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EXISTENCE OF LIMIT CYCLES FOR COUPLED VAN DER POL EQUATIONS

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ABSTRACT. In this paper, we consider the existence of limit cycles of coupled van der Pol equations by using S^1-degree theory.

1. INTRODUCTION

Our purpose in the present paper is to discuss the exsitence of solutions for the system of Lienard differential equations. A second order ordinary differential equation of the form

$$u_{tt} + f(u)u_t + g(u) = e(t)$$

is called a Lienard differential equation, where $f : \mathbb{R}^N \to \mathbb{R}^N$ and $g : \mathbb{R}^N \to \mathbb{R}^N$ are usually assumed to be a continuous function on $\mathbb{R}^N$, and $e(t) : \mathbb{R} \to \mathbb{R}^N$ is a forcing term. Recent 30 years, Lienard equation has been investigated by many authors from various points of view. One of the reason why many mathemticians have been studied this kind of equations is that a broad class of phenomina in Science and Engineering is presented by the Lienard equation. Though the Lienard equation is very simple, the investigation of solutions for the equation is very difficult. One of most interesting problem is to find a nontrivial solution of the autonomous Liearnd equation. Let consider the Lienard problem with $f(t) = \varepsilon(t^2 - 1)$ and $g(t) = t$ for $t \in \mathbb{R}$, where $\varepsilon > 0$ is a given constant. That is we consider the problem

$$u_{tt} + \varepsilon(u^2 - 1)u_t + u = 0 \qquad t \in \mathbb{R} \tag{1.1}$$

Problem (1.1) is known as van del Pol equation. The van der Pol euation has been studied by many authors due to its adoption to wide variety of mechnical, electonical, biological and economical systems, and the behavior of the solutions is now well understood(cf. Guchenheimer and Holmes [[?]]. When $e(t) = 0$, Problem (1.1) has exactly one limit cycle, that is there exists a unique nontrivial periodic solution of (1.1). The period of the solution is determined by $\varepsilon > 0$. The proof of the existence of the limit cycle is basend on the Poincare-Bendixson theorem.

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Since Poincare-Bendixson theorem is valid only in two dimensional Euclidian space, the proof for the exsitence of llimit cycle of (1.1) is not effective in n-dimensional cased($n \geq 2$). We will see in this paper that there is a limit cycle for a system of autonomous van del Pol type equations.

On the other hand, the Lienard equation with a periodic forcing term $e(t)$ has also been studied by many authors. It is known under some conditions, the Liniard problem has multiple periodic solutions and sometimes the dynamics of the solutions are chaotic. We will see that under suitable conditions, n-dimensional Lienard equation has periodic solutions.

2. Lienard equation with periodic forcing terms

In this section, we discuss the Lienard equation of the form

$$u_{tt} + \frac{d}{dt}F(u) + Au = e(t) \qquad t \in \mathbb{R} \tag{V}$$

where $N \geq 2$, $u \in \mathbb{R}^N$, A is $(n,n)-$ matrix, $F : \mathbb{R}^N \to \mathbb{R}^N$ is a C^1 function and $e : \mathbb{R} \to \mathbb{R}^N$ is a continuous T-periodic function with period $T > 0$. In [He], Egami and the author established an existence result for the periodic solution of problem V. To state the result, we need to give some notations In the following, $|\cdot|$ and $\langle \cdot, \cdot \rangle$ stands for the norm and the inner product of $\mathbb{R}^N$, respectively. For each $u \in L^2([0,T];\mathbb{R}^N)$, we put $\|u\| = \left(\int_0^T |u|^2\right)^{1/2}$. We also set

$$H = \left\{ u \in C([0,T];\mathbb{R}^N) : u(0) = u(T), \int_0^T |u|^2 \, dt < \infty, \int_0^T |u_t|^2 \, dt < \infty \right\}.$$

The norm of H is defined by $\|u\|_H = \left(\|u\|^2 + \|u_t\|^2\right)^{1/2}$ for each $u \in H$. We also put

$$\widetilde{H} = \left\{ u \in H : \int_0^{2\pi} u(t)dt = 0 \right\}.$$

We denote by $B_r(0)$ the open ball in H centered at 0 with radius $r > 0$. $\partial B_r(0)$ denotes the boundary of $B_r(0)$. That is $\partial B_r(0) = \{u \in H : \|u\| = r\}$. For each compact mapping $L : H \to H$ and an open set U, we denote by $\deg(I - L, U, 0)$ the Browder degree of L on U with respect to 0. We consider the case that F has the form

$$F(x_1, x_2, \cdots, x_n) = \begin{pmatrix} F_1(x_1) \\ F_2(x_2) \\ \cdots \\ F_N(x_N) \end{pmatrix} \tag{F1}$$

where $F_i : \mathbb{R} \to \mathbb{R}$ is continuous mapping. We put $f_i = F_i'$ for $1 \le i \le N$. We assume that each f_i satisfies

$$f_i(0) < 0 \qquad \text{and} \qquad \frac{f_i(s) - f_i(0)}{s^2} > 0 \qquad \text{for } s \neq 0. \tag{F2}$$

We put $\mu = \min\left\{\frac{f_i(s)-f_i(0)}{s^2} : s \neq 0, 1 \le i \le N\right\}$ and $\nu = \min\{|f_i(0)| : 1 \le i \le N\}$. We also assume that

$$0 < \langle Au, u\rangle \le |u|^2 \qquad \text{for all } u \in \mathbb{R}^N \setminus \{0\}. \tag{A1}$$

Then we have the following existence result[He]:

Theorem 2.1. *Suppose that (F1), (F2) hold. Let $e \in \widetilde{H}$. Then the problem (V) has a T -periodic solution.*

The proof of this theorem is based on the degree theory. For each $\lambda \in [0,1]$, $\delta \in [0,1]$, we define a mapping $T(\lambda,\delta) : \widetilde{H} \to \widetilde{H}$ by $v = T(\lambda,\delta)u$, where $v \in \widetilde{H}$ is the solution of problem

$$\begin{aligned} v_{tt} &= -\delta \tfrac{d}{dt}F(u) - \lambda u + \delta e(t) \qquad \text{on } [0,T] \\ v(0) &= v(T), v_t(0) = v_t(T) \end{aligned}$$

It then easy to see that $T(\lambda,\delta)$ is a compact mapping. Next we define a homotopy of mappings on $\widetilde{H}$ by

$$H(t)u = \begin{cases} T(1-3(1-\lambda_0)t, 1)u & \text{for } t \in [0,1/3] \text{ and } u \in H \\ T(\lambda_0, 2-3t)u & \text{for } t \in [1/3, 2/3] \text{ and } u \in H. \\ T(3\lambda_0(1-t), 0)u & \text{for } t \in [2/3, 1] \text{ and } u \in H. \end{cases}$$

Then $H : [0,1] \times \widetilde{H} \to \widetilde{H}$ is a homotopy of compact mappings. By calculating the degree of the

homotopy H, we can get the existence of periodic solutions. We can derive some properties of the solutions.

A solution u of problem (V) is said to be an attractor if there exists a neighborhood U of the set $\{(u(t), u_t(t)) : t \in T\} \subset \mathbb{R}^N \times \mathbb{R}^N$ such that for each $(u_0, v_0) \in U$,

$$\lim_{t\to\infty} \sup\{|(\tau(t,(u_0,v_0)), \tau_t(u_0,v_0)) - (v,w)| : (v,w) \in \{(u(t),u_t(t)) : t \in T\}\} = 0,$$

where $\tau(t,(u_0,v_0))$ is the solution of initial value problem

$$\begin{aligned} u_{tt} + \frac{d}{dt}F(u) + Au &= e(t) \\ u(0) &= u_0 \\ u_t(0) &= v_0. \end{aligned}$$

On the other hand, a solution u of (V) is said to be a repeller if there exists a neighborhood U of the set $\{(u(t), u_t(t)) : t \in T\}$ such that for each $(u_0, v_0) \in U$, there exists $t_0 > 0$ such that

$$(\tau(t,(u_0,v_0)), \tau_t(t,(u_0,v_0)) \notin U \qquad \text{for all } t \geq t_0.$$

Theorem 2.2. *Suppose that (F1) and (F2) hold. Let $e \in \widetilde{H}$.*
(1) if $\|e\|$ is sufficiently small, there exists a solution $u \in \widetilde{H}$ of (V) which is a repeller;
(2) if $\|e\|$ is sufficiently large, there exists a solution $u \in \widetilde{H}$ of (V) which is an attractor.

The proof of above theorem is also based on the degree theory.

3. Existence of Limit Cycles

For the existence of limit cycle of autonormous Lienard equation, we consider a coupled van del Pol equations. The existence of limit cycles for coupled van der Pol equations is not yet established except some restrictive cases(cf. [6]). In the present paper, we discuss the existence of limit cycles for coupled van der Pol equations by using S^1-degree theory. To avoid unnecessary complexity, we restrict ourselves to the case that $n = 2$. That is we consider the problem

$$\begin{cases} \ddot{u}_1 + \varepsilon_1(u_1^2 - 1)\dot{u}_1 + u_1 + c_2 u_2 = 0 \\ \ddot{u}_2 + \varepsilon_2(u_2^2 - 1)\dot{u}_2 + c_1 u_1 + u_2 = 0 \end{cases} \tag{P}$$

Our argument below remains valid for the case that $n > 2$. We impose that following condition on c_1 and c_2 :

$$c_1 \cdot c_2 \in (0,1) \cup (1, +\infty) \tag{A}$$

We can now state our main result:

Theorem 3.3. *For any α sufficiently large, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that problem (P) has a nontrivial periodic solution $u \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$ with period $2\pi\alpha$.*

The proof of theorem above is based on the theory of S^1- degree. We will explain the frame work of the theory and show how the theory is applied to our problem.

S^1-degree: We denote by Γ_0 the free abelian group generated by $\mathbb{N}$ and let $\Gamma = \mathbb{Z}_2 \oplus \Gamma_0$. Then $\gamma \in \Gamma$ means $\gamma = \{\gamma_r\}$, where $\gamma_0 \in \mathbb{Z}_2$ and $\gamma_r \in \mathbb{Z}$ for $r \in \mathbb{N}$. Let V be a Hilbert space which is a representation of S^1. For each proper subgroup $\mathbb{Q}$ of S^1 and each S^1-equivariant subset X of V, we denote $X^{\mathbb{Q}}$ the subset of fixed points of $\mathbb{Q}$ in X. For each $U \subset V \oplus \mathbb{R}$ and each S^1 equivariant compact mapping $f : U \to V$, we define, by using the fact that there is a one-to-one correspondence between $\mathbb{N}$ and the proper, closed subgroups $\mathbb{Q}$ of S^1, $Deg(I - f, U) = \{\gamma_r\} \in \Gamma$ by $\gamma_0 = \deg_{S^1}(I - f, U)$ and $\alpha_r = \deg_{\mathbb{Q}}(I - f, U)$ if $r = |\mathbb{Q}|$, where $\mathbb{Q}$ is a closed proper subgroup of S^1(cf. [1] and [5]). The S^1- degree theory has been studied by several authors. The following theorem has been formulated and proved in [1] and describe properties of S^1 -degree.

Theorem 3.4 ([1]). *Let V be a Hilbert space which is a representation of S^1, U be an open bounded, invariant subset of $V \oplus \mathbb{R}$ and $f : U \to V$ is a compact S^1-mapping such that $(I - f)(\partial U) \subset V \setminus \{0\}$. Then there exists a $\Gamma-$valued function $Deg(I - f, U)$ called S^1-degree, satisfying the following properties:*

(a) if $\mathrm{Deg}_{\mathbb{Q}}(I - f, U) \neq 0$, then $(I - f)^{-1}(0) \cap U^{\mathbb{Q}} \neq \phi$,
(b) if $U_0 \subset U$ is open, invariant and $(I - f)^{-1}(0) \cap U \subset U_0$, then

$$\mathrm{Deg}(I - f, U) = \mathrm{Deg}(I - f, U_0);$$

(c) if $h : cl(U) \times [0, 1] \to V$ is an S^1-equivariant homotopy of compact mappings such that $(I - h)(\partial U \times [0, 1]) \subset V \setminus \{0\}$. Then

$$\mathrm{Deg}(I - h_0, U) = \mathrm{Deg}(I - h_1, U).$$

To apply S^1-degree theory to our problem, we need some preparations. We denote by $< \cdot, \cdot >_2$ the scalar product of $L^2([0, 2\pi], \mathbb{R}^2)$. Define

$\mathbb{H}_{per} = \{v : \mathbb{R} \to \mathbb{R}^2 : v$ is absolutely continuous, $< \dot{v}, \dot{v} >_2 :< \infty$ and $v(t) = v(t + 2\pi) :: \forall : t \in \mathbb{R}\}$

and scalar products $< \cdot, \cdot >_{\mathbb{H}_{per}} : \mathbb{H}_{per} \times \mathbb{H}_{per} \to \mathbb{R}$ as follows

$$< w, v >_{\mathbb{H}_{per}} = < w, v >_2 + < \dot{w}, \dot{v} >_2 .$$

Let $S^1 = \{z \in \mathbb{C} :::| z |= 1\}$ be a group of complex numbers with an action given by multiplication. For any fixed $m \in \mathbb{N}$ we denote by $\mathbb{Z}_m$ a cyclic group of order m and define homomorphism $\rho_m : S^1 \to GL(2, \mathbb{R})$ as follows

$$\rho_m\left(e^{\sqrt{-1}\theta}\right) = \begin{bmatrix} \cos(m\theta) & -\sin(m\theta) \\ \sin(m\theta) & \cos(m\theta) \end{bmatrix}.$$

It is obvious that $\mathbb{R}[1, m] := (\mathbb{R}^2, \rho_m)$ is a two-dimensional representation of the group S^1. We will denote by $\mathbb{R}[k, m]$ the direct sum of k copies of representation $\mathbb{R}[1, m]$ and by $\mathbb{R}[k, 0]$ $k-$dimensional trivial representation of the group S^1. Define action $\rho : S^1 \times \mathbb{H}_{per} \to \mathbb{H}_{per}$ of the group S^1 as follows

$$\rho(\theta, v(t)) = v(t + \theta) \tag{3.1}$$

In the following fact we collect some well known properties of the space $\mathbb{H}_{per}$.

Under the above assumptions:

Fact 3.1. 1. $(\mathbb{H}_{per}, <\cdot,\cdot>_{\mathbb{H}_{per}})$ *is a separable Hilbert space,*
2. $(\mathbb{H}_{per}, <\cdot,\cdot>_{\mathbb{H}_{per}})$ *is an orthogonal representation of the group* S^1 *with* S^1-*action given by* (3.1) ,
3. $\mathbb{H}_{per} = \bigoplus_{n=0}^{\infty} \mathbb{R}[2,n]$.

Define

$\mathbb{H} = \{v : \mathbb{R} \to \mathbb{R}^2 : v$ is absolutely continuous, $<\dot{v},\dot{v}>_2 < \infty$ and $v(t) = -v(\pi+t)\ \forall\, t \in \mathbb{R}\}$.

In the following fact we collect some well known properties of the space $\mathbb{H}_0$.

Under the above assumptions:

Fact 3.2. 1. $\mathbb{H} = \left((\mathbb{H}_{perp})^{\mathbb{Z}_2}\right)^{\perp}$,
2. $(\mathbb{H}, <\cdot,\cdot>_{\mathbb{H}})$ *is a separable Hilbert space,*
3. $(\mathbb{H}, <\cdot,\cdot>_{\mathbb{H}})$ *is an orthogonal representation of the group* S^1 *with* S^1-*action given by the restriction of* (3.1) ,
4. $\mathbb{H} = \bigoplus_{n=1}^{\infty} \mathbb{R}[2, 2n-1]$.

Let $v = (v_1, v_2)$ be a periodic solution of (P) with period $2\pi\alpha$ for some $\alpha > 1$. Then by putting $t = \alpha\tau$ and $u(\tau) = (u_1(\tau), u_2(\tau)) = (v_1(\alpha\tau), v_2(\alpha\tau))$, we find that $u = (u_1, u_2) \in \mathbb{H}$ is a $2\pi-$ periodic solution of problem

$$\begin{cases} \ddot{u}_1 + \varepsilon_1\alpha(u_1^2-1)\dot{u}_1 + \alpha^2(u_1 + c_2u_2) = 0 \\ \ddot{u}_2 + \varepsilon_2\alpha(u_2^2-1)\dot{u}_2 + \alpha^2(c_1u_1 + u_2) = 0 \end{cases} \tag{3.2}$$

Here we put

$$F(u) = \begin{pmatrix} \varepsilon_1(\frac{1}{3}u_1^3 - u_1) \\ \varepsilon_2(\frac{1}{3}u_2^3 - u_2) \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & c_2 \\ c_1 & 1 \end{pmatrix}.$$

We define a smooth S^1-equivariant function $\theta : \mathbb{H} \to [0,1]$ by the following formula $\theta(u) = \eta\left(\frac{||u||^2}{2}\right)$.

Denote by $\pi : \mathbb{R}[2,0] \oplus \mathbb{H} \to \mathbb{H}$ the S^1-equivariant orthogonal projection.

For each $\alpha > 0$ and $\delta \in [0,1]$, we define a mapping $G(\cdot,\alpha,\delta) : \mathbb{H} \to \mathbb{H}$ by

$$G(v,\alpha,\delta) = -\delta\alpha\pi\left(\int_0^t F(v(\tau))d\tau\right) + \alpha^2\theta(v)L(v)$$

Then each solution $u \in \mathbb{H}$ of problem $G(u,\alpha,\delta) = u$ for some $(\alpha,\delta) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfies

$$\ddot{u} + \delta\alpha\frac{d}{dt}F(u) + \alpha^2\theta(u)Au = 0 \tag{3.3}$$

We will also consider the following family of differential equations

$$\ddot{u} + \delta\alpha\frac{d}{dt}F(u) + \alpha^2 Au = 0 \tag{3.4}$$

Then one can see that there exists a continuous function $m : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|u\| \le m(\alpha)$ for each solution u of 3.3.

We define a bounded operator $E : \mathbb{H} \to \mathbb{H}$ as follows

$$E(v) = \pi \begin{pmatrix} \varepsilon_1 \displaystyle\int_0^t v_1 dt \\ \varepsilon_2 \displaystyle\int_0^t v_2 dt \end{pmatrix} \qquad \text{for each } v = (v_1, v_2) \in \mathbb{H}.$$

For each $\alpha > 0$ and $\delta \in [0,1]$, we define a mapping $H(\cdot,\cdot,\alpha,\delta) : \mathbb{H} \oplus \mathbb{R} \to \mathbb{H}$ by

$$H(u,\lambda,\alpha,\delta) = G(u,\alpha,\delta) + \lambda\alpha E(u).$$

It is easy to see that $H(\cdot,\cdot,\alpha,\delta)$ is an S^1-equivariant compact mapping. One can see that if $u \in \mathbb{H}$ satisfies $u = H(\cdot,\cdot,\alpha,\delta)$ for $(\alpha,\delta) \in \mathbb{R}^+ \times \mathbb{R}^+$ then

$$\ddot{u} + \delta\alpha\frac{d}{dt}F(u) + \alpha^2\theta(u)Au = \lambda\alpha\frac{d^2}{dt^2}E(u) \tag{3.5}$$

That is

$$\begin{cases} \ddot{u}_1 + \delta\varepsilon_1\alpha(u_1^2 - 1)\dot{u}_1 + \alpha^2\theta(u)(u_1 + c_2u_2) = \varepsilon_1\alpha\lambda\dot{u}_1 \\ \ddot{u}_2 + \delta\varepsilon_2\alpha(u_2^2 - 1)\dot{u}_2 + \alpha^2\theta(u)(c_1u_1 + u_2) = \varepsilon_2\alpha\lambda\dot{u}_2 \end{cases} \tag{3.6}$$

If $\theta(u) > 0$ and $1 + \dfrac{\lambda}{\delta} > 0$, we put $\widetilde{\alpha} = \alpha\sqrt{\theta(u)}$ and $w = \dfrac{u}{\sqrt{1 + \dfrac{\lambda}{\delta}}}$. Then (3.6) can be rewritten as

$$\begin{cases} \ddot{w}_1 + \dfrac{\varepsilon_1(\delta+\lambda)}{\sqrt{\theta(u)}}\widetilde{\alpha}(w_1^2 - 1)\dot{w}_1 + \widetilde{\alpha}^2(w_1 + c_2w_2) = 0 \\ \ddot{w}_2 + \dfrac{\varepsilon_2(\delta+\lambda)}{\sqrt{\theta(u)}}\widetilde{\alpha}(w_2^2 - 1)\dot{w}_2 + \widetilde{\alpha}^2(c_1w_1 + w_2) = 0 \end{cases} \tag{3.7}$$

Then one can see that $w = (w_1, w_2)$ is a solution of (P) with $\varepsilon_1, \varepsilon_2$ and α replaced by $\varepsilon_1\dfrac{\delta+\lambda}{\sqrt{\theta(u)}}$, $\varepsilon_2\dfrac{\delta+\lambda}{\sqrt{\theta(u)}}$ and $\widetilde{\alpha}$.

Lemma 3.1. 1. *if* $c_1c_2 \in (0,1)$, *then*

$$DEG_{\mathbb{Q}}(Id - H(\cdot,\cdot,\alpha,0),U) = \begin{cases} 0, & \mathbb{Q} = S^1, \\ 2, & \mathbb{Q} = Z_{2m-1} \text{ for } m \in \{1,\dots,n_0-1\}, \\ 2, & \mathbb{Q} = Z_{2n_0-1} \text{ and } \mu^-_{n_0} > \frac{1}{\alpha^2}, \\ 1, & \mathbb{Q} = Z_{2n_0-1} \text{ and } \mu^-_{n_0} < \frac{1}{\alpha^2}, \\ 0, & \text{otherwise,} \end{cases}$$

where $U = \{u \in \mathbb{H} : m < \|u\| < M\} \times [-1,1]$,

2. *if* $c_1c_2 > 1$, *then*

$$DEG_{\mathbb{Q}}(Id - H(\cdot,\cdot,\alpha,0),U) = \begin{cases} 0, & \mathbb{Q} = S^1, \\ 1, & \mathbb{Q} = Z_{2m-1} \text{ for } m \in \{1,\dots,n_0\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $U = \{u \in \mathbb{H} : m < \|u\| < M\} \times [-1,1]$.

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