数理解析研究所講究録 1338 巻 2003 年 107-121



# New vectors for GSp(4): a conjecture and some evidence

Brooks Roberts *
University of Idaho

Ralf Schmidt
Universität des Saarlandes

In this paper we present and state evidence for a conjecture on the existence and properties of new vectors for generic irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character for $F$ a nonarchimedean field of characteristic zero. To summarize the conjecture, let $\mathcal{O}$ be the ring of integers of $F$ and let $\mathcal{P}$ be the prime ideal of $\mathcal{O}$. We define, by a simple formula, a sequence of compact open subgroups $\mathrm{K}(\mathcal{P}^n)$ of $\mathrm{GSp}(4,F)$ indexed by nonnegative integers $n$. The first group $\mathrm{K}(\mathcal{O})$ is $\mathrm{GSp}(4,\mathcal{O})$. The second group $\mathrm{K}(\mathcal{P})$ is the other maximal compact subgroup of $\mathrm{GSp}(4,F)$, up to conjugacy, and is called the paramodular group. Automorphic forms for the global version of this group have been considered by T. Ibukiyama and his collaborators in a number of papers dealing with a genus two version of Eichler's correspondence and old and new forms. In general, we refer to $\mathrm{K}(\mathcal{P}^n)$ as the paramodular group of level $\mathcal{P}^n$. Given a generic irreducible admissible representation $\pi$ of $\mathrm{GSp}(4,F)$ with trivial central character, we consider the space of vectors fixed by each $\mathrm{K}(\mathcal{P}^n)$. The conjecture for $\pi$ makes three assertions. First, for some nonnegative $n$, the space of $\mathrm{K}(\mathcal{P}^n)$ fixed vectors is nonzero; second, if $N_\pi$ is the smallest such $n$, then the space of $\mathrm{K}(\mathcal{P}^{N_\pi})$ fixed vectors is one dimensional; and third, this one dimensional space contains a vector $W_\pi$ whose Novodvorsky zeta integral gives the Novodvorsky $L$-factor of the representation:

$$Z(s,W_\pi)=L(s,\pi).$$

We call $W_\pi$ the new vector of $\pi$. Zeta integrals depend on a choice of Whittaker model, which depends on a choice of nondegenerate character: we make a choice independent of $\pi$.

Evidently, the conjecture is similar to the theory of new vectors for generic irreducible admissible representations of $\mathrm{GL}(2,F)$ with trivial central character. Just as for $\mathrm{GL}(2,F)$, there is a simple relation between new vectors and

*Partially supported by a NSA Young Investigators Grant

$\epsilon$-factors. Assume the conjecture holds for $\pi$. There exists an Atkin-Lehner type element $u_{N_\pi}$ in GSp$(4, F)$ which normalizes K$(\mathcal{P}^{N_\pi})$ and whose square is in the center. Thus, $\pi(u_{N_\pi})W_\pi = \epsilon_\pi W_\pi$ for some $\epsilon_\pi = \pm 1$. Moreover, it is easy to show that

$$\epsilon(s, \pi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$$

so that $\epsilon(1/2, \pi) = \epsilon_\pi$. Here, $q$ is the order of $\mathcal{O}/\mathcal{P}$, and we use the mentioned nondegenerate character in the definition of the $\epsilon$-factor.

We state three pieces of evidence for the conjecture. First, the first two parts of the conjecture are true for all $\pi$ containing a nonzero vector fixed by the Iwahori subgroup. As evidence for the third part of the conjecture for such $\pi$ one also has

$$\epsilon(s, \varphi_\pi, \psi, \mathrm{d}x_\psi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$$

where $\varphi_\pi$ is the $L$-parameter assigned to $\pi$ by [KL]. Second, the first two parts of the conjecture are true for many $\pi$ induced from the Siegel or Klingen parabolic subgroups, and for these $\pi$, the level $\mathcal{P}^{N_\pi}$ is as expected. Finally, in proving the analogue for GSp(4) of the dihedral case of the global Langlands-Tunnell theorem, [R1] defined certain local $L$-packets $\Pi(\tau)$ and $L$-parameters $\varphi(\tau)$ for GSp$(4, F)$ which depend on a generic tempered irreducible admissible representation $\tau$ of GL$(2, E)$ with trivial central character, where $E$ is either a quadratic extension of $F$, or $F \times F$. The work [R1] gave strong global evidence that $\Pi(\tau)$ is the $L$-packet of $\varphi(\tau)$. Assuming $q$ is odd, we show that if $E/F$ is unramified or $E = F \times F$, then the generic element $\pi$ of $\Pi(\tau)$ contains a nonzero vector $W$ fixed by K$(\mathcal{P}^N)$, where $N$ is defined by $\epsilon(s, \varphi(\tau), \psi, \mathrm{d}x_\psi) = cq^{-N(s-1/2)}$, and $c$ is a constant. Moreover, $Z(s, W) = L(s, \pi)$.

To end this introduction, we emphasis that our conjecture is for generic irreducible admissible representations of GSp$(4, F)$ with trivial central character. In gathering evidence we have encountered various related cases and questions, as mentioned below. But, for example, currently we are not in a position to state a conjecture for the case of nontrivial central character.

## Notation

In this paper GSp$(4, F)$ is the group of $g$ in GL$(4, F)$ such that

$${}^t g \begin{bmatrix} 0 & 1_2 \\ -1_2 & 0 \end{bmatrix} g = \lambda(g) \begin{bmatrix} 0 & 1_2 \\ -1_2 & 0 \end{bmatrix}$$

for some $\lambda(g)$ in $F^\times$. Fix a continuous character $\psi$ of $F$ with conductor $\mathcal{O}$ and a generator $\varpi$ for $\mathcal{P}$. Let $|\cdot|$ be the valuation on $F$ such that if $\mu$ is a Haar measure on $F$, then $\mu(xA) = |x|\mu(A)$ for $x$ in $F$ and measurable sets $A$ in $F$. If $\pi$ is an irreducible admissible representation of a group of td-type [Car], let $\omega_\pi$ denote the central character of $\pi$. Let $\mathrm{L}_F = \mathrm{W}_F \times \mathrm{SU}(2,\mathbb{R})$ be the Langlands group of $F$, where $\mathrm{W}_F$ is the Weil group of $F$. A GSp(4) $L$-parameter over $F$ is a continuous homomorphism $\varphi : \mathrm{L}_F \to \mathrm{GSp}(4,\mathbb{C})$ such that $\varphi(x)$ is semisimple for all $x \in \mathrm{W}_F$ and $\varphi|_{1\times \mathrm{SU}(2,\mathbb{R})}$ is a smooth representation. We denote the $\epsilon$-factor of $\varphi$ with respect to $\psi$ and the Haar measure $\mathrm{d}x_\psi$ self-dual with respect to $\psi$ by $\epsilon(s,\varphi,\psi,\mathrm{d}x_\psi)$. One has $\epsilon(s,\varphi,\psi,\mathrm{d}x_\psi) = cq^{-N(s-1/2)}$ for some nonnegative integer $N$ and constant $c$.

# 1 The conjecture

To state the conjecture we need some definitions and results. First, we recall the fundamentals of the theory of Novodvorsky zeta integrals for $\mathrm{GSp}(4,F)$, as proven in [T-B]. Fix $c_1, c_2 \in F^\times$. Let $\pi$ be an irreducible admissible representation of $\mathrm{GSp}(4,F)$. We say that $\pi$ is **generic** if $\mathrm{Hom}_U(\pi,\psi_{c_1,c_2}) \neq 0$, where $U$ is the group of all elements

$$u = \begin{bmatrix} 1 & u_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & u_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $\psi_{c_1,c_2}(u) = \psi(c_1u_1 + c_2u_2)$. Whether $\pi$ is generic does not depend on the choice of $c_1$ and $c_2$. Assume $\pi$ is generic. Consider the space of functions $W : \mathrm{GSp}(4,F) \to \mathbb{C}$ such that $W(ug) = \psi_{c_1,c_2}(u)W(g)$ for $u$ in $U$ and $g$ in $\mathrm{GSp}(4,F)$, and $W$ is right invariant under some compact open subgroup of $\mathrm{GSp}(4,F)$. There exists a unique $\mathrm{GSp}(4,F)$ subspace $W(\pi,\psi_{c_1,c_2})$ of this space which is isomorphic to $\pi$ [Rod]. This subspace is called the Whittaker model of $\pi$ with respect to $\psi_{c_1,c_2}$. Fix Haar measures on $F^\times$ and $F$. Let $\mu : F^\times \to \mathbb{C}^\times$ be a continuous quasi-character. If $W$ is in $W(\pi,\psi_{c_1,c_2})$, the Novodvorsky **zeta integral** associated to $W$ and $\mu$ is

$$Z(s,W,\mu) = \int_{F^\times}\int_F W(\begin{bmatrix} y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & x & 0 & 1 \end{bmatrix})\mu(y)|y|^{s-3/2}\,\mathrm{d}x\,\mathrm{d}^\times y.$$

The $Z(s,W,\mu)$ for $W$ in $W(\pi,\psi_{c_1,c_2})$ converge absolutely in some right half plane and are elements of $\mathbb{C}(q^{-s})$. There exists $\gamma(s,\pi,\mu,\psi_{c_1,c_2})$ in $\mathbb{C}(q^{-s})$ such that the following **functional equation**

$$Z(1-s,\pi(\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix})W,(\omega_\pi\mu)^{-1}) = \gamma(s,\pi,\mu,\psi_{c_1,c_2})Z(s,W,\mu)$$

holds for $W$ in $W(\pi, \psi_{c_1,c_2})$. This $\gamma$-factor does not depend on the choices of Haar measure on $F$ and $F^\times$. Here,

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The $\mathbb{C}[q^s, q^{-s}]$ module generated by the $Z(s, W, \mu)$ for $W$ in $W(\pi, \psi_{c_1,c_2})$ is a fractional ideal of $\mathbb{C}(q^{-s})$ with generator of the form $1/Q(q^{-s})$ with $Q(0) - 1$, where $Q(X)$ is in $\mathbb{C}[X]$. We define

$$L(s, \pi, \mu) = 1/Q(q^{-s}).$$

This $L$-factor does not depend on the choices of Haar measures or $c_1$ and $c_2$. We also define

$$\epsilon(s, \pi, \mu, \psi_{c_1,c_2}) = \gamma(s, \pi, \mu, \psi_{c_1,c_2}) \frac{L(s, \pi, \mu)}{L(1 - s, \pi, (\omega_\pi \mu)^{-1})}.$$

The function $\epsilon(s, \pi, \mu, \psi_{c_1,c_2})$ is a nonzero monomial in $q^{-s}$ (e.g., see the top of p. 65 of [J]). The work [R2] verifies that $L(s, \pi, \mu) = L(s, \varphi, \mu)$, and $\epsilon(s, \pi, \mu, \psi_{1,-1}) = \epsilon(s, \varphi, \mu, \psi, dx_\psi)$ for the generic element $\pi$ in $\Pi(\chi, \tau)$ and $\varphi = \varphi(\chi, \tau)$, where $\Pi(\chi, \tau)$ and $\varphi(\chi, \tau)$ are the local $L$-packets and parameters defined in [R1]. We take $c_1 = 1$ and $c_2 = -1$ in the remainder of this paper, and write $W(\pi) = W(\pi, \psi_{1,-1})$, $\gamma(s, \pi, \mu) = \gamma(s, \pi, \mu, \psi_{1,-1})$ and $\epsilon(s, \pi, \mu) = \epsilon(s, \pi, \mu, \psi_{1,-1})$. If $\mu = 1$ we drop $\mu$ from our notation.

Next, we define the paramodular group of level $\mathcal{P}^n$. This requires that we first define the Klingen congruence subgroup of level $\mathcal{P}^n$. Let $n$ be a nonnegative integer. The **Klingen congruence subgroup** $\mathrm{Kl}(\mathcal{P}^n)$ of level $\mathcal{P}^n$ is the subgroup of $\mathrm{GSp}(4, F)$ of all elements $k$ such that $\lambda(k)$ is in $\mathcal{O}^\times$ and

$$k \in \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^n & \mathcal{P}^n & \mathcal{O} & \mathcal{P}^n \\ \mathcal{P}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix}.$$

Define the **Atkin-Lehner element** of level $\mathcal{P}^n$ in $\mathrm{GSp}(4, F)$ to be

$$u_n = \begin{bmatrix} 0 & J \\ -\varpi^n J & 0 \end{bmatrix}.$$

Evidently, $u_n^2 = \varpi^n$ is in the center of $\mathrm{GSp}(4, F)$. We now define the **paramodular group** $\mathrm{K}(\mathcal{P}^n)$ of level $\mathcal{P}^n$ to be the subgroup of $\mathrm{GSp}(4, F)$

generated by $\mathrm{Kl}(\mathcal{P}^n)$ and $u_n \mathrm{Kl}(\mathcal{P}^n) u_n^{-1} = u_n^{-1} \mathrm{Kl}(\mathcal{P}^n) u_n$. Equivalently, $\mathrm{K}(\mathcal{P}^n)$ is the subgroup of $\mathrm{GSp}(4, F)$ of all elements $k$ such that $\lambda(k)$ is in $\mathcal{O}^\times$ and

$$k \in \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{P}^{-n} & \mathcal{O} \\ \mathcal{P}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^n & \mathcal{P}^n & \mathcal{O} & \mathcal{P}^n \\ \mathcal{P}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix}.$$

**Conjecture 1.1** *Let $\pi$ be a generic irreducible admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. For each nonnegative integer $n$, let $\pi(\mathcal{P}^n)$ be the subspace of $\pi$ of vectors fixed by $\mathrm{K}(\mathcal{P}^n)$.*

1. *For some nonnegative integer $n$ the space $\pi(\mathcal{P}^n)$ is nonzero.*

2. *If $N_\pi$ is the smallest $n$ such that $\pi(\mathcal{P}^n)$ is nonzero, then*

$$\dim \pi(\mathcal{P}^{N_\pi}) = 1.$$

3. *There exists $W_\pi$ in $\pi(\mathcal{P}^{N_\pi})$ such that*

$$Z(s, W_\pi) = L(s, \pi).$$

In (3) of the conjecture we use the Whittaker model $W(\pi)$ for $\pi$ as defined above. If the conjecture holds for $\pi$, we call $\mathcal{P}^{N_\pi}$ the **level** of $\pi$ and $W_\pi$ the **new vector** of $\pi$.

The reader will note that while the conjecture is quite similar to the theory of new vectors for generic irreducible admissible representations of $\mathrm{GL}(2, F)$ with trivial central character, there is a significant difference: $\mathrm{K}(\mathcal{P}^n)$ is not contained in $\mathrm{K}(\mathcal{P}^{n+1})$! Thus, the theory of old vectors for $\mathrm{GSp}(4, F)$ will not be strictly analogous to that for $\mathrm{GL}(2, F)$. Nevertheless, we have some evidence, which we will not discuss here, that a coherent theory of old vectors for $\mathrm{GSp}(4, F)$ does exist.

# 2 A formal heuristic

Before stating implications for $\epsilon$-factors and our evidence, we will give some formal motivation for the conjecture. As far as we know, there does not exist a conjectural conceptual theory of new vectors for representations of the $F$ points of an arbitrary reductive algebraic group defined over $F$. The situation seems to be that, given a particular group like GSp(4), a theory of new vectors would be useful, but one has no reason to believe it exists. Groups for which new vectors have been considered include $\mathrm{GL}(n)$ (see [Cas], [D], [J-PS-S]) and SL(2) (see [LR]); for GSp(4) see also [S] for the case of square-free level. In our considerations we mostly have been guided by empirical facts. Still, for GSp(4) we can offer the following formal motivation.

Suppose one wants to derive the statement for a conjectural simple theory of new vectors for generic irreducible admissible representations of GSp(4) with trivial central character, and let $\pi$ be one such representation. In $\pi$ one might consider the space of Klingen vectors of level $\mathcal{P}^n$, i.e., the subspace $\pi_{\mathrm{Kl}}(\mathcal{P}^n)$ of vectors fixed by $\mathrm{Kl}(\mathcal{P}^n)$. Alternatively, one might consider vectors fixed by $\Gamma_0(\mathcal{P}^n)$, the Siegel congruence subgroup of level $\mathcal{P}^n$. However, without going into details, examples show that these vectors will not give a simple theory. One might hope, then, that Klingen vectors work, so that if $N$ is the smallest $n$ such that $\pi_{\mathrm{Kl}}(\mathcal{P}^n)$ is nonzero, then $\dim \pi_{\mathrm{Kl}}(\mathcal{P}^N) = 1$, and there exists a $W$ in $\pi_{\mathrm{Kl}}(\mathcal{P}^N)$ such that $Z(s, W) = L(s, \pi)$. One might also hope, as a consequence, that $\epsilon(s, \pi) = cq^{-N(s-1/2)}$ for some constant $c$. Examples show, however, for the smallest $n$ such that $\pi_{\mathrm{Kl}}(\mathcal{P}^n)$ is nonzero one can have $\dim \pi_{\mathrm{Kl}}(\mathcal{P}^n) > 1$: being a Klingen vector at the smallest nontrivial level is not enough to give uniqueness. It seems an enlargement of the Klingen congruence subgroup is required.

How can one arrive at such an enlargement? One might start with a Klingen vector $W$ of level $\mathcal{P}^N$ for which $Z(s, W) = L(s, \pi)$ and $\epsilon(s, \pi) = cq^{-N(s-1/2)}$ and see if $W$ reasonably might be fixed by a natural larger compact open subgroup. Using $Z(s, W) = L(s, \pi)$, the functional equation gives

$$\gamma(s, \pi)L(s, \pi) = Z(1 - s, \pi(\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix})W).$$

Dividing by $L(1 - s, \pi)$, one obtains the $\epsilon$-factor:

$$\epsilon(s, \pi) = Z(1 - s, \pi(\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix})W)/L(1 - s, \pi).$$

Now $\epsilon(s, \pi) = cq^{-N(s-1/2)}$; how can one make the right hand side look like this? A bit of algebra yields

$$\epsilon(s, \pi) = \frac{Z(1 - s, \pi(u_N)W)}{L(1 - s, \pi)} \cdot q^{-N(s-1/2)}.$$

It follows that $Z(s, \pi(u_N)W)$ is a constant multiple of $L(s, \pi)$, or equivalently, $Z(s, \pi(u_N)W)$ is a constant multiple of $Z(s, W)$. What condition on $W$ can guarantee this? It would hold if $\pi(u_N)W$ is a constant multiple of $W$; and if $\pi(u_N)W$ is a constant multiple of $W$, then $\pi(u_N)W$ is fixed by $\mathrm{Kl}(\mathcal{P}^N)$. Thus, one might consider, for nonnegative integers $n$, vectors $W$ such that $W$ and

$u_n W$ are both fixed by $\mathrm{Kl}(\mathcal{P}^n)$, or equivalently, vectors fixed by $\mathrm{K}(\mathcal{P}^n)$. Note that if $W$ is fixed by $\mathrm{Kl}(\mathcal{P}^n)$ then one has no reason to expect $\pi(u_n)W$ to also be fixed by $\mathrm{Kl}(\mathcal{P}^n)$, as $u_n$ does not normalize $\mathrm{Kl}(\mathcal{P}^n)$. On the other hand, $u_n$ does normalize the Borel congruence subgroup $B(\mathcal{P}^n) = \mathrm{Kl}(\mathcal{P}^n) \cap \Gamma_0(\mathcal{P}^n)$ of level $\mathcal{P}^n$, so if $W$ is fixed by $\mathrm{Kl}(\mathcal{P}^n)$, then at least $\pi(u_n)W$ will be fixed by $\mathrm{B}(\mathcal{P}^n)$.

# 3 The connection to $\epsilon$-factors

As mentioned in the introduction, the new vector and level of a representation satisfying the conjecture are closely connected to its $\epsilon$-factor. This is useful in providing evidence for the conjecture.

**Proposition 3.1** *Let $\pi$ be a generic irreducible admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character. Assume (1) and (2) of the conjecture for $\pi$ hold. Then $W_\pi$ is an eigenvector for $\pi(u_{N_\pi})$ with eigenvalue $\epsilon_\pi = \pm 1$:*

$$\pi(u_{N_\pi})W_\pi = \epsilon_\pi W_\pi.$$

*Assume (3) of the conjecture for $\pi$ also holds. Then*

$$\epsilon(s, \pi) = \epsilon_\pi q^{-N_\pi(s-1/2)},$$

*so that $\epsilon_\pi = \epsilon(1/2, \pi)$.*

*Proof.* Assume (1) and (2) of the conjecture for $\pi$ hold. A computation shows $u_{N_\pi}$ normalizes $\mathrm{K}(\mathcal{P}^{N_\pi})$. This implies that $\pi(u_{N_\pi})W_\pi$ is in $\pi(\mathcal{P}^{N_\pi})$; since this space is one dimensional, $\pi(u_{N_\pi})W_\pi = c_\pi W_\pi$ for some $c_\pi \in \mathbb{C}^\times$. As $u_{N_\pi}^2 = \varpi^{N_\pi}$, and $\pi$ has trivial central character, we have $\pi(u_{N_\pi})^2 = 1$, so that $\epsilon_\pi^2 = 1$. Next, assume (3) of the conjecture for $\pi$ also holds. Applying the functional equation to $W_\pi$, we obtain

$$Z(1-s, \pi(\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix})W_\pi) = \gamma(s, \pi) Z(s, W_\pi).$$

The definitions of the zeta integral and $u_{N_\pi}$ imply

$$Z(1-s, \pi(\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix})W_\pi) = \epsilon_\pi q^{-N_\pi(s-1/2)} Z(1-s, W_\pi).$$

Substituting this into the functional equation and using $Z(s, W_\pi) = L(s, \pi)$, we obtain

$$\epsilon_\pi q^{-N_\pi(s-1/2)} L(1-s, \pi) = \gamma(s, \pi) L(s, \pi),$$

so that $\epsilon(s,\pi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$. □

This proposition can be used to supply evidence for the conjecture. For example, suppose $\pi$ is a generic irreducible admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and parts (1) and (2) of the conjecture for $\pi$ are known. To obtain evidence for (3) of the conjecture for $\pi$ we may proceed as follows. Suppose that it is believed that a certain $L$-parameter $\varphi$ is the $L$-parameter associated to $\pi$ via the conjectural local Langlands correspondence, so that it is believed that $\epsilon(s,\varphi,\psi,\mathrm{d}x_\psi) = \epsilon(s,\pi)$ (or even suppose this equality is known). Then, in light of Proposition 3.1, verifying

$$\epsilon(1/2,\varphi,\psi,\mathrm{d}x_\psi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$$

gives evidence that (3) of the conjecture for $\pi$ holds.

# 4 Evidence

We currently have three different pieces of evidence for the conjecture. Our evidence considers a wide variety of generic irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character, and includes all representations of lower level and several families of induced and supercuspidal representations.

To state the first piece of evidence, define the **Iwahori subgroup** $I$ of $\mathrm{GSp}(4,F)$ to be the subgroup of all $k$ in $\mathrm{GSp}(4,F)$ with $\lambda(k)$ in $\mathcal{O}^\times$ and

$$k \in \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{P} \\ \mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O} \end{bmatrix}.$$

Then we have the following theorem. The number $\epsilon_\pi$ is defined in Proposition 3.1.

**Theorem 4.1** *Parts (1) and (2) of the conjecture are true for all generic irreducible admissible representations of* $\mathrm{GSp}(4,F)$ *with trivial central character which contain a nonzero vector fixed by the Iwahori subgroup. Moreover, suppose* $\pi$ *is a generic irreducible admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character which contains a nonzero vector fixed by the Iwahori subgroup, and let* $\varphi$ *be the L-parameter associated to* $\pi$ *by [KL]. Then*

$$\epsilon(1/2,\varphi,\psi,\mathrm{d}x_\psi) = \epsilon_\pi q^{-N_\pi(s-1/2)},$$

*which gives evidence that (3) of the conjecture for* $\pi$ *holds, as explained in section 3.*

In fact, we have computed the spaces of vectors fixed by $\mathrm{K}(\mathcal{P}^0)$, $\mathrm{K}(\mathcal{P}^1)$, $\mathrm{K}(\mathcal{P}^2)$ and $\mathrm{K}(\mathcal{P}^3)$ in all the, possibly nongeneric, irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character which contain a nonzero vector fixed by the Iwahori subgroup. This information is displayed in the table in the next section, which also includes information on how to understand the table. It is interesting to observe that (1) and (2) of the conjecture and $\epsilon(1/2,\varphi,\psi,\mathrm{d}x_\psi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$ hold, with one exception, for all irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character which contain a nonzero vector fixed by the Iwahori subgroup. This exception is the representation VIb, which does not admit a nonzero vector fixed by $\mathrm{K}(\mathcal{P}^0)$, $\mathrm{K}(\mathcal{P}^1)$, $\mathrm{K}(\mathcal{P}^2)$ or $\mathrm{K}(\mathcal{P}^3)$; we would expect a nonzero vector fixed by $\mathrm{K}(\mathcal{P}^2)$. However, the representations VIa and VIb form an $L$-packet, and the conjecture holds for the representation VIa. This suggests that (1) and (2) of the conjecture and the equality $\epsilon(1/2,\varphi,\psi,\mathrm{d}x_\psi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$ may be true for all irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character at the level of $L$-packets.

Our second parcel of evidence concerns certain induced representations. For the representations considered in the following theorem there is a naturally associated $L$-parameter $\varphi$, which should be the $L$-parameter associated to $\pi$ by the conjectural local Langlands conjecture; define the nonnegative integer $N$ by $\epsilon(s,\varphi,\psi,\mathrm{d}x_\psi) = cq^{-N(s-1/2)}$, where $c$ is a constant. We use the notation of [ST] for induced representations.

**Theorem 4.2** *Let $\tau$ be a generic irreducible admissible representation of $\mathrm{GL}(2,F)$. Assume $\omega_\tau$ is unramified.*

1. *(Siegel parabolic) Let $\sigma$ be an unramified quasi-character of $F^\times$ such that $\omega_\tau\sigma^2 = 1$. Assume*
$$\pi = \tau \rtimes \sigma$$
*is irreducible. Then $\pi$ is a generic irreducible admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and (1) and (2) of the conjecture for $\pi$ are true. Moreover, $N_\pi = N$.*

2. *(Klingen parabolic) Assume*
$$\pi = \omega_\tau^{-1} \rtimes \tau$$
*is irreducible. Then $\pi$ is a generic irreducible admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and (1) and (2) of the conjecture for $\pi$ are true. Moreover, $N_\pi = N$.*

Our final piece of evidence considers a broad distribution of representations of $\mathrm{GSp}(4,F)$, including supercuspidals. Recall that [R1] proved an analogue for $\mathrm{GSp}(4)$ of the global Langlands-Tunnell theorem. In doing so, [R1] defined certain local $L$-packets of representations of $\mathrm{GSp}(4,F)$. Let $\Pi(\tau)=\Pi(1,\tau)$ be such a local $L$-packet which happens to occur in a global situation as in Theorem 8.6 of [R1]. Thus, in particular, $\tau$ is a tempered generic irreducible admissible representation of $\mathrm{GL}(2,E)$ with trivial central character, where $E$ is either a quadratic extension of $F$, or $E=F\times F$. The packet $\Pi(\tau)$ has one or two elements, and all elements are tempered irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character. In [R2] it is shown that exactly one element $\pi$ of $\Pi(\tau)$ is generic. The paper [R1] also associates to $\tau$ an $L$-parameter $\varphi(\tau)=\varphi(1,\tau)$, and Theorem 8.6 of [R1] provides evidence that $\Pi(\tau)$ is the $L$-packet associated to $\varphi(\tau)$ by the conjectural local Langlands correspondence for $\mathrm{GSp}(4,F)$. Again, define the nonnegative integer $N$ by $\epsilon(s,\varphi(\tau),\psi,dx_\psi)=cq^{-N(s-1/2)}$, where $c$ is a constant.

**Theorem 4.3** *Assume $q$ is odd. If $E$ is unramified or $E=F\times F$, then $\pi$ contains a vector $W$ fixed by $\mathrm{K}(\mathcal{P}^N)$ such that $Z(s,W)=L(s,\pi)$.*

In writing $Z(s,W)=L(s,\pi)$ we are, as in the conjecture, using the Whittaker model $W(\pi)$ defined in section 1.

# 5 The table

The table gives information relevant to the conjecture about all the irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character which contain a nonzero vector fixed by the Iwahori subgroup.

## The first column

By [Bo], an irreducible admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character contains a nonzero vector fixed by $I$ if and only if it is an irreducible subquotient of a representation of $\mathrm{GSp}(4,F)$ with trivial central character induced from an unramified quasi-character of the Borel subgroup. The basic reference on representations of $\mathrm{GSp}(4,F)$ induced from a quasi-character of the Borel subgroup is section 3 of [ST], and we will use the notation of that paper. Thus, St is the Steinberg representation, $\mathbf{1}$ is the trivial representation, and $\nu=|\cdot|$. The reader will have to consult [ST] for more details. It is also useful to consult section 4.1 of [T-B]. Let $\chi_1$, $\chi_2$ and $\sigma$ be unramified quasi-characters of $F^\times$ with $\chi_1\chi_2\sigma^2=1$, so that the representation $\chi_1 \times \chi_2 \rtimes \sigma$ of $\mathrm{GSp}(4, F)$ induced from the quasi-character $\chi_1 \otimes \chi_2 \otimes \sigma$ has trivial central character. Of course, $\chi_1 \times \chi_2 \rtimes \sigma$ may be reducible. It turns out that by section 3 of [ST], there are six types of $\chi_1 \times \chi_2 \rtimes \sigma$ such that every irreducible admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character which contains a nonzero vector fixed by $I$ is an irreducible subquotient of a representative of one of these six types, and that no two representatives of two different types share a common irreducible subquotient. The first column gives the name of the type. In the table we choose a representative for a type with the notation as below, and in subsequent columns we give information about the irreducible subquotients of that representative. The types are described as follows:

**Type I**

These are the $\chi_1 \times \chi_2 \rtimes \sigma$ where $\chi_1, \chi_2$ and $\sigma$ are unramified quasi-characters of $F^\times$ such that $\chi_1\chi_2\sigma^2 = 1$ and $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible. See Lemma 3.2 of [ST].

**Type II**

These are the $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ where $\chi$ and $\sigma$ are unramified quasi-characters of $F^\times$ such that $\chi^2\sigma^2 = 1$. See Lemmas 3.3 and 3.7 of [ST].

**Type III**

These are the $\chi \times \nu \rtimes \nu^{-1/2}\sigma$ where $\chi$ and $\sigma$ are unramified quasi-characters of $F^\times$ such that $\chi\sigma^2 = 1$. See Lemmas 3.4 and 3.9 of [ST].

**Type IV**

These are the $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$ where $\sigma$ is an unramified quasi-character of $F^\times$ such that $\sigma^2 = 1$. See Lemma 3.5 of [ST].

**Type V**

These are the $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$ where $\xi_0$ and $\sigma$ are unramified quasi-characters of $F^\times$ such that $\xi_0$ has order two and $\sigma^2 = 1$. See Lemma 3.6 of [ST].

**Type VI**

These are the $\nu \times 1 \rtimes \nu^{-1/2}\sigma$ where $\sigma$ is an unramified quasi-character of $F^\times$ such that $\sigma^2 = 1$. See Lemma 3.8 of [ST].

## The second column

Choose a type as in the first column, and choose a representative $\chi_1 \times \chi_2 \rtimes \sigma$ of that type. Then $\chi_1 \times \chi_2 \rtimes \sigma$ admits a finite number of irreducible subquotients, and this number depends only on the type of $\chi_1 \times \chi_2 \rtimes \sigma$. We index the irreducible subquotients by lower case Roman letters. The letter "a" is reserved for the generic irreducible subquotient.

## The third column

This column lists the irreducible subquotients of the representative of the type of the first column. We use the specific notation as in the discussion of the first column.

## The fourth column

Suppose $\pi$ is an entry of the third column, and let $\varphi$ be the $L$-parameter associated to $\pi$ by [KL]. We define $N$ by the equation

$$\epsilon(s, \varphi, \psi, \mathrm{d}x_\psi) = cq^{-N(s-1/2)},$$

where $c$ is a constant.

## The fifth column

Using the notation of the explanation of the fourth column, this is $\epsilon = c = \epsilon(1/2, \varphi, \psi, \mathrm{d}x_\psi)$.

## The sixth, seventh, eighth and ninth columns

The numbers in the columns give the dimensions of the $\mathrm{K}(\mathcal{P}^n)$ fixed vectors for the representations in the third column for $n = 0, 1, 2$ and 3. Note that to save space we have abbreviated $\mathrm{K}(\mathcal{P}^n)$ by $\mathrm{K}(n)$. The signs under the numbers indicate how these spaces of $\mathrm{K}(\mathcal{P}^n)$ fixed vectors split under the action of the Atkin-Lehner operator $\pi(u_n)$. The signs are correct if in the type II case, where the central character of $\pi$ is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial, and in the type IV, V, and IV cases, where the central character of $\pi$ is $\sigma^2$, the character $\sigma$ is trivial. If these assumptions are not met, then the plus and minus signs must be interchanged to obtain the correct signs.

| | | representation | $N$ | $\epsilon$ | $K(0)$ | $K(1)$ | $K(2)$ | $K(3)$ |
|---|---|---|---|---|---|---|---|---|
| I | | $\chi_1 \times \chi_2 \rtimes \sigma$ (irred.) | 0 | 1 | $\underset{+}{\mathbf{1}}$ | $\underset{+-}{2}$ | $\underset{+++-}{4}$ | $\underset{\substack{+++\\---}}{6}$ |
| II | a | $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ | 1 | $-\sigma\chi(\varpi)$ | 0 | $\underset{-}{\mathbf{1}}$ | $\underset{+-}{2}$ | $\underset{++--}{4}$ |
| | b | $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$ | 0 | 1 | $\underset{+}{\mathbf{1}}$ | $\underset{+}{1}$ | $\underset{++}{2}$ | $\underset{++}{2}$ |
| III | a | $\chi \rtimes \sigma\mathrm{St}_{\mathrm{GL}(2)}$ | 2 | 1 | 0 | 0 | $\underset{+}{\mathbf{1}}$ | $\underset{+-}{2}$ |
| | b | $\chi \rtimes \sigma\mathbf{1}_{\mathrm{GL}(2)}$ | 0 | 1 | $\underset{+}{\mathbf{1}}$ | $\underset{+-}{2}$ | $\underset{++-}{3}$ | $\underset{++--}{4}$ |
| IV | a | $\sigma\mathrm{St}_{\mathrm{GSp}(4)}$ | 3 | $-\sigma(\varpi)$ | 0 | 0 | 0 | $\underset{-}{\mathbf{1}}$ |
| | b | $L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GL}(2)})$ | 2 | 1 | 0 | 0 | $\underset{+}{\mathbf{1}}$ | $\underset{+}{1}$ |
| | c | $L(\nu^{\frac{3}{2}}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-\frac{3}{2}}\sigma)$ | 1 | $-\sigma(\varpi)$ | 0 | $\underset{-}{\mathbf{1}}$ | $\underset{+-}{2}$ | $\underset{+--}{3}$ |
| | d | $\sigma\mathbf{1}_{\mathrm{GSp}(4)}$ | 0 | 1 | $\underset{+}{\mathbf{1}}$ | $\underset{+}{1}$ | $\underset{+}{1}$ | $\underset{+}{1}$ |
| V | a | $\delta([\xi_0, \nu\xi_0], \nu^{-\frac{1}{2}}\sigma)$ | 2 | $-1$ | 0 | 0 | $\underset{-}{\mathbf{1}}$ | $\underset{+-}{2}$ |
| | b | $L(\nu^{\frac{1}{2}}\xi_0\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-\frac{1}{2}}\sigma)$ | 1 | $\sigma(\varpi)$ | 0 | $\underset{+}{\mathbf{1}}$ | $\underset{+}{1}$ | $\underset{++}{2}$ |
| | c | $L(\nu^{\frac{1}{2}}\xi_0\mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{\frac{1}{2}}\sigma)$ | 1 | $-\sigma(\varpi)$ | 0 | $\underset{-}{\mathbf{1}}$ | $\underset{+}{1}$ | $\underset{--}{2}$ |
| | d | $L(\nu\xi_0, \xi_0 \rtimes \nu^{-\frac{1}{2}}\sigma)$ | 0 | 1 | $\underset{+}{\mathbf{1}}$ | 0 | $\underset{+}{1}$ | 0 |
| VI | a | $\tau(S, \nu^{-\frac{1}{2}}\sigma)$ | 2 | 1 | 0 | 0 | $\underset{+}{\mathbf{1}}$ | $\underset{+-}{2}$ |
| | b | $\tau(T, \nu^{-\frac{1}{2}}\sigma)$ | 2 | 1 | 0 | 0 | 0 | 0 |
| | c | $L(\nu^{\frac{1}{2}}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-\frac{1}{2}}\sigma)$ | 1 | $-\sigma(\varpi)$ | 0 | $\underset{-}{\mathbf{1}}$ | $\underset{-}{1}$ | $\underset{--}{2}$ |
| | d | $L(\nu, \mathbf{1}_{F^\times} \rtimes \nu^{-\frac{1}{2}}\sigma)$ | 0 | 1 | $\underset{+}{\mathbf{1}}$ | $\underset{+}{1}$ | $\underset{++}{2}$ | $\underset{++}{2}$ |

Table 1: Representations containing a nonzero vector fixed by the Iwahori subgroup. Consult section 5 for definitions and comments.

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