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Characterization of a Chaotic Economic Dynamics by Unstable Periodic Solutions

— An Application to a Generalized Goodwin Model*

Ken-ichi Ishiyama†, Yoshitaka Saiki‡

Department of Mathematical Sciences
Graduate School of Mathematical Sciences, University of Tokyo
3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

Abstract

In this paper we study the properties of the chaotic behavior in a growth cycle model and the unstable periodic solutions found in the attractor, and thereby we point out some similarities between them. This attempt comes from the recent work in physics. The result implies unstable periodic solutions can be the keywords to understand the chaotic dynamics.

1 Introduction

It is no doubt the analysis of the growth trajectory in a phase space is one of the most important themes in economic dynamics. It is often discussed what happens when a growth cycle model has been extended in a direction. In particular, a topic has attracted much attention since 1980s, that is, the controversy surrounding the effects of fiscal policy on a simple trade cycle(Wolfstetter(1982), Goodwin(1990), Takamasu(1995), Yoshida and Asada(2001)). However like as other topics there are also few works concerning this point in which the comparative dynamics involving the chaotic behavior is discussed.

In recent work in physics(Zoldi and Greenside(1998), Kawahara and Kida(2001), Kato and Yamada(2002)), a numerical approach has been attempted using unstable periodic solutions to explain typical chaotic behavior and statistical properties in chaotic solutions. The main purpose of this paper is to propose an idea analogous to physics to understand the dynamics of a chaotic business cycle, which can be expected to be useful to clarify various chaotic phenomena shown in economic dynamics.

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† *E-mail*: ishiyama@ms.u-tokyo.ac.jp

‡ *E-mail*: saiki@ms.u-tokyo.ac.jp

The plan of this paper is as follows. In the next section, we give an example of the nonlinear macro-economic model. Section 3 describes the properties of the dynamics of the model, where the characteristics of a chaotic solution and unstable periodic solutions are illustrated. It is cleared in section 4 that the chaotic behavior in the dynamics of the model is qualitatively and quantitatively related to the unstable periodic solutions found in the attractor. In the final section, we conclude our results and state the possibility of the application.

2 The Model

First, we propose a growth cycle model as an example which represents a chaotic behavior caused by a simple interaction between countries. The model is based on Goodwin(1967) and its extensions. It consists of following assumptions.

(A1) We consider two countries, where there exist respectively three agents: the government, capitalists, and workers. These countries are almost homogeneous, but parameters regarding a stabilization policy assumed next can be different.

(A2) The government controls the public expenditure as a countercyclical policy, while it is financed by income tax and bond selling. The level of the public expenditure is decided on the basis of two factors: the scale of the domestic industry and the domestic employment rate.[1]

(A3) The foreign capital share rate as well as the domestic capital share rate[2] concern the amount of investment in each individual country. This is the only mutual interaction allowed for in the model. The invested capital and the labor employed contribute to the production activity in each country. The level of employment is linearly dependent on the scale of the production.

(A4) The workers bargain with capitalists for their money wages rate taking into consideration of the expected rate of inflation. Moreover the bargaining power is influenced by the employment ratio.[3]

(A5) The workers spend their whole disposable incomes, while the capitalists save their interest incomes besides the better part of their

[1] See Wolfstetter(1982).

[2] If we consider the profit rate instead of the capital share rate with international trade, the discussion becomes extremely difficult.

[3] An expectation-augmented wage Phillips curve is considered as Yoshida and Asada(2001).

profits. The saving is devoted to the investment otherwise purchasing the government bond.

(A6) The population and the productivity per capita grow with constant rates. The price is rigid, that is to say, the gradient of change in price is gentler than that of change in unit labor cost of output.[4]

(A7) The growth of the national output depends on the relationship between the demand and the supply. For convenience, the excess supply is consumed by capitalists, hence the aggregate output is equal to the aggregate income.

From above assumptions, we construct the model which describe the interaction among the labor share rates, the employment ratios, and the expected inflation rates in two countries.

We formulate in advance the six-dimensional simultaneous ordinary differential equations to be derived as follows:

$$\frac{du_i}{dt} = (\hat{w}_i - (\alpha + \hat{p}_i))u_i, \tag{1}$$

$$\frac{dv_i}{dt} = (\hat{Y}_i - (\alpha + \beta))v_i, \tag{2}$$

$$\frac{d\pi_i^e}{dt} = \theta(\hat{p}_i - \pi_i^e), \tag{3}$$

$$i = 1, 2,$$

where the variable u_i means the labor share rate in i-th country, v_i the employment ratio, and π_i^e the expected rate of inflation respectively. The constant γ corresponds to the price rigidity assumed in (A6), α the rate of the technical progress, β the growth rate of labor available, and θ is a parameter with respect to the adaptive behavior of the worker. The symbols $\hat{w}_i$, $\hat{p}_i$, and $\hat{Y}_i$ represent the respective change rates of the money wage, of the price of goods, and of the national output in i-th country. They are summarized or rewritten as follows:

$$\hat{w}_i = f_i(v_i, \pi_i^e); \qquad \frac{\partial f_i}{\partial v_i} > 0, \ \frac{\partial f_i}{\partial \pi_i^e} > 0, \tag{4}$$

$$\hat{p}_i = \gamma(\hat{w}_i - \alpha), \tag{5}$$

[4]We consider the price adjustment equation assumed in Desai(1973) with a constant mark-up factor.

$$\varepsilon \hat{Y}_i = h_i(u_1, u_2) + (1-c)\mu_i(v^* - v_i) + (\delta - 1)(1-c)(1-u_i), \quad (6)$$

where the right-hand side in equation(6) corresponds to the excess demand[5] per output, while ε, c, δ, v^*, and μ_i are respectively an output adjustment coefficient, the consumption coefficient of capitalists, the income tax rate, the target employment rate set so that $f(v^*, 0) = \alpha$, and the parameter of fiscal policy in i-th country. The function h_i determine the effect of the investment on the augmentation of the output.[6] It has the following properties:

$$\frac{\partial h_i}{\partial u_i} < 0, \; \frac{\partial h_i}{\partial u_j} > 0; \qquad i, j = 1, 2 \qquad (j \neq i).$$

In the next section, we will discuss numerically the properties of the solution of the model for a specific case.

3 The Chaotic Solution and the Unstable Periodic Solutions

In this section we consider characteristics of the solutions of the model for a set of parameters and specified functions.[7] Here we give two functions based on Yoshida and Asada(2001):

$$f_i(v_i, \pi_i^e) = 0.1(\frac{1}{1 - v_i} - 4.8) + \pi_i^e, \quad (7)$$

$$h_i(u_1, u_2) = 1.5(1 - u_i)^5 - 10.0(u_i - u_j)^3, \quad (8)$$

while we set constants and parameters as $\alpha = 0.02$, $\beta = 0.01$, $\gamma = 0.5$, $\theta = 0.8$, $\varepsilon = 10.0$, $c = 0.3$, $\delta = \frac{2}{7}$, $v^* = 0.8$; $\mu_1 = 1.2$, and $\mu_2 = 8.0$. Note that the inequality $\mu_1 < \mu_2$ means that the government in country 2 takes more positive stabilizing policy than the other, and the interaction between countries is represented by the second term in equation(8).

The time series after the transition is plotted in Fig.1, where the economy sustained by more positive fiscal policy looks stable for a long time but sometimes disturbed by the other country. The oscillations in two countries seem to be synchronized by the interaction in equation(8). Moreover it is observed in this figure that the business tides, which seem repeated regularly[8], tend to become gradually larger and suddenly get

[5] It includes the government expenditure $G = \delta Y + \mu(v^* - v)Y$.

[6] It is based on Skott(1989).

[7] For some parameter settings, we have derived qualitatively similar results.

[8] It is about 23 years in length for our parameter setting.

smaller. We can view each trend from expansion to contraction as a characteristic cycle in each country. Hereafter we call the individual trend *economic regime.*[9] There appear fourteen[10] *economic regimes* in Fig.1. We notice they have different length. Fig.2 shows a sequence of *economic regimes* as regarding the labor share rate and the employment ratio in country 1. The pattern consisting of them looks like a *trumpet chain.*

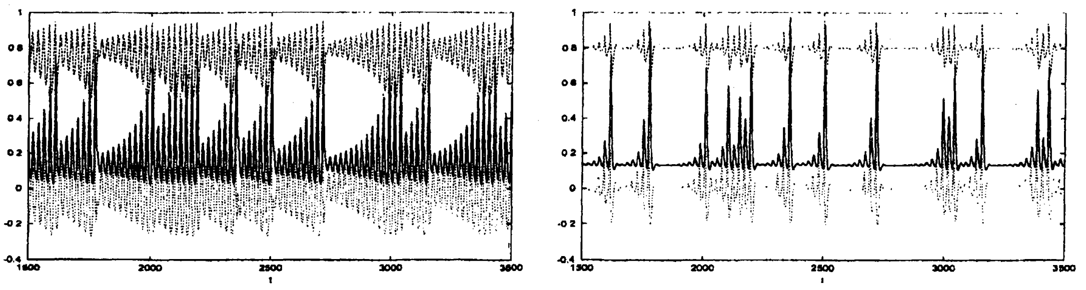

Fig. 1: Time series in country 1(left) and country 2(right)

The solid line is a time series of the labor share rate, the dashed line the employment ratio, and the dotted line the expected rate of inflation respectively. The horizontal axis means the time. In our parameter setting, the unit time approximately equals a year.

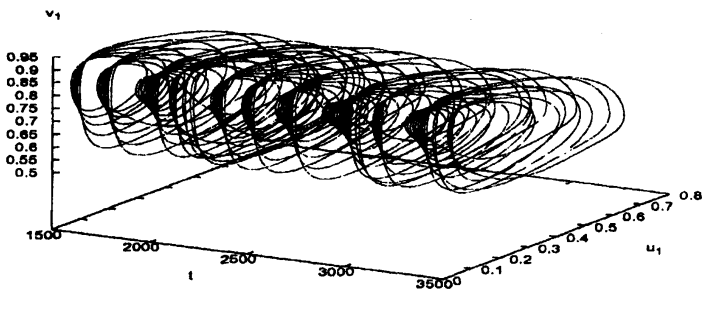

Fig. 2: Trajectory like a *trumpet chain*
A cyclical growth path of the labor share rate u_1 and the employment ratio v_1 is illustrated.

Besides the chaotic solution, we have found[11] thirteen kinds of periodic solutions.[12] These are not stable, however, we presume the economy sufficiently close such a solution grows along the trajectory for a long time. In the literatures of mathematics and physics, they are called *unstable periodic orbits*($UPOs$), and there is an idea that unstable periodic orbits

[9] In meteorology, the typical dynamics are called *weather regime.*
[10] The last process of expansion is not counted because of its incompleteness.
[11] We consider the solution X is periodic if $\|X(T) - X(0)\| \leq 10^{-5}$, where T is the period.
[12] Surprisingly we have observed every *economic regimes* associating every unstable periodic solutions in the simulation(Ishiyama and Saiki(2003)).

densely embedded[13] in chaotic attractor would explain the properties of the chaotic solution. With regard to our case, some of periodic solutions found are drawn in Fig.3. While these orbits have different number of *whirls*[14], all of them look similar in shape to the chaotic attractor(Fig.3 left). Properties of the unstable periodic orbits and their relations with the chaotic solution will be referred to in the next section. Here we note that the economic welfare would be different among on those orbits judging from the figure.[15]

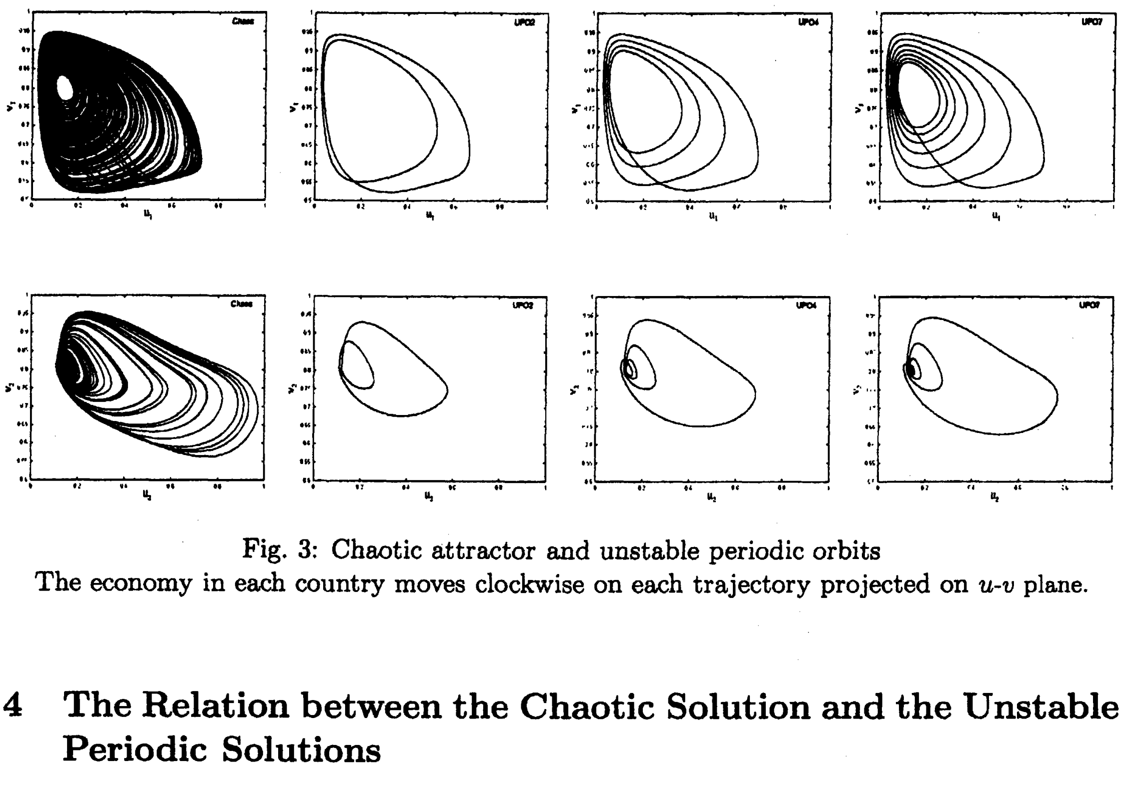

Fig. 3: Chaotic attractor and unstable periodic orbits
The economy in each country moves clockwise on each trajectory projected on u-v plane.

4 The Relation between the Chaotic Solution and the Unstable Periodic Solutions

This section aims at revealing the relationship between the chaotic attractor and the unstable periodic orbits shown in the previous section.

First, we show Table 1, where the statistical data of the chaotic solution and the unstable periodic solutions are listed. In the table, the data of UPO_{ave} is calculated as below:

$$\bar{x}_{UPO_{ave}} = \frac{\sum_{k=1}^{13} \bar{x}_{UPOk}}{13}. \tag{9}$$

The statistical data of *UPO* 7 and UPO_{ave} resemble that of *Chaos* well. It implies that behaviors on unstable periodic orbits would be connected

[13]See Kazantsev(1998).
[14]We define the unstable periodic orbit with n whirls as *UPO n*.
[15]Statistical data presented in the next section will answer this question.

with the chaotic fluctuation over a long time. Now, we will discuss this point further.

Table 1: Mean values of variables of *Chaos* and *UPOs*

The figures in parentheses are the variances corresponding to economic variables of the system.

	$\bar{u}_1$	$\bar{v}_1$	$\bar{\pi}_1^e$	$\bar{u}_2$	$\bar{v}_2$	$\bar{\pi}_2^e$	period
Chaos	0.18415	0.74831	0.00000	0.17848	0.79075	0.00000	
	(0.02151)	(0.01141)	(0.04530)	(0.01508)	(0.00185)	(0.00798)	
UPO 1	0.22396	0.69618	0.00001	0.25123	0.77535	0.00001	23.512
	(0.04269)	(0.02107)	(0.08932)	(0.01751)	(0.00378)	(0.02215)	
UPO 2	0.21772	0.70550	-0.00004	0.22941	0.78064	-0.00003	46.867
	(0.03869)	(0.01923)	(0.08155)	(0.01832)	(0.00323)	(0.01698)	
UPO 3	0.20902	0.71749	-0.00003	0.20830	0.78535	-0.00003	69.995
	(0.03345)	(0.01693)	(0.07150)	(0.01770)	(0.00263)	(0.01271)	
UPO 4	0.20122	0.72737	-0.00002	0.19465	0.78810	-0.00002	92.958
	(0.02916)	(0.01507)	(0.06313)	(0.01648)	(0.00224)	(0.01030)	
UPO 5	0.19444	0.73543	0.00003	0.18526	0.78989	0.00000	115.762
	(0.02576)	(0.01356)	(0.05627)	(0.01526)	(0.00196)	(0.00874)	
UPO 6	0.18863	0.74206	0.00003	0.17844	0.79114	0.00000	138.417
	(0.02303)	(0.01231)	(0.05056)	(0.01414)	(0.00175)	(0.00763)	
UPO 7	0.18364	0.74764	0.00002	0.17325	0.79210	0.00000	160.957
	(0.02080)	(0.01125)	(0.04575)	(0.01316)	(0.00159)	(0.00679)	
UPO 8	0.17937	0.75234	0.00003	0.16918	0.79285	0.00000	183.415
	(0.01895)	(0.01035)	(0.04169)	(0.01229)	(0.00146)	(0.00613)	
UPO 9	0.17565	0.75642	0.00001	0.16588	0.79346	0.00000	205.827
	(0.01738)	(0.00956)	(0.03812)	(0.01153)	(0.00135)	(0.00560)	
UPO 10	0.17253	0.75978	0.00001	0.16325	0.79395	0.00000	228.237
	(0.01611)	(0.00891)	(0.03526)	(0.01088)	(0.00127)	(0.00517)	
UPO 11	0.16983	0.76278	0.00005	0.16098	0.79439	0.00002	250.688
	(0.01500)	(0.00831)	(0.03265)	(0.01030)	(0.00119)	(0.00479)	
UPO 12	0.16742	0.76543	0.00005	0.15909	0.79475	0.00002	273.275
	(0.01401)	(0.00778)	(0.03036)	(0.00981)	(0.00112)	(0.00448)	
UPO 13	0.16528	0.76785	0.00002	0.15757	0.79502	0.00002	296.358
	(0.01314)	(0.00729)	(0.02825)	(0.00948)	(0.00107)	(0.00424)	
UPO_{ave}	0.18836	0.74125	0.00001	0.18435	0.78977	0.00000	160.480
	(0.02370)	(0.01243)	(0.05111)	(0.01360)	(0.00190)	(0.00890)	

To see differences not appearing in low order statistics(mean and variance), we illustrate histograms. Fig.4 upper left shows three histograms calculated from the movements of the labor share rate in country 1. The first one is depicted through the long time movement enough to represent the statistics of chaos, the second is *UPO* 7, and the last one is UPO_{ave}. We find the histogram of UPO_{ave} approximates best that of chaos in the case of variable u_1. In practice, regarding other variables, we obtain the similar results.(See other histograms.) We can consider that the chaotic orbit will pass many times by every unstable periodic orbits in the chaotic attractor, therefore the statistics of the chaotic solution and the unstable periodic solutions are very similar.

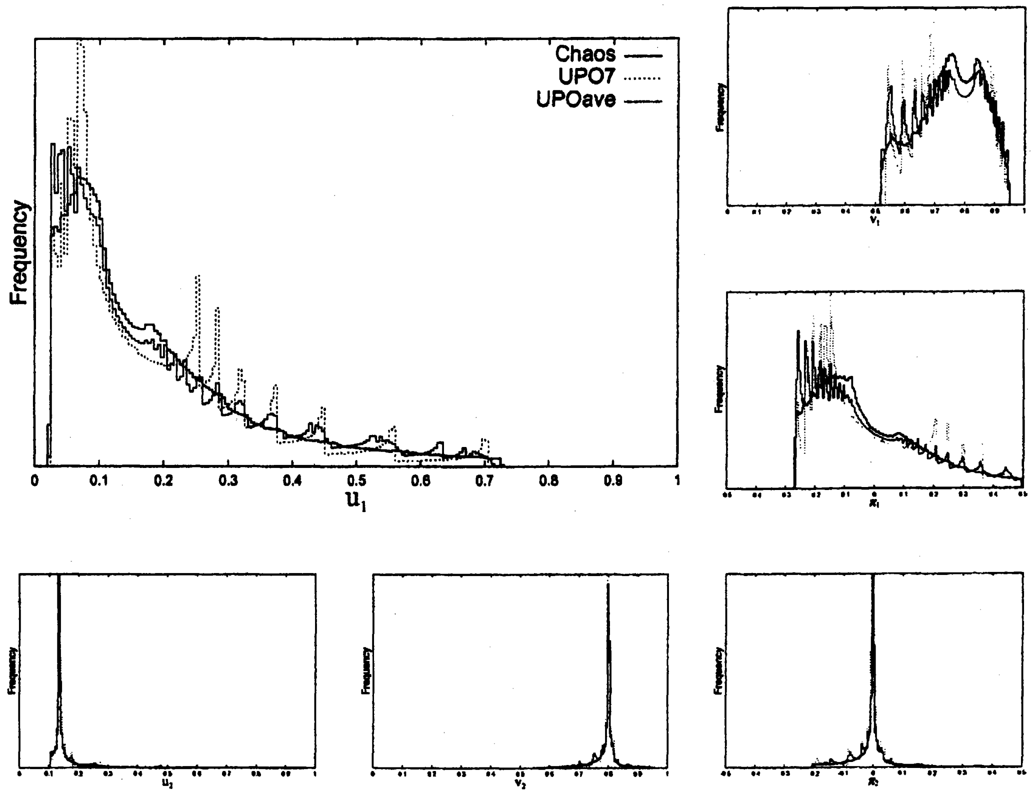

Fig. 4: Histograms of *Chaos*, *UPO* 7, and UPO_{ave}

Another evidence we present to point out the similarity is Fig.5. The Lyapunov dimension calculated for our parameter setting are plotted in this figure. The Lyapunov dimension of the chaotic attractor is denoted by the horizontal dashed line, and that of each unstable periodic orbit found in the attractor corresponds to individual dot in the figure. All dots, except for *UPO* 1, are depicted near the line. Particularly the unstable periodic orbits with 9, 10 and 11 whirls have the similar properties to the chaotic attractor in terms of the Lyapunov dimension. On the other hand, it can be considered the chaotic orbit seldom passes close the shortest periodic orbit, and it means that the *economic regime* corresponding *UPO* 1 is seldom seen in the chaotic fluctuation.[16]

[16]The relation between the period of the unstable periodic orbit and the similarity of the orbit to the chaotic attractor is meaningful, though it is not discussed any more in this paper.

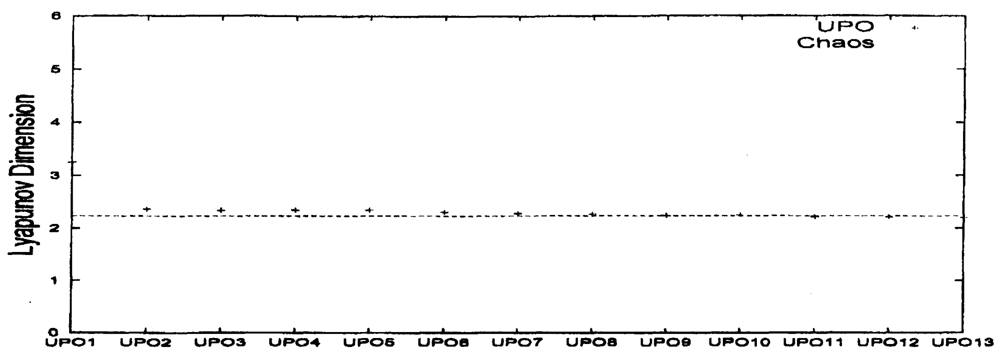

Fig. 5: Lyapunov dimension of *Chaos* and *UPOs*

From the physical point of view, it is well known there are some stable dimensions as regards unstable periodic orbits though they have instability. In fact, the stable directions are considerably abundant in our system. These must be the reasons why typical dynamics of chaotic solutions exist analogous to the unstable periodic orbits in this study. Thus our model displays the various *economic regimes* related to unstable periodic orbits.

Now, we go on with our discussion from the proposition that unstable periodic orbits concern the *economic regimes* experienced on the long term growth trajectory. To be argued are the effects of the variation of each parameter on the growth path where the economy should go. For example, the increase in public spending may improve the domestic business cycles for the time being. If we examine the variation and deformation of unstable periodic orbits by the change in a policy parameter, we can understand the effect on the growth path more clearly. Because the unstable periodic orbits have quite simple structure although they are qualitatively and quantitatively similar to the chaotic orbits. Concerning this point, we will discuss in detail in the next paper.

5 Conclusions

We have given a growth cycle system with international trade and shown some unstable periodic solutions found numerically in the chaotic attractor. It is cleared that they explain a typical long run dynamics consisting of a sequence of short run trade cycles and have the features corresponding to the long run movements classified as the *economic regimes* on the chaotic growth path. In addition, we have shown that the statistical properties of the chaotic fluctuation in the model are approximated to a considerable extent by those of the unstable periodic orbits. It implies unstable periodic solutions can be the keywords to understand the chaotic dynamics. From the above results, we emphasize the importance and usefulness of unstable periodic solutions embedded in the chaotic attractor as objects of studies of chaotic behavior. Above all, this point is important when we discuss the impacts of the economic policy in the chaotic situation.

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