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Skew holomorphic Jacobi forms of general degree

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Introduction

In the study of modular forms of half integral weight, it is well known that Kohnen's plus space (a certain subspace of elliptic modular forms of half integral weight) of weight "even integer minus 1/2" is isomorphic to the space of Jacobi forms of index 1 (cf. Eichler-Zagier[3] Theorem 5.4). Moreover, Skoruppa[14] introduced the notion of skew holomorphic Jacobi forms which satisfy a certain transformation formula like Jacobi forms but not holomorphic functions, and he constructed a linear isomorphism between skew holomorphic Jacobi forms of index 1 and Kohnen's plus space of weight "odd integer minus 1/2" in the case of degree 1. This notion of skew holomorphic Jacobi forms was generalised for higher degree by Arakawa[1]. There are several works about the Jacobi form of general degree (cf. [1],[2],[8],[10],[11],[15],[18] etc), but there are few results about skew holomorphic Jacobi forms of general degree except Arakawa[1].

The purpose of this article is to investigate skew holomorphic Jacobi forms of general degree. This article is a summarisation of three papers of Hayashida[4],[5],[6]. In Section 1 we describe the definition of skew holomorphic Jacobi forms following Arakawa[1]. Skew holomorphic Jacobi forms are not holomorphic functions but vanish under a certain differential operator $\Delta_{\mathcal{M}}$ which will be defined in Section 1. In Section 2 we give an isomorphism between plus space of general degree and the space of skew holomorphic Jacobi forms of index 1 of general degree. In Section 3 we construct Klingen type Eisenstein series of skew holomorphic Jacobi forms. In order to obtain this construction, we used a certain differential operator $\Delta_{\mathcal{M}}$. In Section 4 we give an analogue of the Zharkovskaya's theorem for Siegel modular forms of half integral weight.

1 Skew holomorphic Jacobi forms

We denote $Sp_n(\mathbb{R})$ the real symplectic group of size $2n$. Let $\mathfrak{H}_n$ denote Siegel upper half space of degree n, and let $\mathfrak{D}_{n,l} = \mathfrak{H}_n \times M_{n,l}(\mathbb{C})$.

Skew holomorphic Jacobi forms was first introduced by Skoruppa[14] as function on $\mathfrak{D}_{1,1}$, and he showed the isomorphism between Kohnen's plus space and the space of skew holomorphic Jacobi forms of index 1 in the case of degree 1. This notion of skew holomorphic Jacobi forms was generalised for higher degree by Arakawa[1]. In this section, we would like to describe the definition of skew holomorphic Jacobi forms following Arakawa[1]. We prepare some notations.

Let $G^J_{n,l}$ be the Jacobi group, $G^J_{n,l}$ is a subgroup of $Sp_{n+l}(\mathbb{R})$ as follows,

$$G^J_{n,l} := \left\{ \begin{pmatrix} * & 0 & * & * \\ * & 1_l & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1_l \end{pmatrix} \in Sp_{n+l}(\mathbb{R}) \right\}$$

We put $\Gamma^J_{n,l} = G^J_{n,l} \cap Sp_{n+l}(\mathbb{Z})$.

We denote the action of $Sp_n(\mathbb{R})$ on $\mathfrak{H}_n$ by

$$M \cdot Z \ := \ (AZ+B)(CZ+D)^{-1}$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_n(\mathbb{R})$, and $Z \in \mathfrak{H}_n$.

Let $M > 0$ be a symmetric half integral matrix of size l. Now we describe the definition of the skew holomorphic Jacobi forms.

Definition 1 (skew holomorphic Jacobi forms cf. [1]). *Let $F(\tau, z)$ be a function on $\mathfrak{D}_{n,l}$, holomorphic on $\mathfrak{H}_n$ and real analytic on $M_{n,l}(\mathbb{C})$. We say F is a skew holomorphic Jacobi form of weight k of index $\mathcal{M}$ belongs to $\Gamma^J_{n,l}$, if F satisfies the following two conditions :*

(1) *We put $F_{\mathcal{M}}(Z) := F(\tau, z)e(tr(\mathcal{M}\tau'))$ for $Z = \begin{pmatrix} \tau & z \\ {}^tz & \tau' \end{pmatrix} \in \mathfrak{H}_{n+l}$, then $F_{\mathcal{M}}$ satisfies*

$$F_{\mathcal{M}}(\gamma \cdot Z) \ = \ \overline{\det(CZ+D)}^{k-l} |\det(CZ+D)|^l F_{\mathcal{M}}(Z) \ ,$$

for every $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^J_{n,l}$.

(2) *F has the Fourier expansion as follows :*

$$F(\tau, z) = \sum_{N \in Sym_n, R \in M_{n,l}(\mathbb{Z})} C(N,R)\, e(N\tau - \frac{i}{2}(4N - R\mathcal{M}^{-1}{}^tR) Im\,\tau + {}^t Rz) \,,$$

where we denote by Sym_n the set of all half integral symmetric matrices of size n, and $C(N,R)=0$ unless $4N - R\mathcal{M}^{-1t}R \le 0$.

Moreover, if Fourier coefficients satisfy a condition that $C(N,R)=0$ unless $4N - R\mathcal{M}^{-1t}R < 0$, we say F is a skew holomorphic Jacobi cusp form.

We set differential operators $\frac{\delta}{\delta\tau} := \left(\frac{1+\delta_{s,t}}{2}\frac{\delta}{\delta\tau_{s,t}}\right)$, $\frac{\delta}{\delta z} := \left(\frac{\delta}{\delta z_{i,j}}\right)$ for $(\tau,z)\in\mathfrak{D}_{n,l}$, where $\delta_{s,t}$ is the Kronecker's delta symbol, and $\frac{\delta}{\delta\tau_{s,t}} := \frac{1}{2}\left(\frac{\delta}{\delta x_{s,t}} - i\frac{\delta}{\delta y_{s,t}}\right)$, where $x_{s,t}$ (resp. $y_{s,t}$) is the real part (resp. the imaginary part) of $\tau_{s,t}$. We define a differential operator

$$\Delta_{\mathcal{M}} := 8\pi i\frac{\delta}{\delta\tau} - \frac{\delta}{\delta z}\mathcal{M}^{-1t}\frac{\delta}{\delta z}.$$

We note the following equivalence. If a function F on $\mathfrak{D}_{n,l}$ satisfies the condition (1) of the definition of skew holomorphic Jacobi forms, and if $n>1$, then the condition (2) is equivalent to the following condition

(2') $\Delta_{\mathcal{M}}F = 0_n$.

We denote the vector space of skew holomorphic Jacobi forms (resp. skew holomorphic Jacobi cusp forms) of weight k of index $\mathcal{M}$ by $J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$ (resp. $J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_n)$).

2 Isomorphisms between skew holomorphic Jacobi forms of index 1 and plus spaces

First, we shall describe the definition of Siegel modular forms of half integral weight.

For positive integer q, we put

$$\Gamma_0^{(n)}(q) := \{M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in Sp_n(\mathbb{Z}) \mid C \equiv 0 \,(mod\, q)\}$$

is the congruence subgroup of the symplectic group $Sp_n(\mathbb{Z})$.

We define a character ψ on $\Gamma_0^{(n)}(4)$, ψ is given by $\psi(M) := \left(\frac{-4}{\det D}\right)$ for $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma_0^{(n)}(4)$.

We put the standard theta series $\theta^n(Z)$ and put a function $j(M,Z)$ as follows:

$$\begin{aligned} \theta^n(Z) &:= \sum_{m\in\mathbb{Z}^n} e({}^t mZm), \qquad (Z \in \mathfrak{H}_n) \\ j(M,Z) &:= \frac{\theta^n(M\cdot Z)}{\theta^n(Z)}, \qquad (M \in \Gamma_0^{(n)}(4), Z \in \mathfrak{H}_n), \end{aligned}$$

then this $j(M,Z)$ satisfies

$$j(M,Z)^2 = \psi(M)\det(CZ+D) \text{ for any } M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(4) .$$

Let k be an integer, χ be a Dirichlet character modulo q, and $4|q$. A holomorphic function $F(Z)$ on $\mathfrak{H}_n$ is said to be a *Siegel modular form* of weight $k-1/2$ with character χ belongs to $\Gamma_0^{(n)}(q)$ if F satisfies

$$F(M\cdot Z) = \chi(\det D) j(M,Z)^{2k-1}F(Z) \text{, for any } M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(q),$$

and in the case of $n=1$ we need that the function $F(Z)$ is holomorphic at all cusps of $\Gamma_0^{(1)}(q)$. We denote the set of such functions by $M_{k-1/2}(\Gamma_0^{(n)}(q),\chi)$. If $n=0$ then we denote $M_{k-1/2}(\Gamma_0^{(0)}(q),\chi) = \mathbb{C}$ for $k>0$. Also, we denote the set of cusp forms in $M_{k-1/2}(\Gamma_0^{(n)}(q),\chi)$ by $S_{k-1/2}(\Gamma_0^{(n)}(q),\chi)$.

Next, we define a subspace $M^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u)$ of $M_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u)$ ($u=0$ or 1) by

$$\begin{aligned} &M^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u) \\ :=\ &\Big\{h(\tau)\in M_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u) \mid \text{the coefficients } \ c(T)=0\ , \\ &\quad \text{unless } T \equiv -(-1)^{k+u}\mu^t\mu \mod 4\,Sym_n \text{ for some } \mu\in\mathbb{Z}^n\Big\} . \end{aligned}$$

We also define $S^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u)$ by

$$S^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u) := M^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u)\cap S_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u) .$$

We say that $M^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u)$ is the plus space.

Let $\mathcal{M}>0$ be a half integral symmetric matrix of size l and let $R\in M_{n,l}(\mathbb{Z})$, we put the theta series

$$\vartheta_{R,\mathcal{M}}(\tau,z) = \sum_{\lambda\in M_{n,l}(\mathbb{Z})} e(tr\,(\mathcal{M}\,(\tau[(\lambda+R(2\mathcal{M})^{-1})] + 2^tz(\lambda+R(2\mathcal{M})^{-1})))),$$

where $\tau[(\lambda+R(2\mathcal{M})^{-1})] =^t (\lambda+R(2\mathcal{M})^{-1})\tau(\lambda+R(2\mathcal{M})^{-1})$.

Let $F(\tau,z)\in J^{sk}_{k,\mathcal{M}}(\Gamma_n^J)$, then F satisfies the condition (1) of the definition of skew holomorphic Jacobi forms, we can see

$$F(\tau,z+\tau\lambda+\mu) = e(-\mathrm{tr}(\mathcal{M}({}^t\lambda\tau\lambda+2^t\lambda z)))\,F(\tau,z)$$

for every $\lambda,\mu\in M_{n,l}(\mathbb{Z})$. Hence, we have the following equation,

$$F(\tau,z) = \sum_{R\in M_{n,l}(\mathbb{Z})/(M_{n,l}(\mathbb{Z})(2\mathcal{M}))} F_R(\tau)\,\vartheta_{R,\mathcal{M}}(\tau,z)\ ,$$

where $F_R(\tau)$ are uniquely determined and $F_R(-\overline{\tau})$ are holomorphic functions on $\mathfrak{H}_n$. If we set $F(\tau,z)=\sum_{N,R'}C(N,R')e(N\tau-(4N-R'\mathcal{M}^{-1t}R')Im\,\tau+{}^t R'z)$, then we can write F_R by

$$F_R(\tau)\ =\ \sum_{\substack{N\in sym_n\\ 4N-R\mathcal{M}^{-1t}R\le 0}} C(N,R)\,e(\frac{1}{4}\mathrm{tr}(4N-R\mathcal{M}^{-1t}R)\overline{\tau})\ .$$

In this section, from here, we consider only the case $l=1$, $\mathcal{M}=1$, and we put $\vartheta_r:=\vartheta_{r,1}$.

Let $F(\tau,z)=\sum_{r\in(\mathbf{Z}/2\mathbf{Z})^n}F_r(\tau)\vartheta_r(\tau,z)\in J^{sk}_{k,1}(\Gamma^J_n)$. We define a holomorphic function $\sigma(F)(\tau)=\sum_{r\in(\mathbf{Z}/2\mathbf{Z})^n}F_r(-4\overline{\tau})$, then we have the following theorem.

Theorem 1. *$\sigma(F)$ is an element of $M^+_{k-1/2}(\Gamma^{(n)}_0(4),\psi^{k-1})$. Moreover, the map $\sigma:J^{sk}_{k,1}(\Gamma^J_n)\to M^+_{k-1/2}(\Gamma^{(n)}_0(4),\psi^{k-1})$ induces the linear isomorphism over $\mathbb{C}$. The space of skew holomorphic Jacobi cusp forms corresponds with the space of cusp forms of plus space by this isomorphism. This isomorphism map σ commutes with Hecke operators of both spaces.*

We note that if degree n is odd and integer k is even, then it is easy to see that $M_{k-1/2}(\Gamma^{(n)}_0(4),\psi)=J^{sk}_{k,1}(\Gamma^J_n)=\{0\}$.

We denote the space of holomorphic Jacobi forms of weight k of index 1 of degree n by $J_{k,1}(\Gamma^J_n)$ (cf. Ibukiyama [8]), then the table of linear isomorphisms between the plus space and the holomorphic (or skew holomorphic) Jacobi forms of index 1 is given as follows.

$M^+_{k-1/2}(\Gamma^{(n)}_0(4),\psi^u)\ \cong$

u \ k	even	odd
0	$J_{k,1}(\Gamma^J_n)$	$J^{sk}_{k,1}(\Gamma^J_n)$
1	$J^{sk}_{k,1}(\Gamma^J_n)$	$J_{k,1}(\Gamma^J_n)$

3 Klingen type Eisenstein series

We shall construct Klingen type Eisenstein series of skew holomorphic Jacobi forms. Let r be an integer ($0\le r\le n$). We prepare the following subgroups,

$$\Gamma_{[n,r]}\ :=\ \left\{g=\begin{pmatrix}A_1&0&B_1&B_2\\A_3&A_4&B_3&B_4\\C_1&0&D_1&D_2\\0&0&0&D_4\end{pmatrix}\in Sp_n(\mathbb{Z})\mid A_1,B_1,C_1,D_1\in M_r(\mathbb{Z})\right\},$$

$$\Gamma^J_{[n,r],l}\ :=\ \left\{\begin{pmatrix}A&0&B&0\\0&1_l&0&0\\C&0&D&0\\0&0&0&1_l\end{pmatrix}\begin{pmatrix}1_n&0&0&\mu\\{}^t\lambda&1_l&{}^t\mu&\kappa\\0&0&1_n&-\lambda\\0&0&0&1_l\end{pmatrix}\in\Gamma^J_{n,l}\mid\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_{[n,r]}\ ,\right.$$
$$\left.\lambda=\begin{pmatrix}\lambda_1\\0\end{pmatrix}\in M_{n,l}(\mathbb{Z})\ ,\ \lambda_1\in M_{r,l}(\mathbb{Z})\right\}.$$

Let $F(\tau_1, z_1) \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)$ and let k be an integer satisfies $k \equiv l \mod 2$ (l is the size of $\mathcal{M}$). We define a function F^* on $\mathfrak{D}_{n,l}$ as

$$F^*(\tau, z) := F(\tau_1, z_1) \ , \tag{3.1}$$

where $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ {}^t\tau_2 & \tau_3 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $(\tau_1, z_1) \in \mathfrak{D}_{r,l}$.

We consider the following function

$$E^{sk}_{n,r}(F; (\tau, z)) = \sum_{\gamma \in \Gamma^J_{[n,r],l} \backslash \Gamma^J_{n,l}} (F^*|_{k,\mathcal{M}}\gamma)(\tau, z) \ , \qquad (\tau, z) \in \mathfrak{D}_{n,l}. \tag{3.2}$$

The above sum does not depend on the choice of the representative elements. Because F is a cusp form, we can show the fact that there exists a constant C which satisfies

$$|F(\tau_1, z_1)| \det(Y_1)^{\frac{k}{2}} e(-tr(\mathcal{M}{}^t\beta_1(iY_1)^{-1}\beta_1)) < C,$$

for every $(\tau_1, z_1) \in \mathfrak{D}_{r,l}$, where β_1 and Y_1 are the imaginary part of z_1 and τ_1 respectively. Hence, by the same calculation as Ziegler[18] Theorem2.5, we can show the fact that if $k > n + l + r + 1$ then $E^{sk}_{n,r}$ is uniformly absolutely convergent in the wider sense on $\mathfrak{D}_{n,l}$. It is clear that $E^{sk}_{n,r}(F; (\tau, z))$ satisfies the condition (1) of the definition of skew holomorphic Jacobi forms of weight k of index $\mathcal{M}$ belongs to Γ^J_n

We can show the following equation :

$$\Delta_{\mathcal{M}}\left(E^{sk}_{n,r}(F; (\tau, z))\right) = 0_n. \tag{3.3}$$

Because this equation induces the shape of the Fourier expansion of $E^{sk}_{n,r}(F; (\tau, z))$, and by using Shimura correspondence and Köcher principle, we can show the fact that $E^{sk}_{n,r}(F; (\tau, z))$ satisfies the condition (2) of the definition of skew holomorphic Jacobi forms. Hence, we have the following theorem.

Theorem 2. *Let $\mathcal{M} > 0$ and $F \in J^{sk}_{k,\mathcal{M}}(\Gamma^J_r)$. If $k > n + l + r + 1$ satisfies $k \equiv l \mod 2$, then $E^{sk}_{n,r}(F; (\tau, z))$ is an element of $J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$.*

We note that we can obtain the above theorem under the assumption on $\mathcal{M} \geq 0$ (cf.[4]).

We shall show that the Siegel operator of skew holomorphic Jacobi forms has same properties as holomorphic Jacobi forms case (cf. Ziegler[18]).

For a function $F(\tau, z)$ on $\mathfrak{D}_{n,l}$, we define a function

$$\Phi^n_r(F)(\tau_1, z_1) := \lim_{t \to +\infty} F\left(\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & it1_{n-r} \end{smallmatrix}\right), \left(\begin{smallmatrix} z_1 \\ 0 \end{smallmatrix}\right)\right), \quad (\tau_1, z_1) \in \mathfrak{D}_{n,r}.$$

Then $\Phi^n_r(F)$ is a function on $\mathfrak{D}_{r,l}$. This Φ^n_r is called the Siegel operator.

By direct calculation, we have the following proposition.

Proposition 3. *Let $F(\tau, z) \in J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$ be a skew holomorphic Jacobi form, then $\Phi^n_r(F)$ is also a skew holomorphic Jacobi form in $J^{sk}_{k,\mathcal{M}}(\Gamma^J_r)$.*

Moreover, we have the following theorem.

Theorem 4. *If integer k satisfies $k > n + l + r + 1$ and $k \equiv l \mod 2$, then we have $\Phi^n_r(E^{sk}_{n,r}(F; (\tau, z))) = F(\tau_1, z_1)$ for every $F(\tau_1, z_1) \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)$. Hence, the Siegel operator Φ^n_r induces a surjective map from $J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$ to $J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)$.*

Now, we imitate some Arakawa's work[2]. We assume the following condition on $\mathcal{M} > 0$.

(4.1) If $\mathcal{M}[x] \in \mathbb{Z}$ for $x \in (2\mathcal{M})^{-1}M_{l,1}(\mathbb{Z})$, then necessarily, $x \in M_{l,1}(\mathbb{Z})$.

By the same argument with Arakawa [2] (Proposition 4.1, Theorem 4.2 of [2]), we deduce the following Proposition 5 and Theorem 6.

Proposition 5. *Let $F \in J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$. Under the condition* (4.1) *on $\mathcal{M}$, we have $F \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_n)$ if and only if $\Phi^n_{n-1}(F) = 0$.*

Theorem 6. *Assume that $\mathcal{M}$ satisfies the condition* (4.1). *Let k be a positive integer which satisfies $k > 2n + l + 1$ and $k \equiv l \mod 2$. Then we have the direct sum decomposition $J^{sk}_{k,\mathcal{M}}(\Gamma^J_n) = \bigoplus_{r=0}^n J^{sk,(r)}_{k,\mathcal{M}}(\Gamma^J_n)$, where $J^{sk,(r)}_{k,\mathcal{M}} = \{E^{sk}_{n,r}(F; (\tau, z)) | F \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)\}$.*

In section 2 theorem 1, we obtained the isomorphism between the plus space and the space of skew holomorphic Jacobi forms of index 1. Hence, by using theorem 6, if k is an odd integer which satisfies $k > 2n+2$, we can also obtain a similar decomposition for the plus space of degree n of weight $k - \frac{1}{2}$ with trivial character. Namely, under these conditions, we can deduce the fact that the plus space of weight $k - \frac{1}{2}$ is spanned by Klingen-Cohen type Eisenstein series (which corresponds to the Klingen type Eisenstein series of skew holomorphic Jacobi form of index 1) and cusp forms.

4 Zharkovskaya's theorem

In this section, we give an analogue of the Zharkovskaya's theorem for Siegel modular forms of half integral weight, and quote a conjecture.

Let $q > 0$ be an integer divisible by 4. Let $F \in M_{k-1/2}(\Gamma_0^{(n)}(q), \chi)$ be an eigenfunction for the action of a certain Hecke ring. This F has a Fourier expansion

$$F(Z) = \sum_{N \in Sym^*_n} f(N) e(NZ),$$

where we denote by Sym_n^* the set of all semi positive definite half integral symmetric matrices of size n. From the definition of $M_{k-1/2}(\Gamma_0^{(n)}(q),\chi)$, it follows that $f({}^tUNU)=f(N)$ for every $U\in SL_n(\mathbb{Z})$.

Here, we describe a result of Zhuravlev[17]. Let λ be a completely multiplicative function which grows no faster than some power of argument, and let $N>0$ be a matrix in Sym_n^*.

Theorem 7 (Zhuravlev). *When the real part of s is sufficiently large, The following series has Euler expansion,*

$$(4.1)\qquad \sum_{\substack{M\in SL_n(\mathbb{Z})\backslash M_n^+(\mathbb{Z})\\ (\det M,q)=1}} \frac{\lambda(\det M)f(MN{}^tM)}{(\det M)^{s+k-3/2}} = \prod_{p:prime}\frac{P_{F,p}(N,\lambda,p^{-s})}{Q_{F,p}(\lambda,p^{-s})},$$

where we denote by $M_n^+(\mathbb{Z})$ all positive determinant matrices in $M_{n,n}(\mathbb{Z})$, and $P_{F,p}(N,\lambda,z)$ is the polynomial of z which degree is at most $2n$, $Q_{F,p}(\lambda,z)$ is the polynomial of z which degree is $2n$. Especially $Q_{F,p}(\lambda,z)$ is not depend on the choice of N. The polynomial $Q_{F,p}(\lambda,z)$ was defined as follows,

$$(4.2)\qquad Q_{F,p}(\lambda,z) = \prod_{i=0}^{n}(1-\alpha_{i,p}\chi(p)\lambda(p)z)(1-\alpha_{i,p}^{-1}\chi(p)\lambda(p)z),$$

where $\alpha_{i,p}^{\pm1}$ are the p-parameters of F.

We denote the Siegel operator by Φ. Oh-Koo-Kim [12] showed the existence of a commuting relation between Hecke operators and the Siegel operator acting on the Siegel modular forms of half integral weight, and they showed also the fact that if $F\in M_{k-1/2}(\Gamma_0^{(n)}(q),\chi)$ is a Hecke eigen form then $\Phi(F)\in M_{k-1/2}(\Gamma_0^{(n-1)}(q),\chi)$ is also a Hecke eigen form.

We put $L(s,\lambda,F)=\prod_{(p,q)=1}Q_{F,p}(\lambda,p^{-s+k-3/2})^{-1}$ (see eq(4.1), eq(4.2)), then we obtain the following theorem, this is an analogue of the theorem of Zharkovskaya [16].

Theorem 8. *We assume $\Phi(F)\neq 0$, then we have*

$$L(s,\lambda,F) = L_1(s-n+1,\lambda,E_{2k-2n,\chi^2})L(s,\lambda,\Phi(F)),$$

where we put

$$L_1(s,\lambda,E_{2k-2n,\chi^2}) := \prod_{p,(p,q)=1}(1-\lambda(p)p^{-s})^{-1}(1-\lambda(p)\chi(p)^2p^{2k-2n-1-s})^{-1}.$$

If $k>n+1$ then $L_1(s,\lambda,E_{2k-2n,\chi^2})$ is the L-function of Eisenstein series of degree 1 of weight $2k-2n$ with character χ^2 twisted by λ.

Above theorem was first observed by Hayashida-Ibukiyama [7] in the case of $n = 2$, $\lambda \equiv 1$, and $\chi \equiv 1$. Here, we have the case of higher degree with character.

Let $F \in M_{k-1/2}(\Gamma_0^{(2)}(4))$ be a Hecke eigen form, and we assume $\Phi(F) \neq 0$, then

$$L(s,F) = L(s,\Phi(F))\, L(s,E_{2k-4}),$$

up to Euler 2-factor. Let $f \in M_{2k-2}(SL(2,\mathbb{Z}))$ be a Hecke eigen form which corresponds to $\Phi(F) \in M_{k-1/2}(\Gamma_0^{(1)}(4))$ by Shimura correspondence, then we have

$$L(s,F) = L(s,f)\, L(s,E_{2k-4}) .$$

Similar figure seems valid for the case of cusp forms. We quote a following conjecture from Hayashida-Ibukiyama [7].

Conjecture 1 (cf. [7]). *Let k be an integer, and let $f \in S_{2k-2}(SL(2,\mathbb{Z}))$, $g \in S_{2k-4}(SL(2,\mathbb{Z}))$. We assume both f and g are normalised Hecke eigen forms. Then there exits $F \in S^{+}_{k-1/2}(\Gamma_0^{(2)}(4))$, such that F is a Hecke eigen form and satisfy*

$$L(s,F) = L(s,f)\, L(s-1,g)$$

up to Euler 2-factor, and where $L(s,f)$ and $L(s,g)$ are usual L-functions of f and g respectively.

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