数理解析研究所講究録 1338 巻 2003 年 122-129

Supercuspidal Representations Attached to Symmetric Spaces

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§1. Some motivation.– The purpose of this lecture is to survey some recent results related to harmonic analysis on $H\backslash G$, where (G,H) is a symmetric space over a nonarchimedean local field. Harmonic analysis on symmetric spaces over $\mathbb{R}$ and $\mathbb{C}$ has been developed extensively by many authors over many years. By contrast, the p-adic theory is relatively undeveloped and new.

The impetus for much of the research in this field has come from Jacquet's relative trace formulas (starting with **[15]**) which were designed to study those automorphic representations of a given adelic group which satisfy a specific period condition. Without going into details about the global theory and what we mean by a "period condition," suffice it to say that the set of automorphic representations associated to a period condition tends to be an important set for a variety of reasons. For example, it may be the image of an important (automorphic or theta) lifting. It may be set of representations for which a certain automorphic L-function has a pole. It may be the set which determines when an induced representation is irreducible. Or it may be all of these things (and some other things as well). The original point of developing the local theory was that it described which representations could arise as local components of automorphic representations satisfying a period condition.

At first, most of the results in this area involved a combination of known techniques from: (a) the theory of harmonic analysis on p-adic groups, (b) global theory, and (c) the archimedean theory of symmetric spaces. Recently, more innovative techniques have been developed and we are seeing phenomena which have no archimedean analogues. I have been especially interested in finding techniques which exploit the special features of supercuspidal representations. Below I will indicate various local applications which are similar to the global applications mentioned above.

§2. Basic concepts.– We start by recalling the notion of a "symmetric space over a nonarchimedean field." Let F be a finite extension of some p-adic field $\mathbb{Q}_p$. For simplicity, we assume p is odd. Assume $\mathbf{G}$ is a connected reductive group over a field F and let $G=\mathbf{G}(F)$. Assume τ is an automorphism of $\mathbf{G}$ of order 2 which is defined over F. Let $\mathbf{G}^\tau$ denote the group of fixed points of τ and let $(\mathbf{G}^\tau)^\circ$ be the identity component of $\mathbf{G}^\tau$. Assume $\mathbf{H}$ is an F-subgroup of $\mathbf{G}$ such that $(\mathbf{G}^\tau)^\circ\subset\mathbf{H}\subset\mathbf{G}^\tau$. Now let $H=\mathbf{H}(F)$. Then the pair (G,H) (or the quotient $H\backslash G$) is called a *symmetric space over* F.

The terminology *harmonic analysis on* $H\backslash G$ may mean different things to different people. Classically, one might think of the decomposition of $L^2(H\backslash G)$ or some other induced representation $\mathrm{Ind}_H^G(1)$. For our purposes, it is appropriate to take $\mathrm{Ind}_H^G(1)$ to be the space $C^\infty(H\backslash G)$ of smooth (that is, locally constant) functions on $H\backslash G$.

Suppose $\pi: G\to \mathrm{Aut}(V)$ is an irreducible, admissible complex representation of G. Then we say π is H*-distinguished* if it occurs in $\mathrm{Ind}_H^G(1)$ in the sense that $\mathrm{Hom}_G(\pi,\mathrm{Ind}_H^G(1))$ is nonzero. A specific embedding $\Lambda:\pi\hookrightarrow\mathrm{Ind}_H^G(1)$ will be called an H*-model for* π.

Frobenius Reciprocity gives a canonical bijection between $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(1))$ and the space $\mathrm{Hom}_H(\pi, 1)$ of linear forms $\lambda : V \longrightarrow \mathbb{C}$ satisfying $\lambda(\pi(h)v) = \lambda(v)$, for all $h \in H$ and $v \in V$. Such linear forms λ are called *H-invariant functionals.* The explicit relation between Λ and λ is $\Lambda(v)(g) = \lambda(\pi(g)v)$, where $g \in G$ and $v \in V$.

The relation between H-models and H-invariant functionals is entirely analogous to the relation between Whittaker models and Whittaker functionals. One can hope for an analogue of the uniqueness property of Whittaker models in the symmetric space setting.

Definition. *We say that (G, H) has the multiplicity one property (or is a Gelfand pair) if* $\dim Hom_H(\pi, 1) \leq 1$ *for all irreducible, admissible representations π.*

Note that not everyone uses the terminology "Gelfand pair" in this way.

Definition. *We say (G, H) is a geometric Gelfand pair if there exists an anti-automorphism σ of G of order two such that $Hg^\sigma H = HgH$ for all $g \in G$.*

The Gelfand/Kazhdan Lemma [6]. *If there exists an anti-automorphism σ of G of order two which fixes all bi-H-invariant distributions on G then (G, H) is a Gelfand pair.*

The problem with this result is that, in principle, one needs to study all of the bi-H-invariant distributions on G in order to satisfy the hypotheses of the lemma. However, if (G, H) is a geometric Gelfand pair then the hypothese are automatically satisfied and hence we have the following:

Corollary. *If (G, H) is a geometric Gelfand pair then it is a Gelfand pair.*

§3. **The example** $(GL(n, E), GL(n, F))$.– Assume E is a quadratic extension of F and use the notation $x \mapsto \bar{x}$ for the nontrivial Galois automorphism of E/F. We consider the pair (G, H), with $G = GL(n, E)$ and $H = GL(n, F)$. This is a symmetric space over F. If $g \in G$ let $\bar{g}$ be the matrix obtained by applying $x \mapsto \bar{x}$ to each entry of g. Then τ is an automorphism of G of order two and H is the group of fixed points. It is easy to show $H\bar{g}^{-1}H = HgH$, for all $g \in G$. Hence, (G, H) is a geometric Gelfand pair.

The prototype example is the case in which $n = 2$ which I studied in my Ph.D. thesis and in some subsequent papers motivated by the work of Jacquet/Lai **[15]** and Harder/Langlands/Rapoport **[13]**. Flicker **[2]** generalized some of these results for arbitrary n. In some cases, he arrived at the appropriate conjectures relating distinguishedness with base change from unitary groups and the existence of a pole for the Asai L-function (a.k.a., twisted tensor L-function). For $n = 2$, there are two base change maps from $U(2, E/F)$ to $GL(n, E)$, each characterized by character relations analogous to Shintani's character relations which characterize quadratic base change for $GL(2)$. Flicker showed that the H-distinguished representations of G are precisely the representations which unstable lifts from $U(2, E/F)$. We also note that representations which are base change lifts from $U(2, E/F)$ are characterized by the symmetry condition $\tilde{\pi} \simeq \bar{\pi}$, where $\bar{\pi}(g) = \pi(\bar{g})$. The connection with Asai L-functions for general n has recently been firmly established in unpublished work of Kable **[17]** and, independently, Anandavardhanan and Tandon **[1]**. Their work builds on **[13]** and results developed by Flicker in several papers (starting with [2])

A natural problem, which we will call the "classification problem," is to explicitly determine which irreducible, admissible representations of G are H-distinguished. Assume for a moment longer that $n = 2$. For the nonsupercuspidal representations, it is fairly easy to give explicit conditions on the inducing data for these representations which correspond to distinguishedness. This was probably first done by Clozel in unpublished notes. (See **[2]**, **[4]** and **[9]** for more details.) For supercuspidal representations, a characterization of distinguishedness in terms of Jacquet-Langlands ϵ-factors was given in **[9]**:

Proposition 1 [9]. *Let ψ be a nontrivial character of E which is trivial on F. Then an irreducible, supercuspidal representation π of $G = GL(2,E)$ is H-distinguished if and only if $\epsilon(1/2, \pi \otimes \chi, \psi) = 1$ for all quasicharacters χ of $E^\times$ which are trivial on $F^\times$.*

The result in **[9]** is stated only under the assumption that the central character of π is trivial, however, this assumption is totally unnecessary. Note that the criterion in Proposition 1 is closely related to Corollary 2.4 in Saito's paper **[24]** on Tunnell's formula.

According to the work of Howe **[14]** (in the tame case) and Kutzko (in general), the supercuspidal representations of G may be realized via compactly supported induction from compact-mod-center subgroups. To give a satisfactory solution to the classification problem for distinguished supercuspidal representations requires giving conditions on the inducing data which corresponds to distinguishedness. This is partially done in the tame case for general n in **[12]**. (Note that if $p > n$ then all representations are tame.) According to Howe's construction, each irreducible tame supercuspidal representation π of G corresponds to a certain equivalence class of quasicharacters $\chi : L^\times \to \mathbb{C}^\times$ where L is a degree n tamely ramified extension of E. The quasicharacter χ must be E-admissible in the sense of Kutzko. If $\tilde{\pi} \simeq \bar{\pi}$, as is the case whenever π is H-distinguished, then it is a basic fact that there must exist an automorphism σ of order two of L/F such that $\sigma(x) = \bar{x}$ for all $x \in E$ and $\chi^{-1} = \chi \circ \sigma$. Let L' be the fixed field of σ. We say that the pair $(L/E, \sigma)$ is *odd* if the ramification degree $e(L/E)$ is odd, L/L' is unramified and E/F is ramified. Otherwise, $(L/E,\sigma)$ is *even*. Let $\chi_{L/L'}$ and $\chi_{E/F}$ be the class field theory characters associated to L/L' and E/F, respectively. The following result was proved in collaboration with Fiona Murnaghan:

Theorem 2 [12]. *Assume and $\chi = \chi^{-1} \circ \sigma$ is an E-admissible character of $L^\times$ and π is the associated irreducible, tame supercuspidal representation of G such that $\tilde{\pi} \simeq \pi \circ \tau$. If $(L/E, \sigma)$ is even and $\chi | L'^\times = 1$ or if $(E/F, \sigma)$ is odd and $\chi | L'^\times = \chi_{L/L'}$ then π is H-distinguished. If π is not H-distinguished and χ' is a character of $E^\times$ such that $\chi^L | L'^\times = \chi_{L/L'}$ then $\pi \otimes \chi'$ is H-distinguished. Such characters χ' always exist, for example, one may take any character of $E^\times$ whose restriction to $F^\times$ is $\chi_{E/F}$.*

A closely related result in the case in which E/F is unramified was obtained by Dipendra Prasad **[22]** by totally different methods.

Murnaghan's initial interest in such problems resulted from her joint work with Repka **[21]** on the reduciblity of induced representations of unitary groups, following the approach of Goldberg **[7]** and Shahidi **[25]**. Roughly speaking, G may be embedded as a Levi component of a maximal parabolic subgroup of the quasisplit unitary group $U(2n, E/F)$. If π is an irreducible, admissible representation of G then there is an associated induced representation $I(\pi)$ of $U(2n, E/F)$. Then $I(\pi)$ is irreducible if and only if π is H-distinguished.

When $n = 2$, this is evident in the work of Kazuko Konno [18], where all of the non-supercuspidal representations of the unitary group are computed.

The H-distinguished representations of G also arise in connection with the generic packet conjecture for unitary groups. A relative trace formula approach to this problem is developed for $n = 3$ in [5]. An alternate approach to the generic packet conjecture is given by Takuya Konno [19].

§4. **The example** $(GL(n), U(n))$.– Let E/F be a quadratic extension and $G = GL(n, E)$, as in the previous example. Now fix $\eta \in G$ which is hermitian in the sense that ${}^t\eta = \bar{\eta}$. Let $H = \{h \in G : h\eta{}^t\bar{h} = \eta\}$ be the associated unitary group. One may expect that (G, H) is Gelfand pair, since the analogous pair over a finite field is. Unfortunately, it is not a Gelfand pair, though we will see that it comes very close.

Theorem 3 [11]. *If π is an irreducible, tame supercuspidal representation of G then the dimension of $Hom_H(\pi, 1)$ is at most one.*

Again, it is natural to ask whether distinguishedness can be characterized in terms of a simple condition on the inducing data. We have:

Theorem 4 [11]. *Let L be a tamely ramified degree n extension of E which is embedded, via an E-embedding, in the ring M of n-by-n matrices with entries in E. Assume that ι is the automorphism of M given by applying the nontrivial Galois automorphism of E/F to the entries of each matrix in M. Let $G = M^\times = GL(n, E)$ and $T = L^\times$. Suppose χ is an admissible character of T and let π be the irreducible, supercuspidal representation of G associated to χ by Howe's construction. Let H be a unitary group in G associated to some hermitian matrix $\eta \in G$. Then the following conditions are equivalent:*

i. The space $Hom_H(\pi, 1)$ is nonzero.

ii. $\pi \sim \pi \circ \iota$.

iii. π is a base change lift from $GL(n, F)$.

iv. There exists an automorphism σ of L which agrees with ι on E and satisfies $\theta = \theta \circ \sigma$.

v. θ is trivial $U(1, L/L')$, where L' is the fixed field of an automorphism of L of order two which agrees with ι on E.

The method we use to solve the classification problem for tame supercuspidal representations for $(GL(n), U(n))$ has worked, with some modifications, for other pairs (G, H), as well. Using Jiu-Kang Yu's building theoretic extension [26] of Howe's construction, we hope to extend our methods to essentially arbitrary pairs (G, H).

The situation for $(GL(n), U(n))$ motivates the following:

Definition. *A pair (G, H) is a supercuspidal Gelfand pair if* $\dim Hom_H(\pi, 1) \leq 1$ *for all irreducible supercuspidal representations π of G.*

Fiona Murnaghan has recently found some examples of symmetric spaces which are not supercuspidal Gelfand pairs. Before this, there was a general suspicion that such pairs might not exist.

§5. **The example** $(GL(n), GL(n/2) \times GL(n/2))$.– Assume $n = 2m$ is even and let $G = GL(n, F)$, where we write the elements of G as block matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $a, b, c, d \in$

$M(m,F)$. Let $H \cong GL(m,F)\times GL(m,F)$ be the subgroup of G consisting of block diagonal matrices. Jacquet and Rallis [16] have shown in this case that (G,H) is a Gelfand pair. However, since (G,H) is not a geometric Gelfand pair, it was necessary for Jacquet and Rallis to conduct a very difficult 50-page analysis of the bi-H-invariant distributions on G in order to show that the hypotheses of the Gelfand/Kazhdan Lemma are satisfied.

We have the following block matrix identity:

$$\begin{pmatrix} bd^{-1}c-a & 0 \\ 0 & d-ca^{-1}b \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} -a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which is only valid when a and d are invertible. This shows that $Hg^{-1}H = HgH$ for almost all $g \in G$.

Definition. *(G,H) is almost a Gelfand pair if there exists an anti-automorphism σ of order two such that $Hg^{\sigma}H = HgH$, for almost all $g \in G$.*

Theorem 5 [10]. *Suppose α is an automorphism of order two of G such that $Hg^{\alpha}H = Hg^{-1}H$ for almost all $g \in G$. If π is an irreducible, H-distinguished supercuspidal representation of G then the contragredient $\tilde{\pi}$ of π is equivalent to the representation $\pi^{\alpha}(g) = \pi(g^{\alpha})$ and $\dim Hom_H(\pi,1) = \dim Hom_H(\tilde{\pi},1) = 1$.*

Corollary. *If (G,H) is almost a Gelfand pair then it must be a supercuspidal Gelfand pair.*

So, for $(GL(n), GL(n/2)\times GL(n/2))$, this reduces Jacquet/Rallis' lengthy argument to the above matrix identity. Of course, Jacquet/Rallis' result applies to arbitrary irreducible, admissible representations and not just supercuspidal representations. We will discuss some of the ingredients in the proof in the next section.

In the present context, Murnaghan and I [12] have an analogue of Theorem 2 which gives a weak solution to the classification problem. Since it is rather technical to state, we will not state it here.

We remark that distinguishedness may be correlated to the existence of a pole of the exterior square L-function, in much the same way that distinguishedness for $(GL(n,E), GL(n,F))$ is related to the existence of a pole of the Asai L-function. There also is a relation with reducibility of induced representations of classical groups and it is well known that the self-contragredient representations are expected to be lifts from classical groups. We refer to [12] for details and references for these things.

§6. **Character theory and the proof of Theorem 5.**– If V is the space of π and $\tilde{V}$ is the space of $\tilde{\pi}$, then we note that V embeds in the space $\tilde{V}^*$ of linear forms on $\tilde{V}$. In particular, $v \in V$ corresponds to the linear form $v \mapsto \langle v, - \rangle$ on $\tilde{V}$. The pairing $\langle -,- \rangle$ is the natural pairing on $V \times \tilde{V}$ and it extends in an obvious way to a pairing on $(\tilde{V}^* \times \tilde{V}) \cup (V \times V^*)$. The elements of $\tilde{V}^*$ are sometimes referred to as "generalized vectors" associated to π. Similarly, V^* is the space of generalized vectors for $\tilde{\pi}$. If $f \in C_c^{\infty}(G)$ and $\lambda \in \tilde{V}^*$ then we may define $\pi(f)\lambda \in \tilde{V}^*$ by

$$\langle \pi(f)\lambda, \tilde{v} \rangle = \langle \lambda, \tilde{\pi}(\check{f})\tilde{v} \rangle,$$

where $\check{f}(g) = f(g^{-1})$ and $\tilde{v} \in \widetilde{V}$. In fact, $\pi(f)\lambda$ lies in V. Consequently, given generalized vectors $\lambda \in \widetilde{V}^*$ and $\tilde{\lambda} \in V^*$ there is an associated distribution

$$\Theta_{\lambda,\tilde{\lambda}}(f) = \langle \pi(f)\lambda, \tilde{\lambda} \rangle.$$

It is natural to refer to such distributions as *generalized matrix coefficients* because they generalize the matrix coefficients $f_{v,\tilde{v}}(g) = \langle \pi(g)v, \tilde{v} \rangle$, where $g \in G$, $v \in V$ and $\tilde{v} \in \widetilde{V}$.

For harmonic analysis on $H\backslash G$, the generalized matrix coefficients of most interest are the coefficients $\Theta_{\lambda,\tilde{\lambda}}$ for which $\lambda \in \mathrm{Hom}_H(\tilde{\pi}, 1)$ and $\tilde{\lambda} \in \mathrm{Hom}_H(\pi, 1)$. We call these *spherical matrix coefficients.*

If (G, H) is a Gelfand pair and π and $\tilde{\pi}$ are distinguished then, up to scalar multiples, there is a unique nonzero spherical matrix coefficient of π. This spherical matrix coefficient should be viewed as a symmetric space analogue of the character distributuion $\mathrm{tr}\pi(f)$ of π. One can ask whether these objects enjoy the same analytic properties (such as local integrability and smoothness on the regular set) established for the character distributions by Harish-Chandra (using various results of Howe). Indeed this is the case for pairs of the form $(\mathbf{H}(E), \mathbf{H}(F))$, where $\mathbf{H}$ is a connected reductive F-group and E/F is quadratic. (See **[8]**) However, Rader and Rallis **[23]** have studied this problem for general pairs (G, H) and they have shown the precise extent to which Harish-Chandra's results fail to generlize nicely.

Let us now give a sketch of the formal argument which underlies the proof of the theorem. For the sake of convenience and to simplify our exposition, we now assume that G has trivial center. Assume π is supercuspidal, as in the hypothesis of the theorem. Note that if $f_{v,\tilde{v}}$ is a matrix coefficient of π then, since π is supercuspidal, we have $f_{v,\tilde{v}} \in C_c^\infty(G)$. In addition, $\check{f}_{v,\tilde{v}} = f_{\tilde{v},v}$ is a matrix coefficient of $\tilde{\pi}$. So if π is a supercuspidal H-distinguished representation of G with spherical matrix coefficient $\Theta_{\lambda,\tilde{\lambda}}$ and if $f_{\tilde{v},v}$ is a matrix coefficient of $\tilde{\pi}$ then the quantity $\Theta_{\lambda,\tilde{\lambda}}(f_{\tilde{v},v})$ is well defined. A straightforward generalization of the Schur orthogonality relations shows that

$$\Theta_{\lambda,\tilde{\lambda}}(f_{\tilde{v},v}) = d(\pi)^{-1} \langle \lambda, \tilde{v} \rangle \langle v, \tilde{\lambda} \rangle,$$

where $d(\pi)$ is the formal degree of π.

Unfortunately, $\Theta_{\lambda,\tilde{\lambda}}$ is not a true matrix coefficient, however, it may be realized, in a suitable sense, as a limit of matrix coefficients $f_{w_n,\tilde{w}_n}$. For the moment, in order to provide a formal heuristic, we will pretend that $\Theta_{\lambda,\tilde{\lambda}}$ coincides with a matrix coefficient $f_{w,\tilde{w}}$, where w and $\tilde{w}$ are H-fixed vectors. To legitimize this heuristic, one must engage in various technical manipulations involving approximations of $\Theta_{\lambda,\tilde{\lambda}}$ by matrix coefficients.

Proceeding formally, we now let $\varphi = f_{w,\tilde{w}} f_{\tilde{v},v} \in C_c^\infty(G)$. Rader and Rallis have produced a symmetric space analogue of the Weyl integration formula which formally looks like:

$$\int_G \varphi(g)\, dg = \sum_T \frac{1}{w_T} \int |\Delta(t)|^{1/2} f_{w,\tilde{w}}(t) \Phi^T_{f_{\tilde{v},v}}(t)\, dt,$$

where: (i) we are summing over classes of "Cartan subsets" T of $H\backslash G$, (ii) Δ is a symmetric space analogue of the Weyl discriminant, and (iii) $\Phi^T_{f_{\tilde{v},v}}(t)$ is a type of orbital integral of

$f_{\tilde{v},v}(t)$ which represents an average over the double coset HtH. So we have a fundamental identity

$$d(\pi)^{-1}\langle\lambda,\tilde{v}\rangle\langle v,\tilde{\lambda}\rangle = \sum_T \frac{1}{w_T}\int |\Delta(t)|^{1/2} f_{w,\tilde{w}}(t)\Phi^T_{f_{\tilde{v},v}}(t)\, dt.$$

This identity, though we have obtained it by dubious means, is actually valid if $f_{w,\tilde{w}}$ is interpreted as the smooth function, given by Rader and Rallis, which represents $\Theta_{\lambda,\tilde{\lambda}}$ on the (G,H)-regular set.

Now let σ be the anti-involution $g^\sigma = (g^\alpha)^{-1}$, where α is as in the hypothesis of the theorem. We observe that $f_{\tilde{v},v}(g^\sigma) - \langle v, \tilde{\pi}(g^\sigma)\tilde{v}\rangle - \langle \pi(g^\alpha)v,\tilde{v}\rangle$ is a matrix coefficient of $\pi^\alpha(g) = \pi(g^\alpha)$. Since

$$d(\pi)^{-1}\langle\lambda,\tilde{v}\rangle\langle v,\tilde{\lambda}\rangle = \int_G \varphi(g)\, dg = \int_G \varphi(g^\sigma)\, dg$$

is nonzero for suitable v and $\tilde{v}$ and since this is an average of a matrix coefficient of π against a matrix coefficient of π^α, Schur orthogonality implies that π must be equivalent to $\tilde{\pi}^\alpha$. Thus we may choose a nonzero intertwining operator $I : V \to \tilde{V}$ such that $I(\pi(g)v) = \tilde{\pi}^\alpha(g)I(v)$ for all $g \in G$ and $v \in V$. Consequently,

$$f_{\tilde{v},v}(g^\sigma) = \langle \pi(g^\alpha)v,\tilde{v}\rangle = \langle I^{-1}(\tilde{v}), \tilde{\pi}(g)I(v)\rangle = f_{I(v),I^{-1}(\tilde{v})}(g).$$

It follows that

$$\int_G \varphi(g^\sigma)\, dg = d(\pi)^{-1}\langle\lambda, I(v)\rangle\langle I^{-1}(\tilde{v}),\tilde{\lambda}\rangle.$$

This yields the identity

$$\langle\lambda,\tilde{v}\rangle\langle v,\tilde{\lambda}\rangle = \langle\lambda, I(v)\rangle\langle I^{-1}(\tilde{v}),\tilde{\lambda}\rangle.$$

The theorem follows immediately from this identity, though this may not be obvious. Indeed, fix $\tilde{v}$ such that $\langle\lambda,\tilde{v}\rangle \neq 0$. Since we know that v may be chosen so that $\langle v,\tilde{\lambda}\rangle \neq 0$, we see that $\langle I^{-1}(\tilde{v}),\tilde{\lambda}\rangle \neq 0$. Now letting v vary, we deduce that $I(\ker\tilde{\lambda}) = \ker\lambda$. This seems to contradict the fact that λ and $\tilde{\lambda}$ were chosen independently. The only explanation of this is that both $\mathrm{Hom}_H(\pi,1)$ and $\mathrm{Hom}_H(\tilde{\pi},1)$ have dimension one and thus we essentially have no choice when choosing λ and $\tilde{\lambda}$. This completes the formal argument. The precise details of the proof of the theorem are in [10].

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