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Approximate Maximization Theorems*

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Abstract

There are many economic problems which are difficult for us to handle directly because of the complexities of the problems or of lack of precise data for the models. Many kinds of possible problems can be constructed to approximate one original problem. In this paper we present the sufficient conditions under which approximations are successful.

1 Introduction

According to an objective, economists make a particular model with a particular type of utility functions or with particular production technologies. On the one hand, we know nothing but the economic data and the theories in the abstract form. We expect that the more data we obtain the more precise approximate model we can get. In fact, Shoven-Whalley (1992) developed the CGE models which were expected to describe the rough behaviors of economies, i.e., the approximation to actual economies. Atkinson (1973), Stern (1976) and Tuomala (1984) calculated the optimal income tax by specifying utility functions of agents. Samuelson (1950), Houthakker (1961) and Richter (1966) constructed the revealed preference theory which presented the method to approximate preferences of agents by observing the market data. We can find many other examples of models of approximation in the literature of economics

Suppose that an objective model is given in some way and that we have a series of practical economic models each of which is an approximation to the objective model. We have a series of solutions to the approximate models. Then we must ask whether solutions in approximate models can approximate to that in the objective model or not. In other words, we have to answer the question whether the solutions converge to the one in the objective model or not. It is needless to say that maximization problems play central roles in economics. And thus the question consists in the problem whether we can approximate some maximization problem by the series of maximization problems.

The purpose of this paper is to present three kinds of approximate maximization theorems by which we can judge whether the approximation is successful.

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2 Approximation Problem

An economic problem is often described by a maximization problem such that:

$$\max W^*(u) \text{ subject to } u \in \mathbb{U}^*, \tag{1}$$

where the function $W^*(\cdot)$ is a real valued function defined on a set $\mathbb{F}$ and where $\mathbb{U}^*$ is a constraint set satisfying $\mathbb{U}^* \subset \mathbb{F}$. The set $\mathbb{F}$ is a topological space. The domain $\mathbb{F}$ can be a subset of $\mathbb{R}^n$, a subset of $\mathbb{L}^p$ space or, more generally, a subset of a topological space[1]. In order to approximate the problem (1), we consider a problem for $\nu \in \mathbb{N}$:

$$\max W^\nu(u) \text{ subject to } u \in \mathbb{U}^\nu \tag{2}$$

where $\mathbb{U}^\nu \subset \mathbb{F}$ is a constraint set and $\mathbb{N}$ is the set of all positive integers. The functions $W^\nu(\cdot), \nu = 1, 2, \ldots$ are real valued functions on $\mathbb{F}$.

There are two cases where we must consider the problem (2) instead of the problem (1). One case is that we can not know the exact form of the problem (1). Therefore we are obliged to construct approximate models. In this situation, the problem (1) is defined as a limit problem of (2). The other is not that we can not know the problem (1) but that there are technical difficulties to solve the problem. And thus (2) is the problem for us to know the approximate behavior of the solution to the problem (1).

On the other hand, we are familiar with the problems such as

$$\max W^\nu(u) \text{ subject to } u \in \mathbb{U}, \nu \in \mathbb{N}, \tag{2'}$$

$$\max W(u) \text{ subject to } u \in \mathbb{U}^\nu, \nu \in \mathbb{N}. \tag{2''}$$

These are subproblems of the problem (2). The relationship between (1) and (2′) is obtained in the form of Ascoli-Alzelá's theorem(see, e.g., Royden (1963)). On the other hand, we face with the problems (1) and (2″) frequently when we study the continuity of the demand function, where the function W is interpreted as a utility function and the set $\mathbb{U}^\nu$ as a budget set relative to price p^ν. In this event, the Berge's maximum theorem (see Berge (1963))[2] establishes the continuity of the solutions to (1) and (2″). It must be stressed that not only the objective function W^ν but also the constraint set $\mathbb{U}^\nu$ can vary with ν in our problem (2).

We assume the basic relations between (1) and (2) as follows.

Assumption 1 *The sequence of functions $\{W^\nu(\cdot),\ \nu \in \mathbb{N}\}$ is convergent pointwise to $W^*(\cdot)$, that is, the sequence of real values $W^\nu(u)$ tends to $W^*(u)$ as ν tends to infinity for all $u \in \mathbb{F}$.*

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[1] $\mathbb{R}^n$ denotes the set of all n-dimensional real vectors.

[2] We can show that the maximum theorem implies one of our theorems, Approximate Maximization Theorem III below when the space $\mathbb{F}$ is a regular Hausdorff space.

In other words, the function $W^*(\cdot)$ in the problem (1) can be defined by the limit in the sequences of functions $W^\nu(\cdot)$, $\nu \in \mathbb{N}$. Denote the solution to the problem (2) by u^ν. Furthermore we assume:

Assumption 2 *There exists a limit point u^* of the sequence $(u^\nu)_{\nu \in \mathrm{N}}$ in $\mathbb{F}$.*

Assumption 2 implies that none of the problems in (2) can approximate the problem (1) if the sequence of the solutions $(u^\nu)_{\nu \in \mathrm{N}}$ of (2) has no limit points. And thus Assumption 2 is a natural assumption.

Even if Assumptions 1 and 2 hold, it is not sufficient for a problem in (2) to be an approximation to (1). The solutions of (2) may be far from that of (1). We need some kind of continuity about the solutions. In order for a problem in (2) to be an approximation to (1) we must establish following two properties.

(A) u^* is a solution to the problem (1),

(B) $W^\nu(u^\nu) \to W^*(u^*)$ as $\nu \to \infty$.

We introduce, here, two concepts of semi-uniform convergence. The sequence $(W^\nu)_{\nu \in \mathrm{N}}$ is *lower semi-uniform convergent* to a function W^* *at* $u \in \mathbb{F}$ if for any positive ε there exist a number ν^0 and a neighborhood U of u such that

$$\nu \geqq \nu^0 \quad \Rightarrow \quad (W^*(u') < W^\nu(u') + \varepsilon, \ \forall u' \in U).$$

The sequence $(W^\nu)_{\nu \in \mathrm{N}}$ is *lower semi-uniform convergent* to a function W^* if the sequence is lower semi-uniform convergent to W^* at u for all $u \in \mathbb{F}$.

The sequence $(W^\nu)_{\nu \in \mathrm{N}}$ is *upper semi-uniform convergent* to a function W^* *at* $u \in \mathbb{F}$ if for any positive ε there exist a number ν^0 and a neighborhood U of u such that

$$\nu \geqq \nu^0 \quad \Rightarrow \quad (W^\nu(u') < W^*(u') + \varepsilon, \ \forall u' \in U).$$

The sequence $(W^\nu)_{\nu \in \mathrm{N}}$ is *upper semi-uniform convergent* to a function W^* if the sequence is upper semi-uniform convergent to W^* at u for all $u \in \mathbb{F}$. We call the sequence $(W^\nu)_{\nu \in \mathrm{N}}$ *locally uniform convergent* to W^* if the sequence is upper semi-uniform convergent and lower semi-uniform convergent to W^*.

Furthermore, let us define the semi-continuity of a real valued function. A real valued function W the domain of which is $\mathbb{F}$ is *upper semi-continuous at* $u \in \mathbb{F}$ if for any positive number ε there exists a neighborhood of V of u such that $u' \in V$ implies

$$W(u') < W(u) + \varepsilon.$$

A function W is *upper semi-continuous*[3] if W is upper semi-continuous at u for all $u \in \mathbb{F}$.

[3]Note that the concept of semi-continuous function is different from that of semi-continuous corre-

We define three kinds of sets as possible limits of the sequence of sets $(\mathbb{U}^\nu)_{\nu\in\mathbb{N}}$ as follows[4]:

$$\mathbb{U}^{\mathrm{a}} \stackrel{\mathrm{def}}{=} \cup_{k\in\mathbb{N}} \cap_{\nu\geqq k} \mathbb{U}^\nu, \tag{3}$$

$$\overline{\mathbb{U}}^{\mathrm{a}} : \text{ closure of } \mathbb{U}^{\mathrm{a}} \text{ in } \mathbb{F}, \tag{4}$$

$$\mathbb{U}^{\mathrm{t}} \stackrel{\mathrm{def}}{=} \{u \in \mathbb{F} \mid \text{ for any neighborhood } V \text{ of } u,\ \exists\bar{\nu} : \nu > \bar{\nu} \Rightarrow V \cap \mathbb{U}^\nu \neq \emptyset\}. \tag{5}$$

It is clear that $\mathbb{U}^{\mathrm{a}} \subset \overline{\mathbb{U}}^{\mathrm{a}} \subset \mathbb{U}^{\mathrm{t}}$ and that $\mathbb{U}^{\mathrm{t}}$ is a closed set.

3 Basic Results

We will give some lemmata in order to prove main theorems.

Lemma 1 *Assume that $u^* \in \mathbb{U}^{\mathrm{a}}$. Then it holds that*

$$W^*(u^*) \leqq \sup_{u\in\mathbb{U}^{\mathrm{a}}} W^*(u) \leqq \liminf_{\nu\to\infty} W^\nu(u^\nu) \leqq \limsup_{\nu\to\infty} W^\nu(u^\nu).$$

[Proof] The first and the third inequalities are obvious. Firstly, suppose that $\sup_{u\in\mathbb{U}^{\mathrm{a}}} W^*(u)$ exists. Let ε be an arbitrary positive number. Then there exists a point u^0 in $\mathbb{U}^{\mathrm{a}}$ satisfying

$$\sup_{u\in\mathbb{U}^{\mathrm{a}}} W^*(u) - \varepsilon \leqq W^*(u^0).$$

We find at once that there exists a sufficiently large positive number ν^0 such that

$$(\nu \geqq \nu^0) \Rightarrow \left(W^*(u^0) - \varepsilon \leqq W^\nu(u^0) \leqq W^\nu(u^\nu)\right).$$

It holds therefore that

$$\sup_{u\in\mathbb{U}^{\mathrm{a}}} W^*(u) \leqq W^\nu(u^\nu) + 2\varepsilon, \text{ for all } \nu \geqq \nu^0.$$

Then we have

$$\sup_{u\in\mathbb{U}^{\mathrm{a}}} W^*(u) \leqq \liminf_{\nu\to\infty} W^\nu(u^\nu).$$

Next, suppose that $\sup_{u\in\mathbb{U}^{\mathrm{a}}} W^*(u) = \infty$. Let k be an arbitrary positive integer. Define W_k^η for each $\eta = *, 1, 2, \ldots$:

$$\begin{aligned} W_k^\eta(u) &= W^\eta(u) \quad \text{if} \quad W^\eta(u) \leqq k, \\ &= k \qquad\qquad \text{otherwise} \ . \end{aligned}$$

[4]The symbol '$\stackrel{\mathrm{def}}{=}$' means that the left hand side is defined by the right hand side.

We can apply the theorem for the sequences $(W_k^\nu)_{\nu\in\mathbb{N}}$ and $(u^\nu)_{\nu\in\mathbb{N}}$. Then we have

$$\sup_{u\in\mathbb{U}^a} W_k^*(u) \leqq \liminf_{\nu\to\infty} W_k^\nu(u^\nu).$$

Letting $k \to \infty$, and we can see the lemma holds. ∎

Based on the previous lemma, we obtain:

Lemma 2 *Assume that $u^* \in \overline{\mathbb{U}}^a$ and that W^* is continuous. Then it holds that*

$$W^*(u^*) \leqq \sup_{u\in\overline{\mathbb{U}}^a} W^*(u) \leqq \liminf_{\nu\to\infty} W^\nu(u^\nu) \leqq \limsup_{\nu\to\infty} W^\nu(u^\nu).$$

[Proof] Note that $u^* \in \overline{\mathbb{U}}^a$ implies that $\mathbb{U}^a$ is non-empty. And thus the continuity of W^* implies

$$\sup_{u\in\mathbb{U}^a} W^*(u) = \sup_{u\in\overline{\mathbb{U}}^a} W^*(u).$$

This together with Lemma 1 completes the proof.∎

A real valued function W the domain of which is $\mathbb{F}$ is *lower semi-continuous at $u \in \mathbb{F}$* if and only if for any positive number ε there exists a neighborhood of V of u such that $u' \in V$ implies

$$W(u) < W(u') + \varepsilon.$$

We call a function W *lower semi-continuous* if W is lower semi-continuous at u for all $u \in \mathbb{F}$. Next we are to establish the similar result with respect to the set $\mathbb{U}^t$.

Lemma 3 *Assume that W^* is lower semi-continuous and that the sequence of functions $(W^\nu)_{\nu\in\mathbb{N}}$ is lower semi-uniform convergent to W^*. Then it holds that*

$$W^*(u^*) \leqq \sup_{u\in\mathbb{U}^t} W^*(u) \leqq \liminf_{\nu\to\infty} W^\nu(u^\nu) \leqq \limsup_{\nu\to\infty} W^\nu(u^\nu).$$

[Proof] It is obvious that $\mathbb{U}^t \neq \emptyset$ since $u^* \in \mathbb{U}^t$. It suffices for us to show that the lemma is true for the case that $\sup_{u\in\mathbb{U}^t} W^*(u)$ exists. The first inequality is obvious. The third inequality holds since $u^* \in \mathbb{U}^t$. Let ε be an arbitrary positive number. Then there exists a point u^0 in $\mathbb{U}^t$ satisfying

$$\sup_{u\in\mathbb{U}^t} W^*(u) - \varepsilon \leqq W^*(u^0).$$

By the lower semi-continuity of W^*, there exists a neighborhood V of u^0 satisfying

$$W^*(u^0) - \varepsilon \leqq W^*(u),\ \forall u \in V.$$

By the lower semi-uniform convergence of W^ν to W^*, there exist an integer ν^0 and a neighborhood U of u^0 satisfying

$$W^*(u) \leqq W^\nu(u) + \varepsilon,\ \forall \nu \geqq \nu^0,\ \forall u \in U.$$

Furthermore, it is clear that $(V \cap U) \cap \mathbb{U}^\nu \neq \emptyset$ for all $\nu \geqq \nu^1$ for a sufficiently large number ν^1. And thus there exists a point $\tilde{u}^\nu \in \mathbb{U}^\nu$, $\nu \geqq \max\{\nu^0, \nu^1\}$ satisfying

$$W^*(u^0) \leqq W^*(\tilde{u}^\nu) + \varepsilon \leqq W^\nu(\tilde{u}^\nu) + 2\varepsilon \leqq W^\nu(u^\nu) + 2\varepsilon.$$

This leads us to

$$W^*(u^0) \leqq \liminf_{\nu \to \infty} W^\nu(u^\nu) + 2\varepsilon.$$

Summarizing, we have

$$\sup_{u \in \mathbb{U}^t} W^*(u) \leqq \liminf_{\nu \to \infty} W^\nu(u^\nu) + 3\varepsilon.$$

Then we have

$$\sup_{u \in \mathbb{U}^t} W^*(u) \leqq \liminf_{\nu \to \infty} W^\nu(u^\nu).$$

∎

These lemmata 1, 2, and 3 imply that the property (**A**) holds when we can establish $\limsup_{\nu \to \infty} W^\nu(u^\nu) \leqq W^*(u^*)$. By definition of convergence, the property (**B**) is equivalent to the fact that $\liminf_{\nu \to \infty} W^\nu(u^\nu) = \limsup_{\nu \to \infty} W^\nu(u^\nu) = W^*(u^*)$. This implies the property (**A**) holds when the property (**B**) is true.

Lemma 1 holds without any serious conditions. Lemma 3, on the other hand, holds under assumptions of semi-continuity and semi-uniform convergence. It may be of some interest to find the maximal set $\mathbb{U}^m$ under which the relation

$$\sup_{u \in \mathbb{U}^m} W^*(u) \leqq \liminf_{\nu \to \infty} W^\nu(u^\nu)$$

holds without additional assumptions.

Lemma 4 *It holds that*

$$\mathbb{U}^m = \tilde{\mathbb{U}} \stackrel{\text{def}}{=} \cap_{\varepsilon > 0} \cup_{k \in \mathbb{N}} \cap_{\nu \geqq k} \left\{ u \in \mathbb{F} \;\middle|\; W^\nu(u) < \max_{\hat{u} \in \mathbb{U}^\nu} W^\nu(\hat{u}) + \varepsilon \right\}.$$

[Proof] Firstly we show that $\tilde{\mathbb{U}} \subset \mathbb{U}^m$. The relation $\sup_{u\in\mathbb{U}^m} W^*(u) \leqq \liminf_{\nu\to\infty} W^\nu(u^\nu)$ holds when $\tilde{\mathbb{U}} = \emptyset$ since $\sup_{u\in\emptyset} W^*(u) = -\infty$. Let $\tilde{\mathbb{U}} \neq \emptyset$. For an arbitrary positive number ε there exists $\tilde{u} \in \tilde{\mathbb{U}}$ such that

$$\sup_{u\in\tilde{\mathbb{U}}} W^*(u) \leqq W^*(\tilde{u}) + \varepsilon.$$

The fact $\tilde{u} \in \tilde{\mathbb{U}}$ implies that for every $\varepsilon > 0$ there exists a positive number k_0 such that

$$\nu \geqq k_0 \;\Rightarrow\; W^\nu(\tilde{u}) < \max_{u\in\mathbb{U}^\nu} W^\nu(u) + \varepsilon = W^\nu(u^\nu) + \varepsilon.$$

Redefining k_0 if necessary, we obtain

$$\nu \geqq k_0 \;\Rightarrow\; W^*(\tilde{u}) \leqq W^\nu(\tilde{u}) + \varepsilon.$$

These imply that $\sup_{u\in\tilde{\mathbb{U}}} W^*(u) \leqq W^\nu(u^\nu) + 3\varepsilon$ if $\nu \geqq k_0$. Hence we have

$$\sup_{u\in\tilde{\mathbb{U}}} W^*(u) \leqq \liminf_{\nu\to\infty} W^\nu(u^\nu).$$

Secondly, let $\mathbb{U}$ be a subset of $\mathbb{F}$ satisfying $\mathbb{U} \supset \tilde{\mathbb{U}}$ and $\mathbb{U} \neq \tilde{\mathbb{U}}$. Let $\hat{u}$ be an element of the set $\mathbb{U} \setminus \tilde{\mathbb{U}}$. The there exists a positive number $\hat{\varepsilon}$ such that for any positive integer k there exists a number $\nu_k \geqq k$ satisfying

$$W^{\nu_k}(\hat{u}) \geqq W^{\nu_k}(u^{\nu_k}) + \hat{\varepsilon}.$$

Letting k tend to infinity and we have

$$\sup_{u\in\mathbb{U}} W^*(u) \geqq W^*(\hat{u}) > \liminf_{\nu\to\infty} W^\nu(u^\nu).$$

This implies that $\tilde{\mathbb{U}} = \mathbb{U}^m$.∎

Let us turn to the problem to find sufficient conditions for Lemma 1 to hold in equalities.

Lemma 5 *Suppose that the function $W^*(\cdot)$ is upper semi-continuous at $u^* \in \mathbb{F}$. The relation*

$$\limsup_{\nu\to\infty} W^\nu(u^\nu) \leqq W^*(u^*).$$

holds when one of the following conditions holds:

(i) **USUC** (Upper Semi-Uniform Convergence) at u^*. The sequence $(W^\nu)_{\nu\in\mathbb{N}}$ is upper semi-uniform convergent to W^* at u^*.

(ii) **USEC** (Upper Semi Equi-Continuity) at u^*. For any positive number ε there exist a neighborhood V of u^* and a positive integer ν^0 satisfying

$$(\nu \geqq \nu^0) \;\Rightarrow\; (W^\nu(u) \leqq W^\nu(u^*) + \varepsilon,\ \forall u \in V).$$

[Proof] (i) Let ε be any positive number. Let V and ν^0 be the neighborhood of u^* and the positive integer satisfying **USUC**. From the upper semi-continuity of $W^*(\cdot)$ at u^*, there exists a positive integer ν^1 such that $W^*(u^\nu) \leqq W^*(u^*) + \varepsilon$ for any $\nu \geqq \nu^1$. In addition to this, there exists a positive number ν^2 such that $u^\nu \in V$ when $\nu \geqq \nu^2$. Therefore, we have $W^\nu(u^\nu) \leqq W^*(u^\nu) + \varepsilon$ for any $\nu \geqq \bar{\nu} \stackrel{\text{def}}{=} \max(\nu^0, \nu^1, \nu^2)$. Then it holds that $W^\nu(u^\nu) \leqq W^*(u^*) + 2\varepsilon$ when $\nu \geqq \bar{\nu}$. This implies $\sup\{W^{\bar{\nu}}(u^{\bar{\nu}}), W^{\bar{\nu}+1}(u^{\bar{\nu}+1}), \dots\} \leqq W^*(u^*) + 2\varepsilon$. This leads us to $\limsup_{\nu\to\infty} W^\nu(u^\nu) \leqq W^*(u^*) + 2\varepsilon$ then to $\limsup_{\nu\to\infty} W^\nu(u^\nu) \leqq W^*(u^*)$ since ε is arbitrary. This completes the lemma.
(ii) For any positive ε there exist V and ν^0 satisfying **USEC**. The fact that $W^\nu(u^*)$ tends to $W^*(u^*)$ as ν tends to infinity implies that there exists a number ν^1 such that $W^\nu(u^*) \leqq W^*(u^*) + \varepsilon$ if $\nu \geqq \nu^1$. The fact that the sequence $(u^\nu)_{\nu\in\mathbb{N}}$ converges to u^* implies that there exists a positive integer ν^2 such that $u^\nu \in V$ for any $\nu \geqq \nu^2$. Then we know that we have $W^\nu(u^\nu) \leqq W^*(u^*) + 2\varepsilon$ if $\nu \geqq \max(\nu^0, \nu^1, \nu^2)$.∎

Note that the upper semi-continuity of $W^*(\cdot)$ at u^* is indispensable in Lemma 5. We present here an example that the lemma does not hold without upper semi-continuity at u^*. Let $\mathbb{F}$ be a closed interval of real numbers $[0,2]$. And let $\mathbb{U}^\nu = [1 - 1/\nu, 2]$ for each $\nu \in \mathbb{N}$. Define $W^\nu(\cdot)$ as:

$$\begin{aligned} W^\nu(u) &= -u + 1, && \text{when } 0 \leqq u < 1 \\ &= -u, && \text{when } 1 \leqq u \leqq 2. \end{aligned}$$

Therefore it is clear that $W^*(u) = W^\nu(u)$ for all $u \in \mathbb{F}$ and that $\mathbb{U}^* = \mathbb{U}^{\mathrm{a}} = \mathbb{U}^{\mathrm{t}} = [1,2]$. Hence it holds that that $0 = \limsup_{\nu\to\infty} W^\nu(u^\nu) = \liminf_{\nu\to\infty} W^\nu(u^\nu) > \sup_{u\in\mathbb{U}^{\mathrm{a}}} W^*(u) = W^*(u^*) = -1$.

The condition **USUC** is useful when we can know the concrete functional form of $W^*(\cdot)$. On the other hand, **USEC** is practical conditions when we do not know the functional form of $W^*(\cdot)$. It is noteworthy that the local uniform convergence of the functions W^ν, $\nu \in \mathbb{N}$ to W^* and the continuity of W^* imply **USUC** and **USEC**.

4 Main Theorems

We can establish following three Approximate Maximization Theorems.

Theorem 1 (**Approximate Maximization Theorem I**) *Assume that* $u^* \in \mathbb{U}^* = \mathbb{U}^a$ *and that the function* $W^*(\cdot)$ *is upper semi-continuous at* u^*. *Then the properties* (**A**) *and* (**B**) *hold when either* **USUC** *at* u^* *or* **USEC** *at* u^* *is true.*

Theorem 2 (**Approximate Maximization Theorem II**) *Assume that* $u^* \in \mathbb{U}^* = \overline{\mathbb{U}}^a$ *and that the function* $W^*(\cdot)$ *is continuous. Then the properties* (**A**) *and* (**B**) *hold when either* **USUC** *at* u^* *or* **USEC** *at* u^* *is true.*

Theorem 3 (**Approximate Maximization Theorem III**) *Assume* (i) *that* $\mathbb{U}^* = \mathbb{U}^t$, (ii) *that the functions* W^* *is continuous, and* (iii) *that the sequence of functions* W^ν *is locally uniform convergent to* W^*. *Then the properties* (**A**) *and* (**B**) *hold.*

These theorems are direct consequences of Lemmata thus far developed. In fact, Lemmata 1 and 5 imply Theorem 1. Lemmata 2 and 5 imply Theorem 2. Finally, Lemmata 3 and 5 lead us to Theorem 3.

Let us present a lemma that is very useful in applying Approximate Maximization Theorems to practical economic models.

Lemma 6 *Suppose that the sequence* $\mathbb{U}^\nu, \nu = 1, 2, \ldots$ *is monotonic (i.e., either* $\mathbb{U}^\nu \subset \mathbb{U}^{\nu+1}$ *for all* $\nu \in \mathbb{N}$ *or* $\mathbb{U}^\nu \supset \mathbb{U}^{\nu+1}$ *for all* $\nu \in \mathbb{N}$*), then* $\overline{\mathbb{U}}^a = \mathbb{U}^t$.

[Proof] Suppose that the sequence $\mathbb{U}^\nu$, $\nu \in \mathbb{N}$ is increasing. Let $u \in \mathbb{U}^t$. For any neighborhood V of u there exists ν^V such that $V \cap \mathbb{U}^\nu \neq \emptyset$ for each $\nu \geqq \nu^V$. Therefore there exists a point u^V satisfying $u^V \in V \cap \mathbb{U}^\nu$ for any $\nu \geqq \nu^V$, since the sequence $\mathbb{U}^\nu, \nu \in \mathbb{N}$ is increasing. This implies $u^V \in \mathbb{U}^a$. Let $\mathscr{B}(u)$ be the set of all neighborhoods of u in $\mathbb{F}$. The generalized sequence $\left(u^V\right)_{V \in \mathscr{B}(u)}$ thus obtained in $\mathbb{U}^a$ is convergent to u.

Next let us consider the case that the sequence of sets $\{\mathbb{U}^\nu,\ \nu \in \mathbb{N}\}$ is decreasing, i.e., $\mathbb{U}^\nu \supset \mathbb{U}^{\nu+1}$ for all $\nu \in \mathbb{N}$. In this case $\mathbb{U}^a$ is identical with $\cap_{\nu \in \mathbb{N}} \mathbb{U}^\nu$. Let u be a point not in $\overline{\mathbb{U}}^a$. Then there exists a neighborhood V of u satisfying $V \cap (\cap_{\nu \in \mathbb{N}} \mathbb{U}^\nu) = \emptyset$. Then there exists a number $\bar{\nu}$ such that $V \cap \mathbb{U}^{\bar{\nu}} = \emptyset$. This implies that $V \cap \mathbb{U}^\nu = \emptyset$ when $\nu \geqq \bar{\nu}$. Then we obtain $u \notin \mathbb{U}^t$.∎

Now, let us study the theorems in their relations to the Berge's maximum theorem which is stated as follows.

> **Berge's Maximum theorem.** *If* ϕ *is a continuous real valued function in* $X \times Y$ *and* Γ *is a continuous correspondence of* X *into* Y *such that, for each* x, $\Gamma(x) \neq \emptyset$, *then the numerical function* M *defined by* $M(x) = \max\{\phi(x, y) \mid y \in \Gamma(x)\}$ *is upper semi-continuous in* X *and the mapping* Φ *defined by* $\Phi(x) = \{y \mid y \in \Gamma(x), \phi(x, y) = M(x)\}$ *is a upper semi-continuous correspondence in* X *into* Y.

Note that a correspondence Γ is continuous when Γ is upper and lower semi-continuous. A Necessary and sufficient condition for Γ to be upper semi-continuous is that the set $\Gamma(x)$ is compact for each x and that for each open set G in Y the set $\{x \in X \mid \Gamma(x) \subset G\}$ is open (see Berge(1963, page 110, Theorem 2)).

It is easy to construct the basic problems (1) and (2) by using the function ϕ and the correspondence Γ in Berge's maximum theorem. In fact, let

$$\mathbb{F} = Y, \qquad [0,1] = X \tag{6}$$

$$W^{\nu}(y) \stackrel{\text{def}}{=} \phi\left(\frac{1}{\nu}, y\right), \quad \mathbb{U}^{\nu} \stackrel{\text{def}}{=} \Gamma\left(\frac{1}{\nu}\right), \ \nu \in \mathbb{N} \tag{7}$$

$$W^{*}(y) \stackrel{\text{def}}{=} \phi(0, y), \qquad \mathbb{U}^{*} \stackrel{\text{def}}{=} \Gamma(0), \tag{8}$$

and we can define the problems (1) and (2). Then the following corollary is a direct consequence of Berge's maximum theorem.

Corollary 1 *Assume that the set X, Y are a real interval $[0,1]$ and a topological space respectively and that ϕ is a continuous real valued function in $X \times Y$ and Γ is a continuous mapping of X into Y such that, for each x, $\Gamma(x) \neq \emptyset$. The properties* (**A**) *and* (**B**) *hold when the problems* (1) *and* (2) *are defined by using sets and functions in* (6), (7), and (8).

The continuity of the function ϕ on the set $X \times Y$ implies the pointwise convergence of $W^{\nu}(u)$ to $W^{*}(u)$ for arbitrary given $u \in \mathbb{F} = Y$. This implies the functions obtained in (7) satisfy Assumption 1. The solution u^{ν} to the problem corresponding to (2) exists since Γ is compact valued. Finally, the sequence $u^{\nu}, \nu \in \mathbb{N}$ has a limit point because of the continuity of Γ. And thus Assumption 2 is also satisfied. This result may seem to imply that Berge's maximum theorem presents a new approximate maximization theorem. The corollary is, however, within the scope of our Approximate Maximization theorem III. In fact, we can establish:

Lemma 7 *Assume that the set X, Y are a real interval $[0,1]$ and a regular Hausdorff space respectively and that ϕ is a continuous real valued function in $X \times Y$ and Γ is a continuous mapping of X into Y such that, for each x, $\Gamma(x) \neq \emptyset$. Define functions $W^{*}, W^{\nu}, \nu \in \mathbb{N}$ and sets $\mathbb{U}^{*}, \mathbb{U}^{\nu}, \nu \in \mathbb{N}$ as in* (6), (7), *and* (8). *Then it holds that $\mathbb{U}^{*} = \mathbb{U}^{t}$ and that W^{ν} converges locally uniformly to W^{*}.*

This implies that the sufficient condition in Approximate Maximization Theorem III holds under the conditions of Berge's maximum theorem. Therefore we can say the sufficient condition of Approximate Maximization Theorem III is weaker than that in Corollary 1.

[Proof] We can define functions $W^{\nu}(\cdot), W^{*}(\cdot)$ and sets $\mathbb{U}^{\nu}(\cdot), \mathbb{U}^{*}(\cdot)$ as in (6), (7), and (8).

[Step 1: **(continuity of $\Gamma(x)$) $\Rightarrow$ ($\mathbb{U}^* = \mathbb{U}^t$)**]
The correspondence $\Gamma(x)$ is upper and lower semi-continuous since the correspondence is continuous. Let $y \in \mathbb{U}^* = \Gamma(0)$ and let V be an arbitrary neighborhood of y in $\mathbb{F}$. We have $y \in V \cap \Gamma(0) \neq \emptyset$. The lower semi-continuity of Γ implies that there exists a number ν^0 such that the fact $\nu \geqq \nu^0$ implies $\Gamma(1/\nu) \cap V \neq \emptyset$. Then we have $y \in \mathbb{U}^t$. On the other hand, suppose that $y \notin \Gamma(0)$. The set $\Gamma(0)$ is a closed set since a compact subset in the Hausdorff space Y is closed (see Kelley (1955, p141)). And thus, there exist a neighborhood V_y of y and an open set G containing $\Gamma(0)$ such that $G \cap V_y = \emptyset$, since the space Y is regular. The upper semi-continuity of $\Gamma(\cdot)$ implies that there exists a number ν^0 satisfying that $\Gamma(1/\nu) \subset G$ if $\nu \geqq \nu^0$. We obtain that

$$V_y \cap \mathbb{U}^\nu = V_y \cap \Gamma(1/\nu) \subset V_y \cap G = \emptyset, \ \forall \nu \geqq \nu^0.$$

This relation implies that $y \notin \mathbb{U}^t$. This leads us to $\mathbb{U}^* = \mathbb{U}^t$.

[Step 2: **(continuity of $\phi(x,y)$) $\Rightarrow$ (local uniform convergence of W^ν to W^*)**]
Suppose that $\phi(x,y)$ is continuous. Let ε be an arbitrary positive real number. There exists a neighborhood $U(0) \times V(y_0)$ of the point $(0, y_0) \in [0,1] \times Y$ such that $|\phi(x,y) - \phi(0,y_0)| < \varepsilon$ holds when $(x,y) \in U(0) \times V(y_0)$. There exists a positive integer ν^0 corresponding to $U(0)$ satisfying the property $1/\nu \in U(0)$ if $\nu \geqq \nu^0$. The continuity of $\phi(0,y)$ implies that there exists a neighborhood V' of y_0 such that for any y satisfying $y \in V(y_0) \cap V'$ and for any ν satisfying $\nu \geqq \nu^0$, it holds that

$$\begin{aligned} |W^\nu(y) - W^*(y)| &= |\phi(1/\nu, y) - \phi(0,y)| \\ &\leqq |\phi(1/\nu, y) - \phi(0,y_0)| + |\phi(0,y_0) - \phi(0,y)| < 2\varepsilon. \end{aligned}$$

This fact leads us to the local uniform convergence of W^ν to W^*.

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