)$ is a function satisfying $\int_{-\infty}^{t} K(t,\tau) d\tau = 1$ for any t. In this paper in particular, K is specified as $K(t,\tau) = \beta e^{-\beta(t-\tau)}$ for simplicity of calculation, so that we have

$$A_t = \int_{-\infty}^{t} \beta e^{-\beta(t-\tau)} S_\tau \, d\tau \ . \tag{2}$$

Since the history of the price path $\{S_\tau\}_{\tau<0}$ is given at $t=0$, eq.(2) becomes

$$A_t = e^{-\beta t} A_0 + \int_0^t \beta e^{-\beta(t-\tau)} S_\tau \, d\tau \ , \tag{3}$$

where A_0 is a constant given by $A_0 = \int_{-\infty}^{0} \beta e^{\beta\tau} S_\tau \, d\tau$. By differentiating (3), we get

$$dA_t = \beta(S_t - A_t) dt \ , \tag{4}$$

which is exactly the adaptive adjustment process of the expectation.

On the other hand, the term $\epsilon_{j,t}$ specific to trader j is assumed to be a random variable, independently and identically distributed for each trader and period, according to a smooth, symmetric probability density function ϕ for which mean is 0 and the variance γ^2 is finite. The random term $\epsilon_{j,t}$ can be interpreted as private information, an exogenous shock or a prediction error made by trader j. Hereinafter we call $\epsilon_{j,t}$ the prediction error and ϕ the error density. Consequently, the expectation held by trader j at period t is given by

$$\begin{aligned} S^e_{j,t+\Delta t} &= A_t + \epsilon_{j,t} \\ &= e^{-\beta t} A_0 + \int_0^t \beta e^{-\beta(t-\tau)} S_\tau \, d\tau + \epsilon_{t,j} \quad , \qquad \epsilon_{t,j} \sim i.i.d.\, \phi \ . \end{aligned} \tag{5}$$

Since the mean of ϕ is assumed to be 0, $E[S^e_{j,t+\Delta t}] = A_t$ holds for any j. Hereinafter we call A_t the average expectation at period t, and $A = (A_t)_{t\in\mathcal{T}}$ the average expectation process.

Suppose that trader j demands one unit of the asset when $S^e_{j,t+\Delta t} > S_t$, and supplies when $S^e_{j,t+\Delta t} < S_t$. The probability of $S^e_{j,t+\Delta t} = S_t$ is zero because of the continuity of ϕ. Therefore the probability that trader j will demand the asset is

$$P(S^e_{j,t} > S_t) = P(\epsilon_{t,j} > S_t - A_t) = \int_{S_t - A_t}^{\infty} \phi(s)\, ds = 1 - \Phi(S_t - A_t) \ , \tag{6}$$

where Φ denotes the cumulative distribution function of the prediction error (see Fig.1). On the other hand, the probability of supply is $\Phi(S_t - A_t)$.

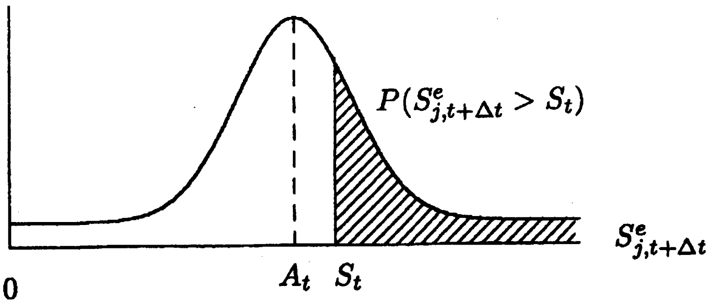

Figure 1: **The error density ϕ and the probability of demand**

Let n_t^+ be the number of traders who demand the asset at period t, and n_t^- the number of traders supplying the asset. Because the prediction error $\epsilon_{j,t}$ is $i.i.d.$ for each $j \in \{1, \cdots, n\}$, n_t^+ obeys a binominal distribution $B(n, 1 - \Phi(S_t - A_t))$. Since the binominal distribution $B(m,p)$ is approximated by the normal distribution $N(mp, mp(1-p))$ when m is large enough (Laplace's Theorem),

$$n_t^+ \sim N\Big(n(1 - \Phi(S_t - A_t)) \ , \ n(1 - \Phi(S_t - A_t))\Phi(S_t - A_t) \Big) \tag{7}$$

if the number of the traders n is large enough.

For the sake of simplicity, we set the price change $\Delta S_t (:= S_{t+\Delta t} - S_t)$ proportional to the excess demand, that is,

$$\Delta S_t = \frac{\rho}{2}\left(n_t^+ - n_t^-\right)\Delta t \ , \tag{8}$$

where ρ denotes price sensitivity to the excess demand per unit of time[1]. By substituting $n_t^- = n - n_t^+$ into eq.(8) we get $\Delta S_t = \rho n \left(n_t^+/n - 1/2\right)\Delta t$. Because n_t^+ is normally distributed, ΔS_t also is normally distributed: that is,

$$\Delta S_t \sim N\left(\rho n(1/2 - \Phi(S_t - A_t))\Delta t, (\rho n)^2 \frac{(1 - \Phi(S_t - A_t))\Phi(S_t - A_t)}{n}\Delta t^2\right) . \tag{9}$$

Accordingly, the discrete-time asset price process is given by

$$\Delta S_t = \mu(1/2 - \Phi(S_t - A_t))\Delta t + \sigma\sqrt{(1 - \Phi(S_t - A_t))\Phi(S_t - A_t)}(W_{t+\Delta t} - W_t) , \tag{10}$$

where $\mu = \rho n$, $\sigma = \rho\sqrt{n\Delta t}$, and $W = (W_t)_{t \in \mathcal{T}}$ is the standard Brownian motion. When Δt is small enough, the discrete process (10) is approximated by a continuous-time process,

$$dS_t = \mu(1/2 - \Phi(S_t - A_t))dt + \sigma\sqrt{(1 - \Phi(S_t - A_t))\Phi(S_t - A_t)}dW_t . \tag{11}$$

[1] **Although the assumption of lineality in price changes is essential to make our model solvable, it makes the model somewhat unrealistic because the price can be negative with positive probability. By taking S_0 sufficiently large, however, the probability of negative S_t becomes neglible for each**

By combining eq.(4) and eq.(11), we get the simultaneous SDEs of the average expectation and the asset price

$$\begin{cases} dA_t = \beta(S_t - A_t)dt \\ dS_t = \mu(1/2 - \Phi(S_t - A_t))dt + \sigma\sqrt{(1 - \Phi(S_t - A_t))\Phi(S_t - A_t)}dW_t \, . \end{cases} \tag{12}$$

This is the model we are interested in.

2 Results

Proposition 2.1 *Define the unexpected shock process $\xi = (\xi_t)_{t\geq 0}$ by*

$$\xi_t = S_t - A_t \, . \tag{13}$$

Then the dynamics of ξ and S are given by

$$\begin{cases} d\xi_t = \Big\{\mu(1/2 - \Phi(\xi_t)) - \beta\xi_t\Big\}dt + \sigma\sqrt{\Big(1 - \Phi(\xi_t)\Big)\Phi(\xi_t)}dW_t \\ S_t = A_0 + \xi_t + \beta\displaystyle\int_0^t \xi_s ds \, . \end{cases} \tag{14}$$

(Proof) By the definition of ξ and eq.(12),

$$\begin{aligned} d\xi_t &= dS_t - dA_t \\ &= \Big\{\mu(1/2 - \Phi(\xi_t)) - \beta\xi_t\Big\}dt + \sigma\sqrt{\Big(1 - \Phi(\xi_t)\Big)\Phi(\xi_t)}dW_t \, . \end{aligned} \tag{15}$$

By integrating both sides of $dA_t = \beta\xi_t dt$, we have $A_t = A_0 + \beta\int_0^t \xi_s ds$, and consequently $S_t = A_t + \xi_t = A_0 + \beta\int_0^t \xi_s ds + \xi_t$. □

Theorem 2.2 (the Fokker-Planck equation) *Suppose an SDE with an initial condition is given as follows:*

$$dX_t = \alpha(t, X_t)dt + \gamma(t, X_t)dW_t \; , \; X_0 = x_0 \, . \tag{16}$$

Let $f(t,x)$ be the density of $X = (X_t)_{t\geq 0}$: that is, f is supposed to satisfy

$$Prob\{X_t \in B \,|\, X_0 = x_0\} = \int_B f(t,x)\, dx \tag{17}$$

for any Borel set B. If the functions α, $\partial_x\alpha$, γ, $\partial_x\gamma$, $\partial_x^2\gamma$, $\partial_t f$, $\partial_x f$, and $\partial_x^2 f$ are continuous for $t > 0$ and $x \in \boldsymbol{R}$, and if α, γ, and their first derivatives are bounded, then $f(t,x)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} f(t,x) = -\frac{\partial}{\partial x}\Big[\alpha(t,x)f(t,x)\Big] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[\gamma(t,x)^2 f(t,x)\Big] \\ \lim_{t\to+0} f(t,x) = \delta(x - x_0) \, , \end{cases} \tag{18}$$

where $\delta(\cdot)$ is the Dirac's delta.

The first equation in (18) is called the Fokker-Planck equation (or Kolmorgorov forward equation). For a proof of this theorem, see *e.g.* Lasota and Mackey (1994), p.360.

Example 2.3 *Consider an SDE with an initial condition,*

$$dX_t = a\,dt + g\,dW_t \ , \ X_0 = 0 \ , \tag{19}$$

where a and g are constant. The solution of eq.(19) and its density are given by $X_t = at + gW_t$ and

$$f(t,x) = \frac{1}{\sqrt{2\pi g^2 t}} \exp\left[-\frac{(x-at)^2}{2g^2 t}\right] .$$

The density f satisfies the Fokker-Planck equation corresponding to eq.(19), that is,

$$\frac{\partial}{\partial t} f(t,x) = -a\frac{\partial}{\partial x} f(t,x) + \frac{g^2}{2}\frac{\partial^2}{\partial^2 x} f(t,x) \ ,$$

and the initial condition $\lim_{t\to+0} f(t,x) = \delta(x)$. □

Example 2.4 *(This example will be used in the proof of Theorem 2.7.)*
Consider an SDE and its initial condition,

$$dX_t = -\beta X_t\,dt + \lambda e^{-\beta t} dW_t \ , \ X_0 = x_0 \ , \tag{20}$$

where β and λ are constant. The solution is

$$X_t = x_0 e^{-\beta t} + \lambda e^{-\beta t} W_t \ , \tag{21}$$

therefore the density of X is given by

$$f(t,x) = \frac{1}{\sqrt{2\pi t \lambda^2 e^{-2\beta t}}} \exp\left[-\frac{(x - x_0 e^{-\beta t})^2}{2t\lambda^2 e^{-2\beta t}}\right] . \tag{22}$$

It can be readily checked that eq.(22) satisfies both of the Fokker-Planck equation,

$$\frac{\partial}{\partial t} f(t,x) = -\frac{\partial}{\partial x}\Big[(-\beta x) f(t,x)\Big] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[(\lambda e^{-\beta t})^2 f(t,x)\Big] \quad , \tag{23}$$

and the initial condition, $\lim_{t\to+0} f(t,x) = \delta(x - x_0)$. □

The Fokker-Planck equation corresponding to the unexpected shock process (15) is given by

$$\begin{aligned}\frac{\partial}{\partial t}f(t,\xi) &= -\frac{\partial}{\partial \xi}\left[\left\{\mu\,(1/2-\Phi(\xi))-\beta\xi\right\}f(t,\xi)\right] \\ &\quad +\frac{1}{2}\frac{\partial^2}{\partial \xi^2}\left[\sigma^2(1-\Phi(\xi))\Phi(\xi)f(t,\xi)\right] . \qquad (24)\end{aligned}$$

Obtaining the explicit solution of the Fokker-Planck equation is generally difficult, however we can study average behavior of ξ by utilizing eq.(24) without solving it.

Theorem 2.5 *Let $f(t,\xi)$ be the solution of eq.(24). If $f(t,\xi)$ satisfies*

$$\int \xi^2 f(t,\xi)\,d\xi < \infty\ , \quad \lim_{\xi\to\pm\infty} \xi^3 f(t,\xi) = 0$$

for any $t>0$, the approximate dynamics of the mean and variance of ξ_t can be given by the following differential equations:

$$\begin{cases} \dfrac{d}{dt}E[\xi_t] = -\left\{\mu\phi(0)+\beta\right\}E[\xi_t] \\[2ex] \dfrac{d}{dt}V[\xi_t] = \dfrac{\sigma^2}{4} - \sigma^2\phi^2(0)E[\xi_t]^2 - \left\{\sigma^2\phi^2(0)+2(\mu\phi(0)+\beta)\right\}V[\xi_t] \quad , \end{cases} \qquad (25)$$

where $E[\xi_t]$ and $V[\xi_t]$ denote, respectively, the mean and variance of ξ_t.

(Proof) Note that the assumption $\int \xi^2 f(t,\xi)\,d\xi < \infty$ implies

$$\lim_{\xi\to\pm\infty} f(t,\xi) = \lim_{\xi\to\pm\infty} \xi f(t,\xi) = \lim_{\xi\to\pm\infty} \xi^2 f(t,\xi) = 0\ ,$$

and

$$\lim_{\xi\to\pm\infty} \frac{\partial}{\partial \xi}f(t,\xi) = \lim_{\xi\to\pm\infty} \xi\frac{\partial}{\partial \xi}f(t,\xi) = \lim_{\xi\to\pm\infty} \xi^2\frac{\partial}{\partial \xi}f(t,\xi) = 0\ .$$

The mean of ξ_t is given by $E[\xi_t] = \int \xi f(t,\xi)\,d\xi$. Differentiating it with respect to time, we have

$$\begin{aligned}\frac{d}{dt}E[\xi_t] &= \int \xi\frac{\partial}{\partial t}f(t,\xi)\,d\xi \\ &= \int \xi\left\{-\frac{\partial}{\partial \xi}\left[\left\{\mu(1/2-\Phi(\xi))-\beta\xi\right\}f(t,\xi)\right]\right. \\ &\qquad \left.+\frac{1}{2}\frac{\partial^2}{\partial \xi^2}\left[\sigma^2(1-\Phi(\xi))\Phi(\xi)f(t,\xi)\right]\right\}d\xi \\ &= -\int \xi\frac{\partial}{\partial \xi}\left[\left\{\mu(1/2-\Phi(\xi))-\beta\xi\right\}f(t,\xi)\right]d\xi \\ &\qquad +\frac{1}{2}\int \xi\frac{\partial^2}{\partial \xi^2}\left[\sigma^2(1-\Phi(\xi))\Phi(\xi)f(t,\xi)\right]d\xi\ .\end{aligned}$$

Integrating by part results in

$$\begin{aligned}
&\int \xi \frac{\partial}{\partial \xi}\left[\left\{\mu(1/2 - \Phi(\xi)) - \beta\xi\right\} f(t,\xi)\right] d\xi \\
&\quad = \left[\xi\left\{\mu(1/2 - \Phi(\xi)) - \beta\xi\right\} f(t,\xi)\right]_{-\infty}^{\infty} - \int \left\{\mu(1/2 - \Phi(\xi)) - \beta\xi\right\} f(t,\xi)\, d\xi \\
&\quad = -E\left[\mu(1/2 - \Phi(\xi_t)) - \beta\xi_t\right]
\end{aligned}$$

since

$$\lim_{\xi\to\pm\infty}\left|\xi\left\{\mu(1/2 - \Phi(\xi)) - \beta\xi\right\} f(t,\xi)\right| \le \frac{\mu}{2}\lim_{\xi\to\pm\infty}\left|\xi f(t,\xi)\right| + \beta \lim_{\xi\to\pm\infty}\left|\xi^2 f(t,\xi)\right| = 0 .$$

In the same way,

$$\begin{aligned}
&\int \xi \frac{\partial^2}{\partial \xi^2}\left[\sigma^2(1 - \Phi(\xi))\Phi(\xi) f(t,\xi)\right] d\xi \\
&\quad = \sigma^2\left[\xi \frac{\partial}{\partial \xi}\left[(1 - \Phi(\xi))\Phi(\xi) f(t,\xi)\right]\right]_{-\infty}^{\infty} + \sigma^2 \int \frac{\partial}{\partial \xi}\left[(1 - \Phi(\xi))\Phi(\xi) f(t,\xi)\right] d\xi \\
&\quad = \sigma^2\left[\xi \frac{\partial}{\partial \xi}\left[(1 - \Phi(\xi))\Phi(\xi) f(t,\xi)\right]\right]_{-\infty}^{\infty} + \sigma^2 \left[(1 - \Phi(\xi))\Phi(\xi) f(t,\xi)\right]_{-\infty}^{\infty} \\
&\quad = 0
\end{aligned}$$

since

$$\begin{aligned}
&\left|\xi \frac{\partial}{\partial \xi}\left[(1 - \Phi(\xi))\Phi(\xi) f(t,\xi)\right]\right| \\
&\qquad \le \left|\xi\Big(1 - 2\Phi(\xi)\Big)\phi(\xi) f(t,\xi)\right| + \left|\xi\Big(1 - \Phi(\xi)\Big)\Phi(\xi)\frac{\partial}{\partial \xi} f(t,\xi)\right| \\
&\qquad \le (\max\phi)\left|\xi f(t,\xi)\right| + \frac{1}{4}\left|\xi \frac{\partial}{\partial \xi} f(t,\xi)\right| \\
&\qquad \to 0 \qquad (|\xi| \to \infty)
\end{aligned}$$

and

$$\left|\Big(1 - \Phi(\xi)\Big)\Phi(\xi) f(t,\xi)\right| \le \frac{1}{4}|f(t,\xi)| \to 0 \qquad (|\xi| \to \infty) .$$

Therefore we get

$$\frac{d}{dt}E[\xi_t] = E\left[\mu\Big(1/2 - \Phi(\xi_t)\Big) - \beta\xi_t\right] \simeq -\left\{\mu\phi(0) + \beta\right\}E[\xi_t]$$

by taking the Taylor expansion around $\xi_t = 0$ up to first order.

The differential equation of the squared mean,

$$\frac{d}{dt}E[\xi_t^2] = 2E\left[\xi_t\left\{\mu\,(1/2 - \Phi(\xi_t)) - \beta\xi_t\right\}\right] + \sigma^2 E\left[(1 - \Phi(\xi_t))\,\Phi(\xi_t)\right],$$

is derived in the same way as the mean equation. By taking the Taylor expansion around $\xi_t = 0$ up to second order, we obtain

$$E\Big[\xi_t\Big\{\mu\,(1/2-\Phi(\xi_t)) - \beta\xi_t\Big\}\Big] \simeq -\Big\{\mu\phi(0)+\beta\Big\}E[\xi_t^2]$$

and

$$E\Big[(1-\Phi(\xi_t))\,\Phi(\xi_t)\Big] \simeq \frac{1}{4} - \phi^2(0)E[\xi_t^2] \quad .$$

From the definition of variance, we can derive the variance equation,

$$\begin{aligned}
\frac{d}{dt}V[\xi_t] &= \frac{d}{dt}\Big[E[\xi_t^2]-E[\xi_t]^2\Big] \\
&= \frac{d}{dt}E[\xi_t^2] - 2E[\xi_t]\frac{d}{dt}E[\xi_t] \\
&\simeq 2\Big[-\Big\{\mu\phi(0)+\beta\Big\}E[\xi_t^2]\Big] + \sigma^2\left[\frac{1}{4}-\phi^2(0)\left\{V[\xi_t]+E[\xi_t]^2\right\}\right] \\
&\qquad +2\Big\{\mu\phi(0)+\beta\Big\}E[\xi_t]^2 \\
&= \frac{\sigma^2}{4} - 2\Big\{\mu\phi(0)+\beta\Big\}V[\xi_t] - \sigma^2\phi^2(0)\left\{V[\xi_t]+E[\xi_t]^2\right\} \ .
\end{aligned}$$

□

Corollary 2.6 *In the steady state the mean and variance of ξ are respectively given by*

$$\overline{E} = 0 \quad , \quad \overline{V} = \frac{1}{4}\left[\phi(0)^2 + \left(\frac{2\mu}{\sigma^2}\right)\phi(0) + \frac{\beta}{\sigma^2}\right]^{-1} \quad . \tag{26}$$

(Proof) Obvious from eq.(25). □

Theorem 2.7 *As $\beta \to \infty$, the asset price process $S = (S_t)_{t\geq 0}$ converges to*

$$dS_t = \frac{\sigma}{2}dW_t \ . \tag{27}$$

Consequently, the price change $\Delta S_t (= S_{t+\Delta t} - S_t)$ obeys a normal distribution of mean 0 and standard deviation $\frac{\sigma\sqrt{\Delta t}}{2}$.

(Proof) By dividing Fokker-Planck equation (24) by β, we get

$$\begin{aligned}
\frac{1}{\beta}\frac{\partial}{\partial t}f(t,\xi) &= -\frac{\partial}{\partial \xi}\left[\left\{\frac{\mu\,(1/2-\Phi(\xi))}{\beta} - \xi\right\}f(t,\xi)\right] \\
&\qquad +\frac{1}{2}\frac{\partial^2}{\partial \xi^2}\left[\frac{\sigma^2(1-\Phi(\xi))\Phi(\xi)}{\beta}f(t,\xi)\right] \ .
\end{aligned}$$

We can assume here $\mu, \sigma^2 \ll \beta$, thus

$$\left|\frac{\mu(1/2-\Phi(\xi))}{\beta}\right| \le \frac{\mu}{2\beta} \simeq 0 \quad and \quad \left|\frac{\sigma^2(1-\Phi(\xi))\Phi(\xi)}{\beta}\right| \le \frac{\sigma^2}{4\beta} \simeq 0$$

hold for any ξ. Therefore, if β is large enough, we can regard $f(t,\xi)$ as the solution of the following problem:

$$\begin{cases} \frac{1}{\beta}\frac{\partial}{\partial t} f(t,\xi) = \frac{\partial}{\partial \xi}[\xi f(t,\xi)] \\ \lim_{t\to+0} f(t,\xi) = \delta(\xi-\xi_0) \,, \end{cases} \tag{28}$$

where ξ_0 is any given initial value of ξ_t.

In order to solve the problem (28), we introduce a perturbation parameter λ and consider a perturbed problem

$$\begin{cases} \frac{1}{\beta}\frac{\partial}{\partial t} f^\lambda(t,\xi) = \frac{\partial}{\partial \xi}\left[\xi f^\lambda(t,\xi)\right] + \frac{\lambda^2}{2\beta}\frac{\partial^2}{\partial \xi^2}\left[e^{-2\beta t} f^\lambda(t,\xi)\right] \\ \lim_{t\to+0} f^\lambda(t,\xi) = \delta(\xi-\xi_0) \end{cases} . \tag{29}$$

As we have shown in Example 2.4, the solution $f^\lambda(t,\xi)$ is given by

$$f^\lambda(t,\xi) = \frac{1}{\sqrt{2\pi t\lambda^2 e^{-2\beta t}}} \exp\left[-\frac{(\xi-\xi_0 e^{-\beta t})^2}{2t\lambda^2 e^{-2\beta t}}\right] .$$

By taking limit $\lambda \to 0$, we get the solution of the original problem (28): that is,

$$f(t,\xi) = \lim_{\lambda\to 0} f^\lambda(t,\xi) = \delta(\xi - \xi_0 e^{-\beta t}) .$$

Moreover, since $f(t,\xi) \to \delta(\xi)$ as β goes to infinity, $\xi_t = S_t - A_t \equiv 0$ holds for any $t > 0$ when β is infinitely large. By substituting $S_t - A_t \equiv 0$ into the first equation in eq.(12), we have

$$dS_t = \mu(1/2 - \Phi(0))dt + \sigma\sqrt{(1-\Phi(0))\Phi(0)}dW_t = \frac{\sigma}{2}dW_t$$

since $\Phi(0) = 1/2$. □

The stationary distribution of unexpected shocks, $\overline{f}(\xi)$, is defined as

$$\overline{f}(\xi) = \lim_{t\to\infty} f(t,\xi)$$

when it exists. Because $\overline{f}(\xi)$ no longer depends on t, $\partial_t \overline{f}(\xi) = 0$ holds and the Fokker-Planck equation (24) is reduced to the following ordinary differential equation:

$$-\frac{d}{d\xi}\left[\left\{\mu(1/2-\Phi(\xi)) - \beta\xi\right\}\overline{f}(\xi)\right] + \frac{\sigma^2}{2}\frac{d^2}{d\xi^2}\left[(1-\Phi(\xi))\Phi(\xi)\overline{f}(\xi)\right] = 0 . \tag{30}$$

Proposition 2.8 *If* $\left|\int \xi \overline{f}(\xi)\, d\xi\right| < \infty$ *holds, then* $\overline{f}(\xi)$ *satisfies the following equation:*

$$\overline{f}'(\xi) = \left\{\left(\frac{\mu}{\sigma^2} - \phi(\xi)\right)\left(\frac{1}{\Phi(\xi)} - \frac{1}{1-\Phi(\xi)}\right) - \frac{2\beta\xi}{\sigma^2}\left(\frac{1}{\Phi(\xi)} + \frac{1}{1-\Phi(\xi)}\right)\right\}\overline{f}(\xi). \quad (31)$$

(Proof) Since $\overline{f}$ satisfies eq.(30), we have

$$\frac{d}{d\xi}\left[-\left\{\mu\,(1/2 - \Phi(\xi)) - \beta\xi\right\}\overline{f}(\xi) + \frac{\sigma^2}{2}\frac{d}{d\xi}\Big((1-\Phi(\xi))\Phi(\xi)\overline{f}(\xi)\Big)\right] = 0.$$

Consequently, there is a constant C s.t.

$$-\left\{\mu\,(1/2 - \Phi(\xi)) - \beta\xi\right\}\overline{f}(\xi) + \frac{\sigma^2}{2}\frac{d}{d\xi}\Big((1-\Phi(\xi))\Phi(\xi)\overline{f}(\xi)\Big) = C\,. \quad (32)$$

Assuming that C is not 0, we have

$$\begin{aligned}
\infty &= \left|\int_{-\infty}^{\infty} C\, d\xi\right| \\
&= \left|\int_{-\infty}^{\infty}\left[-\left\{\mu\,(1/2 - \Phi(\xi)) - \beta\xi\right\}\overline{f}(\xi) + \frac{\sigma^2}{2}\frac{d}{d\xi}\Big((1-\Phi(\xi))\Phi(\xi)\overline{f}(\xi)\Big)\right] d\xi\right| \\
&\le \frac{\mu}{2}\int_{-\infty}^{\infty}\overline{f}(\xi)\, d\xi + \beta\left|\int_{-\infty}^{\infty}\xi\overline{f}(\xi)\, d\xi\right| + \frac{\sigma^2}{2}\left|\Big[(1-\Phi(\xi))\Phi(\xi)\overline{f}(\xi)\Big]_{-\infty}^{\infty}\right| \\
&= \frac{\mu}{2} + \beta\left|\int_{-\infty}^{\infty}\xi\overline{f}(\xi)\, d\xi\right| \quad .
\end{aligned}$$

This contradicts the upper boundness of $\left|\int \xi\overline{f}(\xi)\, d\xi\right|$, therefore C must be 0. By substituting $C = 0$ into eq.(32), we get eq.(31) after some manipulations. □

Corollary 2.9 *The stationary distribution* $\overline{f}(\xi)$ *is symmetric around* 0, *and given by*

$$\begin{aligned}
\overline{f}(\xi) &= N_0 \exp\left[\int_0^{\xi}\left\{\left(\frac{\mu}{\sigma^2} - \phi(\zeta)\right)\left(\frac{1}{\Phi(\zeta)} - \frac{1}{1-\Phi(\zeta)}\right)\right.\right. \\
&\qquad \left.\left. -\frac{2\beta}{\sigma^2}\zeta\left(\frac{1}{\Phi(\zeta)} + \frac{1}{1-\Phi(\zeta)}\right)\right\} d\zeta\right], \quad (33)
\end{aligned}$$

where N_0 *is a normalizing constant.*

(Proof) Eq.(33) can be readily derived from differential equation (31). Because the error density ϕ is assumed to be symmetric, we have $\phi(-\eta) = \phi(\eta)$, $\Phi(-\eta) = 1-\Phi(\eta)$,

$$\begin{aligned}
\overline{f}(-\xi) &= N_0 \exp\left[\int_0^{-\xi}\left\{\left(\frac{\mu}{\sigma^2}-\phi(\zeta)\right)\left(\frac{1}{\Phi(\zeta)}-\frac{1}{1-\Phi(\zeta)}\right)\right.\right. \\
&\qquad \left.\left.-\frac{2\beta}{\sigma^2}\zeta\left(\frac{1}{\Phi(\zeta)}+\frac{1}{1-\Phi(\zeta)}\right)\right\}d\zeta\right] \\
&= N_0 \exp\left[\int_0^{\xi}\left\{\left(\frac{\mu}{\sigma^2}-\phi(-\eta)\right)\left(\frac{1}{\Phi(-\eta)}-\frac{1}{1-\Phi(-\eta)}\right)\right.\right. \\
&\qquad \left.\left.-\frac{2\beta}{\sigma^2}(-\eta)\left(\frac{1}{\Phi(-\eta)}+\frac{1}{1-\Phi(-\eta)}\right)\right\}(-d\eta)\right] \\
&= N_0 \exp\left[\int_0^{\xi}\left\{\left(\frac{\mu}{\sigma^2}-\phi(\eta)\right)\left(-\frac{1}{1-\Phi(\eta)}+\frac{1}{\Phi(\eta)}\right)\right.\right. \\
&\qquad \left.\left.-\frac{2\beta}{\sigma^2}\eta\left(\frac{1}{1-\Phi(\eta)}+\frac{1}{\Phi(\eta)}\right)\right\}d\eta\right] = \overline{f}(\xi)\,.
\end{aligned}$$

□

Corollary 2.10 *The stationary distribution $\overline{f}$ becomes unimodal if*

$$\phi(\xi) < \frac{\mu}{\sigma^2} \tag{34}$$

holds for any ξ.

(Proof) Since ϕ is assumed to be symmetric, $\Phi(\xi)$ is smaller than $1/2$ when $\xi < 0$, and larger when $\xi > 0$. If condition (34) is satisfied, we have

$$\overline{f}'(\xi)\begin{cases} >0 & (\xi<0) \\ =0 & (\xi=0) \\ <0 & (\xi>0) \end{cases}$$

by eq.(14). This indicates that $\overline{f}$ is a unimodal function which reaches its maximum value at $\xi = 0$. □

Theorem 2.11 *Suppose that $\phi(0) > \mu/\sigma^2$. If β is small enough to satisfy*

$$\beta < \sigma^2\phi(0)\left\{\phi(0)-\frac{\mu}{\sigma^2}\right\}, \tag{35}$$

then $\overline{f}$ has at least two relative maximums.

Lemma 2.12 *If both $\phi(0) > \mu/\sigma^2$ and $\beta < \sigma^2\phi(0)\left\{\phi(0)-\frac{\mu}{\sigma^2}\right\}$ hold, then $\overline{f}$ has a relative minimum at 0.*

(Proof) By eq.(31), we have $\overline{f}'(0) = 0$ and

$$\frac{d}{d\xi}\left(\frac{\overline{f}'}{\overline{f}}\right) \left(= \frac{\overline{f}''\overline{f} - (\overline{f}')^2}{\overline{f}^2}\right)$$

$$= -\phi'\left(\frac{1}{\Phi} - \frac{1}{1-\Phi}\right) + \left(\frac{\mu}{\sigma^2} - \phi\right)\left(-\frac{\phi}{\Phi^2} - \frac{\phi}{(1-\Phi)^2}\right)$$

$$-\frac{2\beta}{\sigma^2}\left(\frac{1}{\Phi} + \frac{1}{1-\Phi}\right) - \frac{2\beta\xi}{\sigma^2}\left(-\frac{\phi}{\Phi^2} + \frac{\phi}{(1-\Phi)^2}\right) \quad . \tag{36}$$

By substituting $\xi = 0$ into eq.(36),

$$\frac{\overline{f}''(0)}{\overline{f}(0)} = 8\left\{\phi(0)^2 - \left(\frac{\mu}{\sigma^2}\right)\phi(0) - \left(\frac{\beta}{\sigma^2}\right)\right\} \tag{37}$$

since $\overline{f}'(0) = 0$ and $\Phi(0) = 1/2$. A necessary and sufficient condition for $\overline{f}$ to have a relative minimum at 0 is $\overline{f}'(0) = 0$ and $\overline{f}''(0) > 0$. Since $\overline{f}(0)$ and $\phi(0)$ are both positive,

$$\overline{f}''(0) > 0 \iff \phi(0)^2 - \left(\frac{\mu}{\sigma^2}\right)\phi(0) - \left(\frac{\beta}{\sigma^2}\right) > 0$$

$$\iff \phi(0) > \frac{\mu}{2\sigma^2}\left\{1 + \sqrt{1 + \frac{4\beta\sigma^2}{\mu^2}}\right\} .$$

Therefore, after algebraic manipulation, we get inequality (35). □

(Proof of Theorem 2.11) Because $\lim_{\xi\to\pm\infty}\phi(\xi) = 0$, there is $\overline{\xi} > 0$ s.t. $\mu/\sigma^2 > \phi(\overline{\xi})$ and $\phi'(\overline{\xi}) < 0$. By eq.(31), we get $\overline{f}'(\overline{\xi}) < 0$. On the other hand, by Lemma 2.12, there is $\underline{\xi} \in (0, \overline{\xi})$ s.t. $\overline{f}'(\underline{\xi}) > 0$. Because of the continuity of $\overline{f}'$, there exists $\xi^* \in (\underline{\xi}, \overline{\xi})$ at which $\overline{f}$ has a relative maximum by the Mean Value theorem (see Fig.2). Since we have shown the symmetry of $\overline{f}$ in Corollary 2.9, $\overline{f}$ has a relative maximum also at $\xi = -\xi^*$. □

Figure 2 shows a typical case of Theorem 2.11. The bimodality (or multimodality in some cases) of $\overline{f}$ indicates that discrepancy between the asset price and the average expectation frequently occurs, and that the price move rapidly at $\xi(= S - A) = 0$ to keep the likelihood of $\xi = 0$ relatively minimum. In other words, two (or more) peaks of $\overline{f}$ are interpreted as locally stable points of the dynamics. Stochastic shocks, however, prevent the system from remaining at either point, and transition from one peak to the other causes large-scale fluctuations. The bimodality of the unexpected-shock distribution therefore implies variability of the asset price.

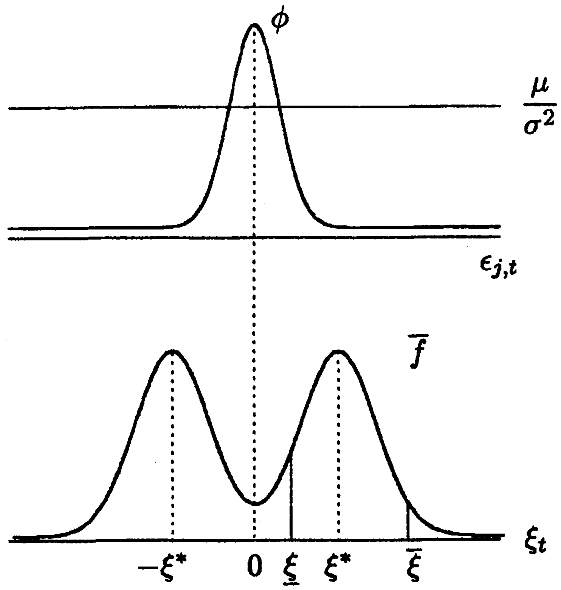

Figure 2: **The error density ϕ and the stationary distribution of unexpected shocks $\overline{f}$ in a typical case of Theorem 2.11.**

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