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Shimura correspondence for Maass wave forms and Selberg zeta functions

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0 Introduction

Shimura in [Shm] established a significant correspondence from holomorphic modular forms of even integral weight $2k-2$ to modular forms of half integral weight $k-1/2$ which is consistent with the actions of Hecke operators. The converse correspondence was given by Shintani [Shn] in terms of period integrals. After these results, Kohnen ([Koh]) showed that this correspondence yields a bijection from the space S_{2k-2} of holomorphic modular forms of weight $2k-2$ on $SL_2(\mathbb{Z})$ to the plus space $S_{k-1/2}^{+}$ of modular cusp forms of weight $k-1/2$ on $\Gamma_0(4)$. On the other hand the plus space corresponds bijectively to the space $J_{k,1}^{cusp}$ of holomorphic Jacobi cusp forms (resp. the space $J_{k,1}^{sk,\,cusp}$ of skew holomorphic Jacobi cusp forms ([Sk1], [Sk2])) of weight k and index 1 on $SL_2(\mathbb{Z})$ if k is even (resp. odd). We exhibit here the isomorphisms in the case of $k>1$ being odd:

$$S_{2k-2}\cong S_{k-1/2}^{+}\cong J_{k,1}^{sk,\,cusp}. \tag{0.1}$$

As for the Maass wave forms Katok-Sarnak in [KS] formed the Shimura correspondence from the space of even Maass wave forms to a certain plus space consisting of automorphic forms of weight $1/2$. This work is understood to give an analogue of Shintani's converse correspondence to the case of Maass wave forms.

A purpose of this article is to explain an analogue of the right correspondence in the above (0.1) in the case of Maass wave forms. Another purpose is to interpret this Shimura correspondence for Maass wave forms from viewpoints of Selberg zeta functions and resolvent Selberg trace formulas. Finally we discuss some arithmetic aspects of Selberg zeta functions and also some applications.

We explain a little more in details. Let $\Gamma=SL_2(\mathbb{Z})$ and $\mathcal{H}_0^{even}$ denote the space of even functions $f\in\mathcal{H}_0=L^2(\Gamma\backslash\mathfrak{H})$ satisfying $f(-\overline{z})=f(z)$. It is known by Katok-Sarnak [KS] that to each Hecke eigen Maass wave form $f\in\mathcal{H}_0^{even}$ there corresponds an automorphic form g in the plus space of weight $1/2$ having reasonable properties. The whole plus space corresponds to the space $\mathcal{H}_{-1/4,\chi}$ of automorphic forms attached to the theta multiplier system χ defined by (1.2). This space plays an alternative role of the space of skew holomorphic Jacobi cusp forms in (0.1). We have computed the resolvent trace formula for $\mathcal{H}_0^{even}$ and that of $\mathcal{H}_{-1/4,\chi}$. There attached to the space $\mathcal{H}_0^{even}$ the Selberg zeta function $Z_{even}(s)$ is introduced, while associated to the multiplier system χ we have the Selberg zeta function $Z_\chi(s)$ (see (2.1), (2.3)). By comparing the both resolvent trace formulas for $\mathcal{H}_0^{even}$ and $\mathcal{H}_{-1/4,\chi}$ the conjectural bijectivity of the Katok-Sarnak correspondence will be reduced to some simple relationship of the two Selberg zeta functions concerned, which will be presented as a new conjecture (Conjecture 4). Towards the solution of our conjecture we discuss an explicit arithmetic expression of the Selberg zeta function $Z_\chi(s)$. The explicit espression of $Z_{even}(s)$ can easily be obtained similarly from that of $Z(s)$, the original Selberg zeta function for $SL_2(\mathbb{Z})$.

Finally as an application of the trace formula for $\mathcal{H}_0^{even}$ the prime geodesic theorem ((4.4), Theorem 6) for $GL_2(\mathbb{Z})$ will be given. This will be a refinement of the original result for the group $SL_2(\mathbb{Z})$ due to Sarnak [Sa].

1 Shimura correspondence for Maass wave forms

We use the symbol $e(w)$ for $\exp(2\pi i w)$. Throughout this article Γ denotes the modular group $SL_2(\mathbb{Z})$. Let $\mathfrak{H}$ denote the upper half plane. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathfrak{H}$, $J(A,z) := cz + d$ denotes the usual factor of automorphy for $SL_2(\mathbb{R})$. For a non-zero complex number w, $\arg w$ is chosen so that $-\pi < \arg w \leqq \pi$ and the branch of a holomorphic function $w^s = \exp(s \log w)$ $(w \neq 0)$ is fixed once and for all. For A, $B \in SL_2(\mathbb{R})$, the cocycle $\sigma_{2k}(A,B)$ is given by

$$\sigma_{2k}(A,B) = \exp\bigl(2ik\{\arg J(A,Bz) + \arg J(B,z) - \arg J(AB,z)\}\bigr)$$

(note here that the right hand side is independent of z).

Following [Fi], we give a definition of a multiplier system of Γ. Let V be a finite dimensional $\mathbb{C}$-vector space equipped with a positive definite hermitian scalar product $\langle v, w\rangle$ $(v, w \in V)$ and let $\mathcal{U}(V)$ denote the group of unitary tansformations of V with respect to the scalar product. A map $\chi : \Gamma \longrightarrow \mathcal{U}(V)$ is called a *multipier system* of Γ of weight $2k$ $(k \in \mathbb{R})$, if it satisfies

(i) $\chi(-1_2) = e^{-2\pi i k} id_V$, id_V being the identity map of V.

(ii) $\chi(AB) = \sigma_{2k}(A,B)\chi(A)\chi(B)$ for all A, $B \in \Gamma$.

We set, for $A \in SL_2(\mathbb{R})$ and a function f on $\mathfrak{H}$,

$$f|[M,k](z) := j_M(z)^{-1} f(Mz)$$

with $j_M(z) = \exp(2ik \arg J(M, z))$. Let $\mathcal{H}_{k,\chi}$ denote the space of V-valued measurable functions on $\mathfrak{H}$ with the properties

(i) $f|[M, k] = \chi(M)f$ for all $M \in \Gamma$,

(ii) $(f, f) := \int_{\Gamma\backslash\mathfrak{H}} \langle f(z), f(z)\rangle d\omega(z) < +\infty$.

Then $\mathcal{H}_{k,\chi}$ forms a Hilbert space with respect to the scalar product

$$(f, g) = \int_{\Gamma\backslash\mathfrak{H}} \langle f(z), g(z)\rangle d\omega(z), \qquad (f,\ g \in \mathcal{H}_{k,\chi}).$$

The differential operator Δ_k which is consistent with the action $f|[A, k]$ is given by

$$\Delta_k := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2iky\frac{\partial}{\partial x}.$$

A fundamental subspace $\mathcal{D}$ of $\mathcal{H}_{k,\chi}$ consists of C^2-class functions f satisfying $(\Delta_k f, \Delta_k f) < \infty$. Since $-\Delta_k$ is symmetric on $\mathcal{D}$, it is known by [Ro],I, Satz3.2 that there exists the unique self-adjoint extension $-\widetilde{\Delta}_k : \widetilde{\mathcal{D}} \longrightarrow \mathcal{H}_{k,\chi}$, where $\widetilde{\mathcal{D}}$ denotes the domain of definition of $-\widetilde{\Delta}_k$. By the self-adjointness of $-\widetilde{\Delta}_k$, eigen values of $-\widetilde{\Delta}_k$ are all real numbers. So we let

$$\lambda_n = \frac{1}{4} + r_n^2 \quad (\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots)$$

denote all distinct eigen values of $-\widetilde{\Delta}_k$. We may choose r_n so that $r_n \in i(0, \infty) \cup [0, \infty)$. Denote by $\mathcal{H}_{k,\chi}(s)$ the space of C^2-class functions $f \in \mathcal{H}_{k,\chi}$ satisfying $-\Delta_k f = s(1-s)f$. It is known that $\mathcal{H}_{k,\chi}(s)$ is a finite dimensional $\mathbb{C}$-vector space. Moreover

$$d_n := \dim \mathcal{H}_{k,\chi}\left(\frac{1}{2} + ir_n\right)$$

gives the multiplicity of $\lambda_n = \frac{1}{4} + r_n^2$ of $-\widetilde{\Delta}_k$. Let $s,\ a \in \mathbb{C}$. The spectral series attached to the multiplier system (Γ, χ) is defined by

$$S_{\Gamma,\chi}(s, a) := \sum_{n=0}^{\infty} \left(\frac{d_n}{(s-1/2)^2 + r_n^2} - \frac{d_n}{(a-1/2)^2 + r_n^2}\right). \tag{1.1}$$

It is known that the infinite series is absolutely convergent for s, a with $s \neq \frac{1}{2} \pm ir_n$, $a \neq \frac{1}{2} \pm ir_n$. Then $S_{\Gamma,\chi}(s, a)$ indicates a meromorphic function of s whose poles are located at $s = \frac{1}{2} \pm ir_n$. They are simple poles except for $s = 1/2$ $(r_n = 0)$.

In this note we exclusively concider the following two cases. First let $k=0$, $V=\mathbb{C}$ and χ be the trivial character of Γ. Then

$$\mathcal{H}_0 := \mathcal{H}_{0,\chi} = L^2(\Gamma\backslash\mathfrak{H}).$$

A function f of $\mathcal{H}_0$ is called an even function if it satisfies $f(-\bar{z}) = f(z)$. Let $\mathcal{H}_0^{even}$ (resp. $\mathcal{H}_0^{even}(s)$ $(s \in \mathbb{C})$) be the subspace of $\mathcal{H}_0$ consisting of even functions (resp. even C^2-class functions with $-\Delta_k f = s(1-s)f$). We denote by $S_\Gamma^{even}(s,a)$ the spectral series attached to the space $\mathcal{H}_0^{even}$ and the differential operator $\Delta_0 = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ which is similarly defined as in (1.1).

Another one is the multiplier system obtained from the theta transformation formula. Let $\theta_i(\tau,z)$ $(i=0,\ 1)$ be the usual theta series defined by

$$\theta_i(\tau,z) = \sum_{n\in\mathbf{Z}} e\big((n+i/2)^2\tau + (2n+i)z\big).$$

The theta transformation law for these theta series is described as follows:

$$\begin{pmatrix} \theta_0(M(\tau,z)) \\ \theta_1(M(\tau,z)) \end{pmatrix} = e\Big(\frac{cz^2}{J(M,\tau)}\Big) J(M,z)^{1/2} U(M) \begin{pmatrix} \theta_0(\tau,z) \\ \theta_1(\tau,z) \end{pmatrix} \quad \Big(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\Big),$$

where $U(M)$ is a unitary matrix of size two. For the convenience we consider the complex conjugate χ of U:

$$\chi(M) = \overline{U(M)} \quad (M \in \Gamma). \tag{1.2}$$

Since we have $\chi(-1_2) = e^{\pi i/2} 1_2$, χ forms a multiplier system of Γ with weight $-1/2$. Let $\mathcal{H}_{-1/4,\chi}$ and $\mathcal{H}_{-1/4,\chi}(s)$ be the spaces defined as above for this multiplier χ and Γ.

We explain here the Maass wave form version of the correspondences in (0.1). Denote by $j(M,\tau)$ $(M \in \Gamma_0(4))$ Shimura's factor of automorphy on $\Gamma_0(4)$ given by

$$j(M,\tau) = \theta(M\tau)/\theta(\tau),$$

$\theta(\tau)$ being the theta series $\theta_0(\tau,0) = \sum_{n\in\mathbf{Z}} e(n^2\tau)$. Katok-Sarnak defined a certain plus space consisting of Maass wave forms of weight $1/2$. For $s \in \mathbb{C}$ let T_s^+ denote the space consisting of C^2-class functions $g : \mathfrak{H} \longrightarrow \mathbb{C}$ satisfying the following two conditions:

(i) $g(Mz) = g(z)j(M,z)|cz+d|^{-1/2}$ for all $M \in \Gamma_0(4)$ and $\displaystyle\int_{\Gamma_0(4)\backslash\mathfrak{H}} |g(z)|^2\, d\omega(z) < +\infty$,

(ii) g has a Fourier expansion of the form:

$$g(z) = \sum_{n\in\mathbf{Z}} B(n,y)e(nx),$$

where we impose the condition that if $n \equiv 2, 3 \bmod 4$, then necessarily $B(n, y) = 0$. Moreover we assume the Fourier coefficients $B(n, y)$ for $n \neq 0$ are given of the form

$$B(n, y) = b(n) W_{\mathrm{sign} n/4,\, s-1/2}(4\pi y|n|), \tag{1.3}$$

where $W_{\alpha,\beta}$ denotes the usual Whittaker function.

Then a modified version of the second isomorphism in (0.1) generalized to Maass wave forms is given by

Proposition 1 *There exists the following anti $\mathbb{C}$-linear isomorphism*

$$\mathcal{H}_{-1/4,\chi}(s) \cong T_{\bar{s}}^{+} \tag{1.4}$$

given by $\mathcal{H}_{-1/4,\chi}(s) \ni g = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \mapsto G(\tau) = \overline{g_0(4\tau)} + \overline{g_1(4\tau)} \in T_{\bar{s}}^{+}$.

Remark. We note that, if s is real or of the form $s = \frac{1}{2} + ir$ with r real, then $T_s^+ = T_{\bar{s}}^+$, and moreover that $T_s^+ = \{0\}$, otherwise. In particular if $s = 1/4$, then the space $T_{1/4}^+ = T_{3/4}^+$ is nothing but $M_{1/2}^+(\Gamma_0(4))$.

For the proof of the proposition we refer to [Ar4].

An analogue of the correspondences in (0.1) to Maass wave forms is described as follows:

$$\mathcal{H}_0^{even}(2s - \frac{1}{2}) \sim T_s^+ \cong \mathcal{H}_{-1/4,\chi}(s).$$

Here the symbol "$\sim$" means that there exists a certain correspondence from $\mathcal{H}_0^{even}(2s - \frac{1}{2})$ to T_s^+ described as in the following theorem due to Katok-Sarnak [KS].

Theorem 2 ([KS]) *Let $s \in \mathbb{C}$ and let f be an even Hecke eigen Maass wave form of $\mathcal{H}_0^{even}(2s - 1/2)$. Then there exists $g = \sum_{n \in \mathbb{Z}} B(n, y) e(nx) \in T_s^+$ whose Fourier coefficients satisfy the relation*

$$b(-n) = n^{-3/4} \sum_{T,\ \det 2T = n} f(z_T) |AutT|^{-1} \qquad (n \in \mathbb{Z}_{>0}),$$

where T runs through all the $SL_2(\mathbb{Z})$-equivalence classes of positive definite half-integral symmetric matrices T with $\det 2T = n$ and $|AutT|$ denotes the order of the unit group of T. Moreover z_T is the point in $\mathfrak{H}$ corresponding to T; namely if we write $T = {}^t g^{-1} g^{-1}$ with $g \in GL_2^+(\mathbb{R})$, then $z_T = g(i)$.

Remark. It is expected that for each Hecke eigen Maass wave form f there exists at least one non-zero g corresponding to f. Under this expectation

$$\dim \mathcal{H}_0^{even}(2s - 1/2) \leqq \dim T_s^+ \qquad (?). \tag{1.5}$$

2 Selberg zeta functions concerned

The Selberg zeta functions $Z_{even}(s)$ has been introduced in [Ar3] to describe the trace formula for $\mathcal{H}_0^{even}$. Let $Prm^+(\Gamma)$ be the set of primitive hyperbolic elements P of Γ with $\mathrm{tr}P > 2$ and $Prm^+(\Gamma)^I$ the set consisting of $P \in Prm^+(\Gamma)$ that are primitive even in $GL_2(\mathbb{Z})$. Set $\widetilde{\Gamma} = GL_2(\mathbb{Z}) - SL_2(\mathbb{Z})$. An elemnet of $\widetilde{\Gamma}$ is called primitive hyperbolic, if $\mathrm{tr}P \neq 0$ and P cannot be represented as any power of any element of $\widetilde{\Gamma}$. Let $Prm^+(\widetilde{\Gamma})$ be the set of primitive hyperbolic elements P of $\widetilde{\Gamma}$ with $\mathrm{tr}P > 0$. For any element $P \in Prm^+(\Gamma)$ (or $P \in Prm^+(\widetilde{\Gamma})$) let $N(P)$ denote the square of the eigen value (> 1) of P. For any subset S of $GL_2(\mathbb{Z})$ which is stable under the $SL_2(\mathbb{Z})$-conjugation we denote by S/Γ the set of Γ-conjugacy classes in S. We define $Z_{even}(s)$ by

$$(2.1)\quad Z_{even}(s) = \prod_{\{P_0\}_\Gamma} \prod_{m=0}^{\infty} \left(1 - (-1)^m N(P_0)^{-s-m}\right)^2 \times {\prod_{\{P\}_\Gamma}}^{I} \prod_{m=0}^{\infty} \left(1 - N(P)^{-s-m}\right),$$

where $\{P_0\}_\Gamma$ is taken over $Prm^+(\widetilde{\Gamma})/\Gamma$ and the product $\prod_{\{P\}_\Gamma}^I$ indicates that $\{P\}_\Gamma$ runs through $Prm^+(\Gamma)^I/\Gamma$. The zeta function $Z_{even}(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Moreover it is immediate to see from (2.1) that the logarithmic derivative of $Z_{even}(s)$ is given by

$$(2.2)\quad \frac{Z'_{even}}{Z_{even}}(s) = \sum_{\{P\}_\Gamma} \sum_{m=1}^{\infty} \frac{\log N(P)}{1 - N(P)^{-m}} N(P)^{-ms} + \sum_{\{P_0\}_\Gamma} \sum_{\substack{n>0 \\ odd}} \frac{\log N(P_0)^2}{1 + N(P_0)^{-n}} N(P_0)^{-ns}$$

In [Ar1] we obtained the resolvent trace formula for the space $\mathcal{H}_{-1/4,\chi}$ involving the zeta function $Z_\chi(s)$ given by

$$(2.3)\qquad Z_\chi(s) = \prod_{\{P\}_\Gamma \in Prm^+(\Gamma)/\Gamma} \prod_{m=0}^{\infty} \det\left(1_2 - \chi(P) N(P)^{-s-m}\right).$$

On the other hand in [Ar3], [Ar4] we computed the resolvent trace formula for the space $\mathcal{H}_0^{even}$ and compared the both trace formulas for $\mathcal{H}_{-1/4,\chi}$ and $\mathcal{H}_0^{even}$ in an explicit manner. As an important consequence of this comparison we have the following fundamental theorem which connects the spectral series with the Selberg zeta functions concerned.

Theorem 3 *Let* $s' = 2s - 1/2$ *and* $a' = 2a - 1/2$ *with* $\mathrm{Re}(s) > 1$, $\mathrm{Re}(a) > 1$. *Then*

$$(2.4)\qquad \begin{aligned} & S_{\Gamma,\chi}(s,a) - \left(\frac{1}{2s-1}\frac{Z'_\chi}{Z_\chi}(s) - \frac{1}{2a-1}\frac{Z'_\chi}{Z_\chi}(a)\right) \\ &= 4\left(S_\Gamma^{even}(s',a') - \left(\frac{1}{2(2s'-1)}\frac{Z'_{even}}{Z_{even}}(s') - \frac{1}{2(2a'-1)}\frac{Z'_{even}}{Z_{even}}(a')\right)\right). \end{aligned}$$

For the proof we refer to [Ar4].

In (2.4) we expect that the hyperbolic contributions of the both hand sides should coincide. Therefore we may present the following conjecture.

Conjecture 4 *We have*

$$\frac{Z'_\chi}{Z_\chi}(s) = \frac{Z'_{even}}{Z_{even}}(2s-1/2) \quad \textit{or eqivalently,} \quad Z_\chi(s)^2 = Z_{even}(2s-1/2).$$

Towards the solution of the conjecture it will be necessary to obtain explicit arithmetic expressions of the zeta functions $Z_\chi(s)$ and $Z_{even}(2s-1/2)$; in particular that of $Z_\chi(s)$.

3 Arithmetic forms

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ $(M \neq \pm 1_2)$, we write

$$\widetilde{M} = \begin{pmatrix} b & (d-a)/2 \\ (d-a)/2 & -c \end{pmatrix} \quad \text{and} \quad n(M) = \frac{1}{\beta}\widetilde{M},$$

where $\beta = \gcd(b, d-a, c)$ (β is often denoted by $\beta(M)$). By a straightforward computation it is not dificult to see that, for $P \in GL_2(\mathbb{Z})$,

$$n(PMP^{-1}) = (\det P)^{-1} P n(M)^t P.$$

Let $t := a+d$ be the trace of M. The trace of $U(M)$ for $M \in \Gamma$, $t > 2$ is given by

$$\text{(3.1)} \qquad \mathrm{tr}U(M) = \frac{1}{(t-2)^{3/2}} \sum_{\lambda,\mu \in \mathbb{Z}/(t-2)\mathbb{Z}} e\Big(\frac{1}{t-2}(\lambda,\mu)\widetilde{M}\begin{pmatrix} \lambda \\ \mu \end{pmatrix}\Big).$$

The matrix entries of $U(M)$ have been computed by Skoruppa-Zagier [SZ] in terms of Gaussian sums. The formula above is easily derived from their results. We note that this trace depends only on Γ-conjugacy class of M:

$$\mathrm{tr}U(P^{-1}MP) = \mathrm{tr}U(M) \qquad (P \in \Gamma).$$

Let D range over all positive discriminants and $C_{pr}(D)$ denote the set of all primitive half integral symmetric matrices $N = \begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix}$ with $n_2^2 - 4n_1n_3 = D$. Denote by C_{pr} the collection of such $N \in C_{pr}(D)$ with D varying in all positive discriminants D. The modular group Γ acts on C_{pr} (also on $C_{pr}(D)$) in a usual manner by $N \longmapsto PN^tP$

$(N \in C_{pr},\ P \in \Gamma)$. Denote by $C_{pr}(D)//\Gamma$ (resp. $C_{pr}//\Gamma$) the set of the Γ-equivalence classes in $C_{pr}(D)$ (resp. C_{pr}) and by $h(D)$ the cardinality of this finite set $C_{pr}(D)//\Gamma$; namely $h(D)$ is the class number of primitive binary integral quadratic forms with discriminant D. Let $\epsilon_D = \frac{t+\beta\sqrt{D}}{2}$ denote the minimal solution of the Pell equation $t^2 - \beta^2 D = 4$ with $t,\ \beta \in \mathbb{Z}_{>0}$. Moreover we denote by $\epsilon_D^0 = \frac{t_0+\beta_0\sqrt{D}}{2}$ the minimal solution of the Pell equation $t_0^2 - \beta_0^2 D = -4$ with $t_0,\ \beta_0 \in \mathbb{Z}_{>0}$ if it exists (in this case $\epsilon_D = (\epsilon_D^0)^2$).

It is known that there exists a bijection from $Prm^+(\Gamma)$ to C_{pr}:

$$Prm^+(\Gamma) \ni P \longmapsto n(P) \in C_{pr}. \tag{3.2}$$

and that it induces a bijective map from the set $Prm^+(\Gamma)/\Gamma$ of all the Γ-conjugacy classes in $Prm^+(\Gamma)$ onto $C_{pr}//\Gamma$. For each $N = \begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix} \in C_{pr}(D)$ the opposite map is given by

$$N \longmapsto P = \begin{pmatrix} (t-\beta n_2)/2 & \beta n_1 \\ -\beta n_3 & (t+\beta n_2)/2 \end{pmatrix} \in Prm^+(\Gamma).$$

We define, for each positive discriminant D and a positive integer m,

$$C_{\chi,m}(D) := \sum_{N \in C_{pr}(D)//\Gamma} \mathrm{tr}(\chi(P)^m),$$

where P corresponds to N by the above bijective map, namely, $n(P) = N$. Then we have another expression of $(Z'_\chi/Z_\chi)(s)$:

$$\frac{Z'_\chi}{Z_\chi}(s) = \sum_{D>0} \sum_{m=1}^{\infty} C_{\chi,m}(D) \frac{\log(\epsilon_D^2)}{1-\epsilon_D^{-2m}} \epsilon_D^{-2ms},$$

where $\epsilon_D = \dfrac{t+\beta\sqrt{D}}{2}$ with (t,β) denoting the minimal solution of the Pell equation $t^2 - D\beta^2 = 4$, $t,\ \beta \in \mathbb{Z}_{>0}$. To obtain this expression we note that $\mathrm{tr}(\chi(P^m)) = \mathrm{tr}(\chi(P)^m)$. Since $\chi(P)^m$ are unitary matrices of size two, the values which $\mathrm{tr}(\chi(P)^m)$ can take are rather limited. We have tried to compute $C_{\chi,m}(D)$, but at present we have got only partial results.

Proposition 5 *Let D be a positive discriminant with $D \equiv 1 \bmod 4$. Assume that there exists a fundamental unit $\epsilon_D^0 = \dfrac{t_0+\beta_0\sqrt{D}}{2}$ $(t_0,\ \beta_0 \in \mathbb{Z}_{>0})$ with (t_0, β_0) giving the minimal solution of the Pell equation $t_0^2 - D\beta_0^2 = -4$ (namely, $N(\epsilon_D^0) = -1$) and* *moreover assume that t_0 is odd. For each $N \in C_{pr}(D)$, chosse $P \in Prm^+(\Gamma)$ which corresponds to N by $n(P) = N$. Then we have*

$$\mathrm{tr}(\chi(P)^m) = 2\cos\frac{m\pi}{3} \quad (m \in \mathbb{Z}_{>0}).$$

Accordingly,

$$C_{\chi,m}(D) = 2h(D)\cos\frac{m\pi}{3}.$$

Proof. Let ϵ_D, ϵ_D^0, P and N be the same as above. We note that $\epsilon_D = (\epsilon_D^0)^2$, $\widetilde{P} = \beta N$, from which we have $t-2 = t_0^2$ and $\beta = t_0\beta_0$. The expression (3.1) implies that

$$\mathrm{tr}U(P) = \frac{1}{(t-2)^{3/2}} \prod_{p\,|\,t-2} J_p \tag{3.3}$$

where for each prime p dividing $t-2$ we set

$$J_p := \sum_{\lambda,\mu\in\mathbb{Z}/p^e\mathbb{Z}} e\Big(\frac{1}{t-2}(\lambda,\mu)\widetilde{P}\begin{pmatrix}\lambda\\ \mu\end{pmatrix}\Big) = \sum_{\lambda,\mu\in\mathbb{Z}/p^e\mathbb{Z}} e\Big(\frac{\beta_0}{t_0}(\lambda,\mu)N\begin{pmatrix}\lambda\\ \mu\end{pmatrix}\Big)$$

with $p^e \| t-2$ (this means that p^e divides $t-2$ and p^{e+1} does not). For each prime p the function $e(x)$ restricted to $\mathbb{Q}$ extends to a continuous function $e_p(x)$ on $\mathbb{Q}_p$ in such a manner that $e_p(x) = e(x)$ for $x \in \mathbb{Q}$. Let a prime p divide $t-2$. By the assumption on t_0, p is an odd prime. We may assume that N is $SL_2(\mathbb{Z}_p)$-equivalent to $\begin{pmatrix}u & 0\\ 0 & -u^{-1}D\end{pmatrix}$ with $u \in \mathbb{Z}_p^\times$. Then,

$$J_p = \Big(\sum_{\lambda \bmod p^e} e_p\Big(\frac{\beta_0}{t_0}u\lambda^2\Big)\Big)\Big(\sum_{\mu \bmod p^e} e_p\Big(-\frac{\beta_0}{t_0}u^{-1}D\mu^2\Big)\Big).$$

If we write $t_0 = p^f t_0'$ with $(t_0', p) = 1$, then $e = 2f$ and

$$J_p = p^e G_{p^f}\Big(\frac{\beta_0 u}{t_0'}\Big) G_{p^f}\Big(-\frac{\beta_0 u^{-1} D}{t_0'}\Big),$$

where we put, for $a \in \mathbb{Z}_p^\times$,

$$G_{p^f}(a) = \sum_{\lambda \bmod p^f} e_p\Big(\frac{a\lambda^2}{p^f}\Big).$$

It is well-known and easy to see that

$$G_{p^f}(a) = \begin{cases} p^{f/2} & \text{if } f \text{ is even,}\\ p^{(f-1)/2}\psi_p(a)G(\psi_p) & \text{if } f \text{ is odd,}\end{cases}$$

where ψ_p is the non-trivial quadratic character modulo p (ψ_p is extended to $\mathbb{Z}_p^\times$) and $G(\psi_p)$ is the usual Gaussian sum associated to ψ_p:

$$G(\psi_p) = \sum_{x \bmod p} \psi_p(x) e_p(x).$$

Using the identity $G(\psi_p)^2 = \psi_p(-1)p$, one can compute J_p in an explicit manner:

$$J_p = \begin{cases} p^{3e/2} & \text{if } f \text{ is even,} \\ p^{3e/2}\psi_p(D) & \text{if } f \text{ is odd.} \end{cases}$$

Since $t_0^2 - \beta_0^2 D = -4$, we have $\psi_p(D) = 1$. Therefore by (3.3) we conclude that

$$\mathrm{tr}U(P) = 1, \qquad \text{namely,} \quad \mathrm{tr}\chi(P) = 1.$$

Set, for any $M \in \Gamma$,

$$\omega(M) = \det U(M).$$

Then ω forms a character of Γ. We now borrow some notations and results from [Ar2]. We may assume $N = \begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix} \in C_{pr}(D)$ to be reduced; namely, n_1, $n_3 > 0$ and $n_2 > n_1 + n_3$. Set

$$\alpha = \frac{n_2 + \sqrt{D}}{2n_1}.$$

Then N is reduced, if and only if α satisfies the condition

$$\alpha > 1 \quad \text{and} \quad 0 < \alpha' < 1, \tag{3.4}$$

which amounts to saying that α has a purely periodic continued fraction expansion:

$$\alpha = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots \, b_r - \cfrac{1}{b_1 - \ddots}}} \qquad (b_j \in \mathbb{Z},\ b_1, \ldots, b_r \geqq 2).$$

This expansion is denoted by

$$\alpha = [[\, \overline{b_1, b_2, \ldots, b_r} \,]] \tag{3.5}$$

(for this type of continued fraction expansion and the relationship with quadratic forms we refer to Zagier [Za]). Here r is called the period of α. Let B denote the Γ-equivalence class in $C_{pr}(D)$ represented by N. Then the period r depends only on the class B and is denoted by $r(B)$. Let B^* be the class of $C_{pr}(D)$ represented by $N^* = -{}^tQNQ$ with $Q = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$. Then we know in the proof of Proposition 5.1 of [Ar2] that

$$\omega(P) = i^{r(B)-r(B^*)}.$$

Moreover it is known that if there exists ϵ_D^0 with norm -1, then $r(B) = r(B^*)$. Therefore $\det U(P) = \omega(P) = 1$. This means that $U(P)$ is $GL_2(\mathbb{C})$-conjugate to some $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ with $\theta \in \mathbb{R}$. Then $\mathrm{tr}U(P) = 2\cos\theta = 1$, which implies $\theta = \pm\pi/3 + 2n\pi$ $(n \in \mathbb{Z})$. Thus,

$$\mathrm{tr}U(P^m) = \mathrm{tr}(U(P)^m) = 2\cos m\theta = 2\cos\frac{m\pi}{3}.$$

We have completed the proof of Proposition 5. ∎

Let $Z(s)$ denote the ordinary Selberg zeta function for Γ:

$$Z(s) = \prod_{\{P\}_\Gamma \in Prm^+(\Gamma)/\Gamma} \prod_{m=0}^{\infty} \left(1 - N(P)^{-s-m}\right).$$

It is well-known ([Sa], [He]) and easy to see from the bijection (3.2) that $Z(s)$ has the following arithmetic expression:

$$\begin{aligned} Z(s) &= \prod_{D>0} \prod_{m=0}^{\infty} \left(1 - \epsilon_D^{-2(s+m)}\right)^{h(D)}, \\ \frac{Z'}{Z}(s) &= \sum_{D>0} \sum_{m=1}^{\infty} h(D) \frac{\log(\epsilon_D^2)}{1 - \epsilon_D^{-2m}} \epsilon_D^{-2ms}. \end{aligned}$$

For each positive discriminant D let $C_{pr}^-(D)$ be the subset of $C_{pr}(D)$ consisting of N for which there exists a $P \in \widetilde{\Gamma}$ with $PN\,{}^tP = -N$. Denote by C_{pr}^- the union of all $C_{pr}^-(D)$ with D varying in all positive discriminants. We see easily that for each D only the case of either $C_{pr}^-(D) = \phi$ or $C_{pr}^-(D) = C_{pr}(D)$ occurs and moreover that $C_{pr}^-(D) = C_{pr}(D)$ if and only if ϵ_D^0 with norm -1 exists.

Therefore one can consider the set $C_{pr}^-(D)//\Gamma$ (or $C_{pr}^-//\Gamma$) of Γ-equivalence classeis in $C_{pr}^-(D)$ (in C_{pr}^-). Then it is easy to show in a similar manner that there exists a bijection from $Prm^+(\widetilde{\Gamma})$ onto C_{pr}^- via the map $Prm^+(\Gamma) \ni P \longmapsto n(P) \in C_{pr}^-$ and that it induces a bijective map from the set $Prm^+(\widetilde{\Gamma})/\Gamma$ onto $C_{pr}^-//\Gamma$.

Consequently by (2.1), (2.2), we have the expression for $Z_{even}(s)$:

$$Z_{even}(s) = \prod_{D>0}^{\sharp}\prod_{m=0}^{\infty}\left(1-(-1)^m\epsilon_D^{-(s+m)}\right)^{2h(D)} \times \prod_{D>0}^{I}\prod_{m=0}^{\infty}\left(1-\epsilon_D^{-2(s+m)}\right)^{h(D)},$$

$$\frac{Z'_{even}}{Z_{even}}(s) = \sum_{D>0}\sum_{m=1}^{\infty}\frac{2h(D)\log\epsilon_D}{1-(\epsilon_D)^{-2m}}(\epsilon_D)^{-2ms} + \sum_{D>0}^{\sharp}\sum_{\substack{n>0\\ odd}}\frac{2h(D)\log\epsilon_D}{1+(\epsilon_D)^{-n}}(\epsilon_D)^{-ns},$$

where $^\sharp$ (resp. ^{I}) indicates that D runs over all positive discriminants for which ϵ_D^0 with norm -1 exist (resp. for which ϵ_D^0 do not exist).

4 Prime geodesic theorem

It is known originally by Sarnak [Sa] that

$$\sum_{\substack{\{P\}_\Gamma\\ N(P)\leqq X}} \log N(P) = X + O(X^{\frac{3}{4}+\epsilon}). \tag{4.1}$$

and hence that

$$\sum_{\substack{D>0\\ \epsilon_D\leqq X}} h(D)\log((\epsilon_D)^2) = X^2 + O(X^{\frac{3}{2}+\epsilon}) \tag{4.2}$$

(note that (4.2) is easily derived from (4.1) with the help of the bijection from $Prm^+(\Gamma)/\Gamma$ onto $C_{pr}//\Gamma$). The best possible error term in the right hand side of (4.2) is $O(X^{\frac{7}{8}+\epsilon})$ which is given by Luo-Sarnak [LS].

Similarly by using the Selberg trace formula for the space $\mathcal{H}_0^{even}$ and by a general procedure (cf. [Iw], [He]) the following estimate follows:

$$\frac{1}{2}\Bigg(\sum_{\substack{\{P\}_\Gamma\\ N(P)\leqq X}} \log N(P) + \sum_{\substack{\{P_0\}_\Gamma\\ N(P_0)\leqq X}} \log N(P_0)^2\Bigg) = X + O(X^{\frac{3}{4}+\epsilon}) \qquad (\epsilon>0), \tag{4.3}$$

where the summations indicate that $\{P\}_\Gamma$ and $\{P_0\}_\Gamma$ run through $Prm^+(\Gamma)/\Gamma$ and $Prm^+(\tilde{\Gamma})/\Gamma$ with the conditions $N(P)\leqq X$ and $N(P_0)\leqq X$, respectively. Then by comparing (4.1) and (4.3) we have

$$\sum_{\substack{\{P_0\}_\Gamma\\ N(P_0)\leqq X}} \log N(P_0)^2 = X + O(X^{\frac{3}{4}+\epsilon}). \tag{4.4}$$

Therefore in the arithmetic terminology we have

Theorem 6 *Assume $\varepsilon > 0$. We have*

$$\sum_{\substack{D>0\\ \epsilon_D^0 \leqq X}}{}^{\sharp} h(D)\log((\epsilon_D^0)^2) = \frac{X^2}{2} + O(X^{\frac{3}{2}+\epsilon}) \tag{4.5}$$

and

$$\sum_{\substack{D>0\\ \epsilon_D \leqq X}}{}^{I} h(D)\log((\epsilon_D)^2) = X^2 + O(X^{\frac{3}{2}+\epsilon}),$$

where the second summation indicates that D runs through all positive discriminants for which fundamental units with norm -1 do not exist.

Proof. The former identity is a direct consequence of (4.4) and the bijectivity of the map from $Prm^+(\widetilde{\Gamma})/\Gamma$ onto $C_{pr}^-//\Gamma$, while the latter one is derived from (4.2) and (4.5).

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