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Note on homotopy and index in incomplete markets

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Abstract

The purpose of this paper is to present a method to compute the index in an incomplete market economy. We show that generically in endowments and the asset structure the index theorem holds when the deficiency of markets, $S-J$, is even. The result is based on the indices of homotopies which have been presented for the computation of an equilibrium in incomplete market economies.

1 Introduction

The index theorem was first introduced to economics by Dierker (1972) who applied it to the Arrow-Debreu exchange model (GE model). Though the index theorem is a mathematical concept, it has important economic implications as Mas-Colell, Whinston and Green (1995) [p.593] point out for example. In particular, it implies that by knowing the index at one equilibrium of an economy, we can get some information as for existence of other equilibria of the economy. The existence of an equilibrium with index -1 implies that the economy has at least two other equilibria with index +1. This information cannot be obtained from the modulo-2 degree theorem.

The general equilibrium model with incomplete markets (GEI model) is an extension of the GE model so that it describes the trading mechanism for uncertainty more precisely. Thus it is a natural question to ask whether the index theorem holds or not in GEI models.

Very little is known about the index theorem in GEI models. Hens (1991) proved the index theorem for a GEI model with a single commodity. Schmedders (1999) proved it for restrictive cases where his homotopy algorithm for computation of an equilibrium is effective. To the best of our knowledge so far, no general result to this problem has been

It is well known that the fixed point index is homotopy invariant. Actually, a simple proof of the index theorem for the GE economy is given by considering a homotopy between excess demand function of a single agent economy and that of the total economy as shown in Garcia and Zangwill (1981), Mas-Colell (1985), Mas-Colell, Winston and Green (1995). Our approach is a variation of this approach.

It is also well known that the above mentioned simple proof cannot be applied directly to GEI models because of discontinuity of demand functions. The problem that arises is the same as that in computation of an equilibrium based on the homotopy method. Brown, DeMarzo and Eaves (1996b) and DeMarzo and Eaves (1996) presented homotopy methods to compute an equilibrium in GEI models overcoming this difficulty. We exploit these results. Actually, what we show in this paper is how the indices of these homotopies are related. As a result, we can prove the index theorem in the case where $S-J$ is even, where S and J are the number of states and securities, respectively.

2 The GEI model

The basic GEI model describes an exchange economy over two time periods ($T=0,1$) with uncertainty over the state of nature in period 1.

At time $T=0$, each of the I agents ($i=1,2,\ldots,I$) in the economy know the state of nature ($s=0$) in time $T=0$, but not know which of the S possible states ($s=1,2,\ldots,S$) at time $T=1$ will occur. In each state $s=0,1,\ldots,S$ there are N goods ($n=1,\ldots,N$). Thus $M=(S+1)N$ is the total number of goods. For each of the goods, there exists a spot market in every state.

The vector of spot prices is denoted by $p=(p_0,\ldots,p_S)\in P\equiv\{p\in R^M_{++}|p_{SN}=1\}$, where $p_s=(p_{s1},\ldots,p_{sN})$ is the vector of prices of N goods at state s. Note that prices are normalized so that $p_{SN}=1$. Though we adapt this normalization for simplicity of the statement, this specification does not affect our result in this paper. We often use notation $p_{\mathbf{1}}$ to denote the prices of goods at states in period 1; $p_{\mathbf{1}}=(p_1,\ldots,p_S)$.

The consumption of agent i is denoted by $x^i=(x^i_0,\ldots,x^i_S)^T\in R^M_{++}$ where $x^i_s=(x^i_{s1},\ldots,x^i_{sN})^T\in R^N_{++}$ denotes the consumption at state s and the symbol T denotes the transpose of vector or matrix. We often write $x^i_{\mathbf{1}}=(x^i_1,\ldots,x^i_S)^T$ to denote the consumption in period 1.

The characteristics of agent i are given by her initial endowment and her preference ordering. The initial endowment is denoted by $\omega^i=(\omega^i_0,\ldots,\omega^i_S)^T\in R^M_{++}$, where $\omega^i_s=(\omega^i_{s1},\ldots,\omega^i_{sN})^T$ denotes the initial endowment at state s. We also write $\omega^i_{\mathbf{1}}$ to denote $(\omega^i_1,\ldots,\omega^i_S)^T$. The preference ordering is represented by a utility function $u^i:R^M_{++}\to R$

having standard properties: u^i is smooth ($u^i \in C^\infty$) and differentiably strictly monotone ($Du^i(x) \in R^M_{++}$ for all $x \in R^M_{++}$), satisfies a boundary condition ($\{x \in R^M_{++} | u^i(x) \geq u^i(\bar{x})\}$ is closed in R^M_{++} for all $\bar{x} \in R^M_{++}$), and represents differentiably strictly convex preferences ($h^T D^2 u^i(x) h < 0$ for all $h \neq 0$ such that $Du^i(x)h = 0$).

Agents face a separate budget constraint in every state of nature. In order to transfer income between states, agents hold assets. There are $J(\leq S)$ real assets traded on financial markets at asset prices $q = (q_1, \ldots, q_J) \in R^J$. One unit of asset j promises a bundle of goods $A^j_s = (A^j_{s1}, \ldots, A^j_{sN})^T \in R^N$ in state s, $s = 1, \ldots, S$. Therefore $A^j = (A^j_1, \ldots, A^j_S)^T \in R^{SN}$ characterizes asset j and $A = (A^1, \ldots, A^J) \in R^{SNJ}$ characterizes the total assets in the economy.

For given spot prices $p \in P$ and asset prices $q \in R^J$ the payoff of one unit of asset j, which is bought at price q_j at $s = 0$, is $p_s A^j_s$ for each $s = 1, \ldots, S$. We define

$$A(p) = \begin{bmatrix} p_1 A^1_1 & \ldots & p_1 A^J_1 \\ \vdots & & \vdots \\ p_S A^1_S & \ldots & p_S A^J_S \end{bmatrix},$$

and call this $S \times J$ matrix $A(p)$ the payoff matrix. Asset holding of agent i is denoted by $\theta^i = (\theta^{i1}, \ldots, \theta^{iJ})^T$, where θ^{ij} denotes her asset holding of asset j.

Under given prices, the utility maximization problem of each agent i is given by

$$\max_{\theta, x} u^i(x) \tag{1}$$

s.t.

$$p_0(x_0 - \omega^i_0) \leq -q\theta$$

$$p_{\mathbf{1}} \Box (x_{\mathbf{1}} - \omega^i_{\mathbf{1}}) \leq A(p)\theta,$$

where $p_{\mathbf{1}} \Box (x_{\mathbf{1}} - \omega_{\mathbf{1}}) = (p_1(x_1 - \omega_1), \ldots, p_S(x_S - \omega_S))^T \in R^S$.

In this paper we fix utility functions of agents and parameterize the economy by the initial endowments $\omega = (\omega^1, \ldots, \omega^I) \in R^{MI}_{++}$ of agents and the assets structure $A = (A^1, \ldots, A^J) \in R^{SNJ}$. Therefore we call (ω, A) an economy.

Definition 1. An equilibrium for an economy (ω, A) is a pair of prices $(\bar{q}, \bar{p})$ and actions $(\bar{\theta}^i, \bar{x}^i)_{i=1,\ldots,I}$ satisfying

1. $(\bar{\theta}^i, \bar{x}^i)$ solves (1) given $(\bar{q}, \bar{p})$,
2. $\sum_{i=1}^I \bar{\theta}^i = 0$,
3. $\sum_{i=1}^I \bar{x}^i = \sum_{i=1}^I \omega^i$.

As shown in Magill and Shafer (1991) for example, equilibrium is essentially equivalent to the effective equilibrium defined as follows.

Definition 2. An effective equilibrium for an economy (ω, A) is a pair $((\bar{x}^i)_{i=1,\dots,I}, \bar{p})$ such that

1. $\bar{x}^1$ solves $\max u^1(x)$ subject to $\bar{p}(x-\omega^1)=0$
2. $\bar{x}^i$, $i \geq 2$ solves

$$\begin{aligned} &\max u^i(x) \\ &s.t. \\ &\bar{p}(x-\omega^i)=0, \\ &\bar{p}_{\mathbf{1}}\square(x_{\mathbf{1}}-\omega^i_{\mathbf{1}}) \in \langle A(\bar{p})\rangle \end{aligned} \tag{2}$$

where $\langle A(p)\rangle$ denotes the subspace of R^S spanned by the column vectors of $A(p)$.

3. $\sum_{i=1}^I \bar{x} = \sum_{i=1}^I \omega^i$.

We use this definition as the equilibrium concept. We call agent 1 the unconstrained agent and often use the superscript u instead of 1. All of the other agents $i = 2, \dots, I$ are called constrained. We write the individual excess demand function as $z^i : p \to z^i(p) = x^i(p) - \omega$, where $x^i(p)$ is the solution of the maximization problem of each agent in Definition 2 and write the (effective equilibrium) aggregate excess demand function $Z : P \to R^M$ as

$$Z(p) = Z^u(p) + Z^c(p),$$

where $Z^u(p) = z^1(p)$ and $Z^c(p) = \sum_{i\geq 2} z^c(p)$. Note that the demand functions of constrained agents, hence $Z(p)$, are not continuous at prices where the payoff matrix $A(p)$ drops its rank. We define $P^g \equiv \{p \in P | A(p)$ has full column rank$\}$ and $P^b \equiv P \setminus P^g$. That is, P^g is the set of "good prices" on which the excess demand function is continuous and P^b is the set of "bad prices" where the excess demand is not continuous.

Though Z consists of M equations $(Z_{01}, \dots, Z_{0N}, \dots, Z_{S1}, \dots, Z_{SN})$ one equation is redundant from Walras' law. We let $\hat{Z}$ denote the excess demand function deleting the last element Z_{SN}. We also use this notation for Z^u and Z^c. It is clear that an effective equilibrium price is $\bar{p}$ satisfying $\hat{Z}(\bar{p}) = 0$.

3 Index theorem in the GEI model

When $\bar{p}$ is an effective equilibrium price for an economy (ω, A), we define the index at $\bar{p}$ as

$$index\hat{Z}(\bar{p}) = (-1)^{M-1} sign\left(det\left[\partial_p \hat{Z}(\bar{p})\right]\right),$$

where ∂_p denotes the derivatives with respect to $(p_{01}, \dots, p_{0N}, \dots, p_{S1}, \dots, p_{S(N-1)})$, hence $[\partial_p \hat{Z}(\bar{p})]$ is the $(M-1) \times (M-1)$ Jacobian matrix of $\hat{Z}$ at $\bar{p}$.

The index of the economy is defined by

$$\sum_{\bar{p} \in \hat{Z}^{-1}(0)} index \hat{Z}(\bar{p}),$$

where $\hat{Z}^{-1}(0) = \{p \in P | \hat{Z}(p) = 0\}$ denotes the set of effective equilibrium prices for the economy (ω, A). Thus the index of an economy (ω, A) is well defined only when $\hat{Z}^{-1}(0)$ is a finite set and the derivatives of $\hat{Z}$ is well defined at every $\bar{p} \in \hat{Z}^{-1}(0)$. As, for example, Duffie and Shafer (1985) have shown, this condition is satisfied for generic (ω, A).

Note that this definition of the index is given at an effective equilibrium using the artificial excess demand function arising from Definition 2. However, we can show that this definition does not loss generality (see Momi (2003)).

Theorem. When $S - J$ is even, for generic $(\omega, A) \in R^{MI}_{++} \times R^{SNJ}$, the index of the economy is $+1$, that is,

$$\sum_{\bar{p} \in \hat{Z}^{-1}(0)} index \hat{Z}(\bar{p}) = +1.$$

4 Index and homotopy

Having in mind that $Z^c(p)$ is continuous on P^g, we define a homotopy $H : P^g \times [0,1] \to R^{M-1}$ by

$$H(p,t) = \hat{Z}^u(p) + t\hat{Z}^c(p). \tag{3}$$

Though $H(p,t)$ is defined for a given (ω, A), we omit this specification for simplicity and this should not induce any confusions.

At $t = 0$, this system reduces to a single agent economy, whose equilibrium price is uniquely given by the supporting price p^u at the unconstrained agent's initial endowment. Therefore $(p^u, 0)$ is the unique solution to $H(p, 0) = 0$. On the other hand at $t = 1$ this system reduces to the total economy and solutions p to $H(p, 1) = 0$ are the equilibrium prices we are interested in. Thus, we have to analyze the solution set $H^{-1}(0)$ of the homotopy (3).

As for the property of $H^{-1}(0)$, it is easy to obtain the following results.

- The prices in $H^{-1}(0)$ are in some compact set bounded away from zero.

- For almost all (ω, A), each connected part of $H^{-1}(0)$ is either a segment-like part, which we also call a path, having (i) its open end point on $P^b \times [0,1]$ or (ii)its closed end point on the boundary $P^g \times \{0,1\}$, or a circle-like part.
- For almost all (ω, A), it is not the case that $H^{-1}(0)$ is tangent to the boundary $P^g \times \{0,1\}$.

For a proof of these results, see Brown, DeMarzo and Eaves (1996b).

Since our purpose is to investigate the index of the economy, we briefly recall the relation between the homotopy path following method and the index. We define index of H at $(\bar{p}, \bar{t}) \in H^{-1}(0)$ as

$$indexH(\bar{p}, \bar{t}) = (-1)^{M-1} sign\left(det\left[\partial_p H(\bar{p}, \bar{t})\right]\right). \tag{4}$$

Note that, at $(\bar{p}, 1) \in H^{-1}(0)$, $indexH(\bar{p}, 1) = index\hat{Z}(\bar{p})$, where the right hand side is the index of the economy at equilibrium price $\bar{p}$. Suppose $H^{-1}(0)$ gives smooth paths and consider to follow a path. Then, as shown in Garcia and Zangwill (1981), Mas-Colell (1985), as long as we move in the same direction with respect to t, the index of H does not change its sign. By combining this rule with the fact that p^u is the unique solution to $H(p, 0) = 0$ and index of H at $(p^u, 0)$ is $+1$, $indexH(p^*, 1) = index\hat{Z}(p^*)$ would be determined to be $+1$, if the path is a continuous path between $(p^u, 0)$ and $(p^*, 1)$. If the other paths are also away from $P^g \times [0,1]$, the indices at end points of each of these paths are in opposite signs and the index of the economy would be $+1$.

This story is, unfortunately, not the typical case in our incomplete market economy. The problem is, of course, that the paths could have their open end points on bad prices in P^b. Then $H^{-1}(0)$ does not present us good continuous paths useful for computation of an equilibrium price and for calculation of index.

5 Sketch of proof

For computation of an equilibrium price in the GEI model, Brown, DeMarzo and Eaves (1996b) and DeMarzo and Eaves (1996) presented new methods to overcome the difficulty we mentioned in the last section. (See also Brown, DeMarzo and Eaves (1996a).)

Brown, DeMarzo and Eaves (1996b) introduced artificial new J homotopies H_j, $j = 1, \ldots, J$ in addition to the original homotopy H. Homotopy H_j is defined by replacing the payoff vector $[p_1 A_1^j, \ldots, p_S A_S^j]^T$ of asset j in the utility maximization problems of constrained agents by $[p_1 z_1^u(p), \ldots, p_S z_S^u(p)]^T$. Each of these homotopies is defined on the domain where the replaced "payoff matrix" has full rank. They showed that, generically, the solution sets $H^{-1}(0)$, $H_j^{-1}(0)$, $j = 1, \dots, J$, of these homotopies coincide in the overlap of their respective domains, and the union $H^{-1}(0) \bigcup (\bigcup_j H_j^{-1}(0))$ induces good paths in total space $P \times [0,1]$. Therefore path-following is possible by changing paths to another path given by another homotopy in a neighborhood of a "bad point" where the original homotopy does not give a connected path.

Our idea is essentially the same as Brown, DeMarzo and Eaves (1996b). When we follow a path given by the original homotopy H, we know the index of H. We have to change the path to another one given by another homotopy H_j before the original path encounter a bad point. Suppose we can compute the index of H_j from the index of H when we change the paths. Then we can continue to follow the path given by H_j computing the index of H_j. After passing through the bad point, we can change the path to the original one. Suppose again we can compute the index of H from the index of H_j at the path change point. Repeating this process for each bad point, we can reach to an equilibrium price and know the index at the equilibrium. Therefore, the calculation of the index seems to be possible if we are able to know the index of a homotopy from the index of another homotopy at overlapped points.

In this paper, we change paths between the original path given by $H^{-1}(0)$ and the path given by $\tilde{K}^{-1}(0)$ where $\tilde{K}$ is defined by DeMarzo and Eaves (1996) as a homotopy on the price simplex and the Grassmannian G of the J dimensional vector subspaces of R^S. $\tilde{K}$ consists of two parts, $\tilde{K}^1$ and $\tilde{K}^2$. $\tilde{K}^1(p, L, t)$ is defined by replacing the budget constraints of the constrained agents of period 1 with $p_1 \Box (x_1 - \omega_1) \in L$, where L is an element of G. $\tilde{K}^2(p, L)$ is defined so that $\tilde{K}^2(p, L) = 0$ requires $\langle A(p) \rangle \in L$. Then $\tilde{K}$ is a continuous function of p and L and the discontinuity at bad price is solved. $\tilde{K}^{-1}(0)$ typically provides good paths in $P \times G \times [0,1]$ which we can follow.

We change paths between $H^{-1}(0)$ and $\tilde{K}^{-1}(0)$ and show that the computation of indices between H and $\tilde{K}$ is possible. Keep in mind following two points. First, as easily known from Brown DeMarzo and Eaves (1996b) and DeMarzo and Eaves (1996), though $\tilde{K}^{-1}(0)$ is a set in $P \times G \times [0,1]$, projection of $K^{-1}(0)$ to $P \times [0,1]$ is typically equal to $H^{-1}(0) \bigcup (\bigcup_j H_j^{-1}(0))$. Therefore our path change has the same effect as the path change of Brown, DeMarzo and Eaves (1996b). Second, the index of $\tilde{K}$ has to be defined via local coordinate systems for G. Thus in this paper, for simplicity of calculation, we define the homotopies directly on local coordinate systems.

6 The pseudo-path

Let G denote the set of J dimensional vector spaces in R^S. We define a function $\tilde{Z}^c : P \times G \to R^M$ as

$$\tilde{Z}^c(p, L) = \sum_{i \geq 2} \tilde{z}^i(p, L)$$

where $\tilde{z}^i(p, L) = \tilde{x}^i(p, L) - \omega^i$ and

$$\begin{aligned} \tilde{x}^i(p, L) = \arg\max u^i(x) & \\ & s.t. \\ & p(x - \omega^i) = 0, \\ & p_1 \Box (x_1 - \omega_1^i) \in L. \end{aligned}$$

G is the smooth compact manifold of dimension $(S-J)J$ called the Grassmannian manifold. We use essentially the same atlas as one used in Duffie and Shafer (1985). The only difference is that, for simplicity of calculations, we use permutation matrix which works to a column vector from left. Σ denotes the set of permutation of $\{1, \ldots, S\}$ and Π_σ is the permutation matrix such that $\Pi_\sigma \begin{bmatrix} X_1 \\ \vdots \\ X_S \end{bmatrix} = \begin{bmatrix} X_{\sigma(1)} \\ \vdots \\ X_{\sigma(S)} \end{bmatrix}$ for $\sigma \in \Sigma$. As in Duffie and Shafer (1985), by some $\sigma \in \Sigma$ and $Y = \begin{bmatrix} y_{11} & \cdots & y_{1J} \\ \vdots & & \vdots \\ y_{(S-J)1} & \cdots & y_{(S-J)J} \end{bmatrix} \in R^{(S-J)J}$, any $L \in G$ can be written as

$$L = \{w \in R^S | [I|Y]\Pi_\sigma w = 0\}, \tag{5}$$

where I denotes the $(S-J) \times (S-J)$ identity matrix. For each $\sigma \in \Sigma$, let $\varphi_\sigma : W_\sigma \to R^{(S-J)J}$ be defined by $L = \{w \in R^N | [I|\varphi_\sigma(L)]\Pi_\sigma w = 0\}$ where $W_\sigma = \{L \in G |$there is $y \in R^{(S-J)J}$ such that $L = \{w \in R^S | [I|Y]\Pi_\sigma w = 0\}\}$. Then $\{W_\sigma, \varphi_\sigma\}_{\sigma \in \Sigma}$ is actually an atlas for G.

A pair $(\bar{p}, \bar{L})$ satisfying $Z^u(\bar{p}) + \tilde{Z}^c(\bar{p}, \bar{L}) = 0$ and $\langle A(\bar{p}) \rangle \in \bar{L}$ is called a pseudo-equilibrium. As known from the definition of $\tilde{Z}^c$, if $A(\bar{p})$ has full column rank at a pseudo-equilibrium $(\bar{p}, \bar{L})$, then $\bar{p}$ is an effective equilibrium price. As shown in Duffie and Shafer (1985), a pseudo-equilibrium exists for any (ω, A) and a pseudo-equilibrium price generically satisfies the full rankness of the payoff matrix, that is, it is generically an effective equilibrium price.

DeMarzo and Eaves (1996) presented an homotopy method defined on the price simplex and the Grassmannian manifold to compute a pseudo-equilibrium. Their homotopy path serves as the auxiliary path for us. However, for simplicity of calculations which we need later, we define our homotopy as follows.

For each $\sigma \in \Sigma$, we define a homotopy $K_\sigma = (K_\sigma^1, K_\sigma^2) : P \times R^{(S-J)J} \times [0,1] \to R^{M-1} \times R^{(S-J)J}$ by

$$K_\sigma^1(p,Y,t) = \hat{Z}^u(p) + t\hat{\hat{Z}}^c(p, \varphi_\sigma^{-1}(Y)),$$

$$K_\sigma^2(p,Y) = [I|Y]\Pi_\sigma A(p).$$

In the rest of this section, we present properties of $H^{-1}(0)$ and $K^{-1}(0)$, which is almost evident from Brown DeMarzo and Eaves (1996) and DeMarzo and Eaves (1996b), and show how we can evade bad points and continue the path following.

Lemma 1. (i) If $(p,t) \in H^{-1}(0)$, then $(p, \varphi_\sigma(\langle A(p)\rangle), t) \in K_\sigma^{-1}(0)$ for $\sigma \in \Sigma$ satisfying $\langle A(p)\rangle \in W_\sigma$. (ii) If $(p,Y,t) \in K_\sigma^{-1}(0)$ for some σ and $A(p)$ has full column rank, then $Y = \varphi_\sigma(\langle A(p)\rangle)$ and $(p,t) \in H^{-1}(0)$.

Proof. Evident from the definitions of $\tilde{Z}^c$ and K_σ. ∎

Lemma 2. For each $\sigma \in \Sigma$, for generic (ω, A), (i) $K_\sigma^{-1}(0)$ is a one dimensional smooth submanifold of $P \times R^{(S-J)J} \times [0,1]$ and (ii) the dimension of $\{(p,Y,t) \in K_\sigma^{-1}(0) | A(p)$ is not full column rank$\}$ is 0.

Proof. Immediate from Duffie and Shafer (1985) and DeMarzo and Eaves (1996). ∎

For each σ, let the generic set E_σ be the set of (ω, A) which supports the claim of Lemma 2. Note that, for any σ, the claim is true for $(\omega, A) \in \bigcap_{\sigma\in\Sigma} E_\sigma$ and $\bigcap_{\sigma\in\Sigma} E_\sigma$ is a generic set because Σ is a finite set.

We show that $det[\partial_Y K_\sigma^2] \neq 0$ is typical on $K_\sigma^{-1}(0)$, where $\partial_Y K_\sigma^2$ denotes the derivatives of K_σ^2 with respect to Y.[1]

From $K_\sigma^2(p,Y) = [I|Y]\Pi_\sigma A(p)$, $[\partial_Y K_\sigma^2]$ is calculated straight as

$$\left[\partial_Y K_\sigma^2\right] = \begin{bmatrix} \left[Q_\sigma\right]^T & & 0 \\ & \ddots & \\ 0 & & \left[Q_\sigma\right]^T \end{bmatrix} \tag{6}$$

where $[Q_\sigma]$ is the $J \times J$ matrix that consists of the last J rows of $\Pi_\sigma A(p)$, hence,

$$\left[Q_\sigma\right]^T = \begin{bmatrix} p_{\sigma(S-J+1)}A^1_{\sigma(S-J+1)} & \cdots & p_{\sigma(S)}A^1_{\sigma(S)} \\ \vdots & & \vdots \\ p_{\sigma(S-J+1)}A^J_{\sigma(S-J+1)} & \cdots & p_{\sigma(S)}A^J_{\sigma(S)} \end{bmatrix},$$

[1]Both K_σ^2 and Y are of the form of $(S-J) \times J$ matrix. When we have to introduce an order in the elements of a matrix, we assume the (i,j) element is before the (i',j') element when $i < i'$ or $i = i'$ and

and $[\partial_Y K_\sigma^2]$ is the $(S-J)J \times (S-J)J$ matrix that consists of $[Q_\sigma]^T$'s in number $(S-J)$ lined diagonally.

Therefore

$$det\left[\partial_Y K_\sigma^2\right] = \left(det\left(\left[Q_\sigma\right]^T\right)\right)^{S-J} = \left(det\left[Q_\sigma\right]\right)^{S-J}, \tag{7}$$

and $det[\partial_Y K_\sigma^2] = 0$ if and only if $det[Q_\sigma] = 0$.

Lemma 3. For each $\sigma \in \Sigma$, the dimension of $\{(p,x,t) \in K_\sigma^{-1}(0) | det[\partial_Y K_\sigma^2] = 0\}$ is 0, for generic (ω, A) .

Proof. See Momi (2003). ∎

Let us call a path given by $K_\sigma^{-1}(0)$ as a pseudo-path while a path given by $H^{-1}(0)$ is called original. Now, we can continue the path following of the original path evading bad points as follows. See Figure 1.

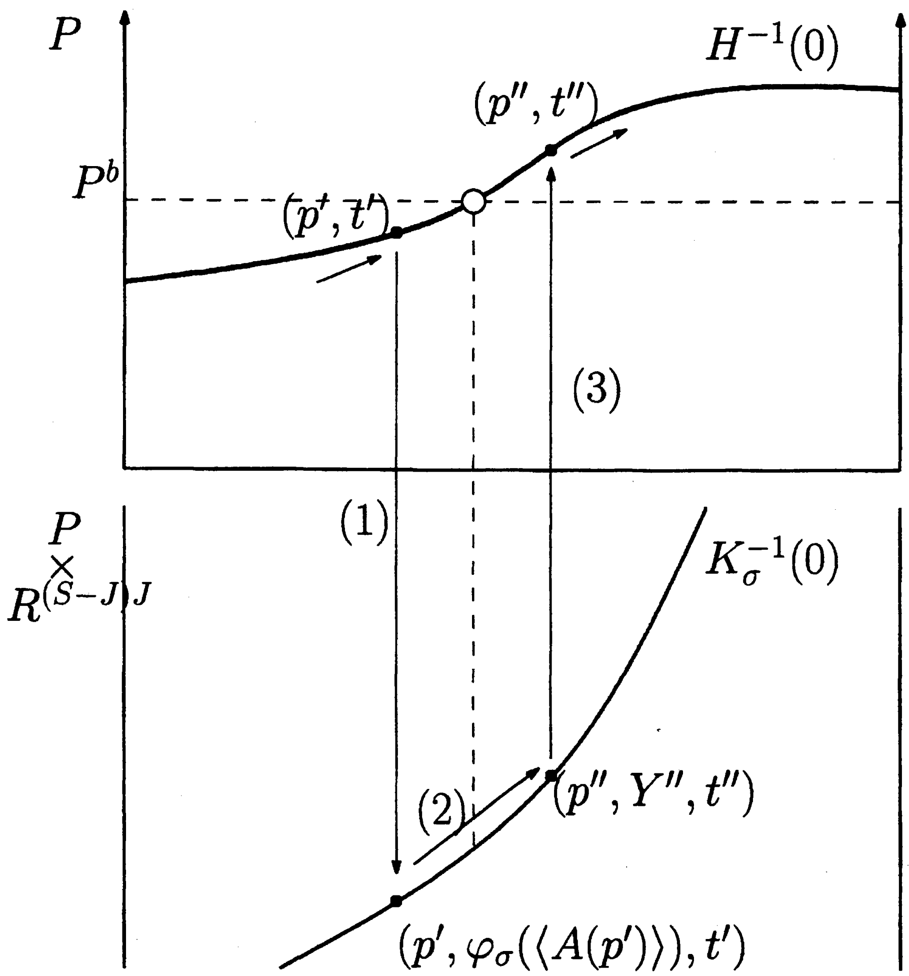

Figure 1: Bad point evading process

(1) Just before encountering a bad point, we jump, from a point (p', t') on the original path in $H^{-1}(0)$, onto the corresponding point $(p', \varphi_\sigma(\langle A(p')\rangle), t')$ on a pseudo-path in $K_\sigma^{-1}(0)$, Lemma 1 (i).

(2) We can follow the pseudo-path in $K_\sigma^{-1}(0)$, Lemma 2 (i), and pass through the "bad" (not bad, of course, for the pseudo-path) point.

(3) We jump back from a point (p'', Y'', t'') on the pseudo-path in $K_\sigma^{-1}(0)$ to the corresponding point (p'', t'') on the original path in $H^{-1}(0)$. This is possible for almost all time, Lemma 2 (ii) and Lemma 1 (ii).

Note that in this process (1)-(3), we need the pseudo-path of only one homotopy K_σ because, after passing through a bad point, we can immediately go back to the original path, Lemma 2 (ii) and Lemma 1 (ii).

7 Indices of H and K_σ

Our purpose is to know the index of H at $(p, 1) \in H^{-1}(0)$.Thus we want to know how the index of H changes between (p', t') and (p'', t'') in Figure 1. This is known from the relation between the indices of H and K_σ.

Proposition 1. Suppose $(p, t) \in P^g \times [0, 1]$. For $\sigma \in \Sigma$ satisfying $\langle A(p)\rangle \in W_\sigma$, if $det[\partial_Y K_\sigma^2] \neq 0$ at $(p, \varphi_\sigma(\langle A(p)\rangle), t)$, then

$$det\left[\partial_p H\right] = det\left(\left[\partial_Y K_\sigma^2\right]^{-1}\right) det\begin{bmatrix}\partial_p K_\sigma^1 & \partial_Y K_\sigma^1\\ \partial_p K_\sigma^2 & \partial_Y K_\sigma^2\end{bmatrix}, \tag{8}$$

where left and right hand sides are evaluated at (p, t) and $(p, \varphi(\langle A(p)\rangle), t)$ respectively, and the symbol -1 means the inverse of matrix.

Proof. See Momi (2003). ■

Recall the definition (4) of the index of H at $(\bar{p}, \bar{t}) \in H^{-1}(0)$. The index of K_σ at $(\bar{p}, \bar{Y}, \bar{t}) \in K_\sigma^{-1}(0)$ is naturally defined as

$$indexK_\sigma(\bar{p}, \bar{Y}, \bar{t}) = (-1)^{M-1+(S-J)J} sign\left(det\begin{bmatrix}\partial_p K_\sigma^1(\bar{p}, \bar{Y}, \bar{t}) & \partial_Y K_\sigma^1(\bar{p}, \bar{Y}, \bar{t})\\ \partial_p K_\sigma^2(\bar{p}, \bar{Y}, \bar{t}) & \partial_Y K_\sigma^2(\bar{p}, \bar{Y}, \bar{t})\end{bmatrix}\right).$$

What Proposition 1 means is evident. From (8), we can obtain the relation between indices of H and K_σ. That is, when $det[\partial_Y K_\sigma^2] \neq 0$,

$$indexH(p, t) = (-1)^{(S-J)J} sign\left(det\left(\left[\partial_Y K_\sigma^2\right]^{-1}\right)\right) indexK(p, \varphi_\sigma(\langle A(p)\rangle), t). \tag{9}$$

Thus, when we change paths between a original path in $H^{-1}(0)$ and a pseudo-path in $K_\sigma^{-1}(0)$, the index is determined by equation (9). We need the path changes twice for evading a bad point ((1) and (3) in Figure 1). Therefore we have the effect of $sign(\det([\partial_Y K_\sigma^2]^{-1}))$ in (9) twice, i.e., before and after a bad point. Thus, if $\det([\partial_Y K_\sigma^2]^{-1})$ has the same sign before and after the bad point, the effects are canceled out when we come back to the original path after evading a bad point. If it has opposite signs before and after the bad point, it induces the index change of the original homotopy. See Figure 2. (i) draws the case where $det([\partial_Y K_\sigma^2]^{-1})$ does not change its sign at any bad points. (ii) draws the case where $det([\partial_Y K_\sigma^2]^{-1})$ changes its sign at every bad point.

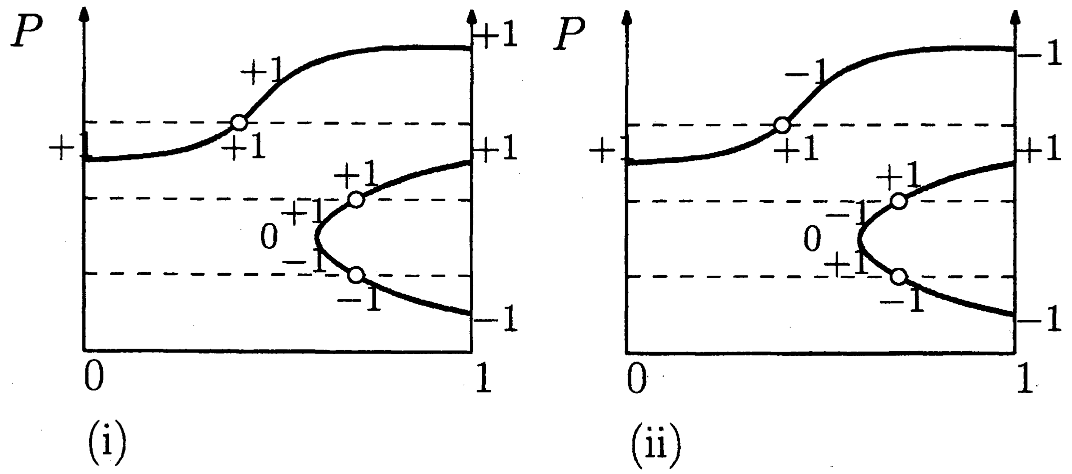

Figure 2: Examples of homotopy paths

From Lemma 3, at almost all points on pseudo paths, the equation (9) is true. Now all we have to do is to see the sign of $det([\partial_Y K_\sigma^2]^{-1})$, which is straight from (6), or (7).

$$sign\left(det\left(\left[\partial_Y K_\sigma^2\right]^{-1}\right)\right) = sign\left(\frac{1}{\left(det\left[Q_\sigma\right]\right)^{S-J}}\right).$$

Note that $det([\partial_Y K_\sigma^2]^{-1}) \neq 0$ for all $(p,x,t) \in K_\sigma^{-1}(0)$, because $det[Q_\sigma] < \infty$ for any $(p,x,t) \in K_\sigma^{-1}(0)$. Also note that $det([\partial_Y K_\sigma^2]^{-1})$ is not well defined ($+\infty$ or $-\infty$) when $det[Q_\sigma] = 0$, or equivalently when $det[\partial_Y K_\sigma^2] = 0$. Especially $det([\partial_Y K_\sigma^2]^{-1})$ is not well defined at bad points. We proved in Lemma 3 that such case is exceptional on $K_\sigma^{-1}(0)$

Now it is clear that, if $S-J$ is even, $sign(det([\partial_Y K_\sigma^2]^{-1})) = +1$ for all $(p,x,t) \in K_\sigma^{-1}(0)$ except the exceptional points where $det([\partial_Y K_\sigma^2]) = 0$. Therefore, in a small segment on a path in $\overline{H^{-1}(0)}$ including a bad point, $sign(det([\partial_Y K_\sigma^2]^{-1}))$ is constant. Then, from the above discussions, we can follow the each path in $H^{-1}(0)$ neglecting the bad points.

In other words, we can follow the paths in $\overline{H^{-1}(0)}$ and count the index as if these are standard homotopy paths.

For generic (ω, A), as we showed in the previous section, $\overline{H^{-1}(0)}$ and $H^{-1}(0)\bigcup(\bigcup_j H_j(0)$ produce the same paths, which are one dimensional smooth manifolds with regular end points at the boundaries $t = 0, 1$ as shown in Brown DeMarzo and Eaves (1996). In $\overline{H^{-1}(0)}$, one path whose one endpoint is $(p^u, 0)$ has another endpoint at $(p^*, 1)$ and $index\hat{Z}(p^*) = +1$. Each of other paths has both endpoints at boundary $t = 1$, hence indices at these points are in opposite signs. This ends the proof of Theorem.

8 The case where $S - J$ is odd: comments

Though we have proved that the index of the GEI model where $S - J$ is even is typically $+1$, the case where $S - J$ is odd is still an open problem.

It is quite certain that $det[Q_\sigma]$ could change its sign at bad points. When we parameterize (p, x, t) on a path in $K^{-1}(0)$ as $(p(r), x(r), t(r))$, that $det[Q_\sigma]$ does not change sign at a bad point requires that $\partial_r det[Q_\sigma] = 0$ at the bad point where we consider $det[Q_\sigma]$ as a function of r. Thus the change of the sign of $det[Q_\sigma]$ at bad points seems to be more likely than keeping the sign of it, though we cannot prove the latter is not the case for generic (ω, A).

As shown in Section 7, when $det[Q_\sigma]$ changes its sign and $S - J$ is odd, the index of H should change its sign at the bad point. Thus Figure 2 (ii) would occur, for example. Extremely speaking, it seems that any odd number could be the index of an economy. We can draw a picture of homotopy paths of $H^{-1}(0)$ where the index of the economy is arbitrary odd number by adequately deciding the number of homotopy paths in $H^{-1}(0)$ and the number of bad points on these paths at each of which the sign of $det[Q_\sigma]$ changes. Is this true?

If the above story is true, it means $S - J$ plays the crucial role for the index of the GEI economy. It would be very surprising that the index, which would be the value intrinsic for economies, depends of $S - J$. Though we do not have an example of the economy whose index is not $+1$, our result in this paper suggests this would be the case.

If the story is not true, this is also a very surprising result because this means the paths in $H^{-1}(0)$ and the number of the bad points on the paths satisfy some restriction. For example, if each path certainly has even number of bad points where $det[Q_\sigma]$ changes its sign, then the effects cancel out on each path and the index of the economy would be $+1$. However, it is hard to believe that the homotopy paths always have such property.

References

[1] Brown, D.J., DeMarzo, P.M., Eaves, B.C., 1996a. Computing zeros of sections of vector bundles using homotopies and relocalization. Mathematics of Operations Research 21, 26-43.

[2] Brown, D.J., DeMarzo, P.M., Eaves, B.C., 1996b. Computing equilibria when asset markets are incomplete. Econometrica 64, 1-27.

[3] DeMarzo, P.M., Eaves, B.C., 1996. Computing equilibria of GEI by relocalization on a grassmann manifold. Journal of Mathematical Economics 26, 479-497.

[4] Dierker, E., 1972. Two remarks on the number of equilibria of an economy. Econometrica 40, 951-953.

[5] Duffie, D., Shafer, W., 1985. Equilibrium in incomplete markets I. Journal of Mathematical Economics 14, 285-300.

[6] Garcia, C., Zangwill, W., 1981. Pathways to Solutions, Fixed Points, and Equilibria. Prentice-Hall, Englewood Cliffs, NJ.

[7] Hens, T., 1991. Structure of General Equilibrium Models with Incomplete Markets. Dissertation, Rheinische-Friedrich-Wilhelms-Universität, Bonn.

[8] Hirsh, M.W., Smale, S., 1974. Differential Equations, Dynamical Systems, and Linear Algebra. Academic Press, New York.

[9] Magill, M.J.P., Shafer, W., 1991. Incomplete markets. In: Hildenbrand, W., Sonnenschein, H. (Eds.), The Handbook of Mathematical Economics, Volume IV. North-Holland, New York, pp. 1523-1614.

[10] Mas-Colell, A., 1985. The Theory of General Economic Equilibrium: A Differentiable Approach. Cambridge Univ. Press, New York.

[11] Mas-Colell, A., Whinston, M.D., Green, J.R., 1995. Microeconomic Theory. Oxford Univ. Press, New York.

[12] Momi, T., 2003. The index theorem for a GEI economy when the degree of incompleteness is even. forthcoming in Journal of Mathematical Economics.

[13] Schmedders, K., 1999. A homotopy algorithm and an index theorem for the general equilibrium model with incomplete asset markets. Journal of mathematical economics 32, 225-241.