数理解析研究所講究録 1339 巻 2003 年 1-5



# A few comments on turbulent pair diffusion's dependence on straining stagnation points

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August 5, 2003

## 1. Introduction

Fung *et al.* (1992) conjectured that turbulent pair diffusion is controlled by hyperbolic points in the turbulence and that, as a result, pairs travel together for a long time and separate suddenly when they meet such points in the flow. The laboratory experiment of Jullien *et al.* (1999) confirmed this view on the history of pair trajectories. However, Jullien *et al.* (1999) did not attempt to determine whether and to what extent hyperbolic points are responsible for the sudden separation of fluid element pairs. Fung & Vassilicos (1998) proposed a schematic topological picture for the instantaneous multiple-scale streamline structure of a planar turbulence with energy spectrum $E(k) \sim k^{-p}$ where $1 < p < 3$. Their streamline picture is one of cat's eyes within cat's eyes and is suggestive of the way that hyperbolic points are spatially distributed on a planar fractal set. Davila & Vassilicos (2003) quantified this multiple-scale structure by determining, both in planar and three-dimensional homogeneous isotropic turbulence, that the number density $n_s$ of straining stagnation points (i.e. hyperbolic points in planar turbulence but any stagnation points with non-zero local straining action in three-dimensional turbulence) has a power law dependence on the ratio of the inner ($\eta$) to the outer ($L$) length-scales of the turbulence (these length-scales characterise the range over which the energy spectrum is a power law $k^{-p}$, and in Kolmogorov turbulence the ratio $L/\eta$ scales as the 3/4 power of the Reynolds number). Specifically, $n_s(L/\eta) = C_s(L/\eta)^{D_s}$ where, in two dimensions, $D_s + p = 3$ and $C_s$ is a number density per integral-scale area and, in three dimensions, $p + \frac{2}{3}D_s = 3$ and $C_s$ is a number density per integral-scale volume.

To test the dependence of pair diffusion on straining stagnation points, Davila & Vassilicos

(2003) added a constant velocity $\mathbf{V}$ to a turbulent velocity field $\mathbf{u}(\mathbf{x},t)$ obtained by Kinematic Simulation and integrated particle trajectories $\mathbf{x}(t)$ by solving $\frac{d}{dt}\mathbf{x} = \mathbf{u} + \mathbf{V}$. Note that this is *not* a Galilean transformation as the particles are advected past the turbulence $\mathbf{u}(\mathbf{x},t)$ which is itself not advected by $\mathbf{V}$. The closest analogue would be the addition of a terminal fall velocity to the particles' motions. Davila & Vassilicos (2003) found that $C_s$ decreases as $V = |\mathbf{V}|$ increases, which means that the number density of straining stagnation points decreases as $V$ increases; they also found that, in parallel, pairs separate more slowly as $V$ increases, thus establishing a connection, or at least a correlation, between pair separation and number density of straining stagnation points. Another correlation between pair separation and number density of straining stagnation regions was established by Fung & Vassilicos (1998) by varying the power $p$ of the energy spectrum in Kinematic Simulations. Nicolleau & Vassilicos (2003) gave an argument based on the divergence of the acceleration field which suggests that it is persistent regions of high strain rate and low vorticity which separate pairs initially very close together.

In the present paper we firstly propose a simple explanation for why $C_s$ is a decreasing function of $V$ and for why pairs separate more slowly as $V$ increases in Kinematic Simulations. Secondly, we discuss the Galilean transformation properties of turbulent pair diffusion.

## 2. $C_s$ is a decreasing function of $V$

In the Kinematic Simulations used by Davila & Vassilicos (2003) the homogeneous and isotropic turbulent velocity field is simulated as a sum of random incompressible Fourier modes of wavenumber $k$ with a prescribed energy spectrum $E(k) \sim k^{-p}$ where $1 < p$, and an unsteadiness frequency $\omega(k)$ characterising the oscillatory time dependence of each mode $k$. The velocity field is therefore a sum over wavevectors $\mathbf{k}$ ($k = |\mathbf{k}|$) of cosine and sine functions of $\mathbf{k}\cdot\mathbf{x} - \omega(k)t$.

When the uniform velocity $\mathbf{V}$ is superposed in the way described above, a critical wavenumber $k_V$ is defined by $Vk_V = \sqrt{k_V^3 E(k_V)}$ where $\sqrt{k^3E(k)}$ is the eddy turnover time related to wavenumber $k$. It might be expected that the superposed uniform velocity $\mathbf{V}$ erases from the velocity field all stagnation points related to wavenumbers $k$ such that $Vk > \sqrt{k^3E(k)}$ and simply displaces the other stagnation points related to wavenumbers $k$ such that $Vk < \sqrt{k^3E(k)}$. Note here that we are tagging a length-scale to every stagnation point, so that some stagnation points correspond to large-scale features and others to small-scale features. The schematic fractal picture of streamlines within streamlines given by Fung & Vassilicos (1998) makes this tagging clear to the eye.

Quantitatively, we might conclude that the number density of straining stagnation points of $\mathbf{u} + \mathbf{V}$ is equal to $C_s(L/\eta_V)^{D_s}$ where $\eta_V = 2\pi/k_V$ and is therefore smaller than $C_s(L/\eta)^{D_s}$

when $\eta_V > \eta$ ($\sqrt{k^3E(k)} \sim k^{\frac{3-p}{2}}$ and $1 < p$ guarantee that the straining stagnation points displaced but not erased correspond to wavenumbers $k < k_V$). If $V$ is so small that $\eta_V \leq \eta$, that is when $V \leq u(\eta)$ where $u(\eta)$ is defined by $u(\eta)\frac{2\pi}{\eta} = \sqrt{(\frac{2\pi}{\eta})^3E(\frac{2\pi}{\eta})}$, then the number density of straining stagnation points is unaffected. If $V$ is so large that $\eta_V > L$, that is when $V > u(L)$ where $u(L)$ is defined by $u(L)\frac{2\pi}{L} = \sqrt{(\frac{2\pi}{L})^3E(\frac{2\pi}{L})}$, then the number density of straining stagnation points is zero.

Davila & Vassilicos (2003) measured $C_s$ as a function of $V$ from $n_s = C_s(V)(L/\eta)^{D_s}$. It follows that $C_s(V) = C_s(0)$ when $V \leq u(\eta)$ and that $C_s(V)(L/\eta)^{D_s} = C_s(0)(L/\eta_V)^{D_s}$ when $u(L) \geq V > u(\eta)$. Hence, $C_s(V)$ is a decreasing function of $V$ as observed in the Kinematic Simulations of Davila & Vassilicos (2003) because $\eta_V$ is an increasing function of $V$ (as long as $p > 1$) and $C_s(V) = C_s(0)(\eta_V/\eta)^{-D_s}$ for $u(\eta) < V < u(L)$.

To establish the dependence of $\eta_V$ on $V$ in the Kinematic Simulations of Davila & Vassilicos (2003) note that $\eta_V \sim \eta[V/u(\eta)]^{\frac{2}{p-1}}$ for $u(\eta) \leq V \leq u(L)$. Hence, $C_s(V) \sim C_s(0)[V/u(\eta)]^{-\frac{2D_s}{p-1}}$ for $u(\eta) < V < u(L)$. For completeness, note that $C_s(V) = C_s(0)$ when $V \leq u(\eta)$ and $C_s(V) = 0$ when $V > u(L)$. Considering the relation between $p$ and $D_s$ given above, and that in two dimensions $0 < D_s < 2$ whereas in three dimensions $0 < D_s < 3$, the power $\frac{2D_s}{p-1}$ which determines the dependence of $C_s(V)$ on $V$ is a monotonically increasing function of $D_s$ and equivalently a monotonically decreasing function of $p$. This means that, when the irregularity of the velocity field is increased, the number density of straining stagnation points decreases faster with increasing $V$.

## 3. Pair separation and Galilean transformations

The frequency $\omega(k)$ characterises the degree of unsteadiness or (inversely) persistence of spatial velocity fluctuations related to $k$ and it is natural to expect it to be an increasing function of $k$ in a model of turbulence such as Kinematic Simulation. In fact, $\omega(k)$ is taken to be proportional to the eddy-turnover frequency $\sqrt{k^3E(k)}$ in the Kinematic Simulations of Davila & Vassilicos (2003) and in some of the Kinematic Simulations of Fung *et al* (1992) and Fung & Vassilicos (1998) too. The eddy-turnover frequency $\sqrt{k^3E(k)}$ is an increasing function of $k$ for $E(k) \sim k^{-p}$ and $p < 3$.

When a uniform velocity $\mathbf{V}$ is superposed on the turbulence as was done by Davila & Vassilicos (2003), then it might be expected that wavenumbers $k$ such that $Vk > \omega(k), \sqrt{k^3E(k)}$ do not affect pair diffusion because particle pairs simply fly over such small-scale flow features without allowing them to affect their flights. Hence, straining stagnation points related to such high wavenumbers do not affect pair diffusion and it might be expected that, as a consequence, pairs separate more slowly as $V$ increases which is what Davila & Vassilicos (2003) observed.

If the transformation $\mathbf{u}' = \mathbf{u} + \mathbf{V}$ used by Davila & Vassilicos (2003) is supplemented by $\mathbf{x}' = \mathbf{x} + \mathbf{V}t$ it then becomes a Galilean transformation. The Kinematic Simulation's velocity field in the Galilean transformed frame is a sum over wavevectors $\mathbf{k}$ ($k = |\mathbf{k}|$) of cosine and sine functions of $\mathbf{k}\cdot\mathbf{x}' - \mathbf{V}\cdot\mathbf{k}t - \omega(k)t$ and of $\mathbf{V}$. The cosine and sine modes are therefore translated in space with the same velocity and in the same direction as fluid element trajectories. As a consequence, pair separation is unaffected as fluid element pairs cannot fly over small-scale flow features and erase their effects on their separation rates and statistics. However, straining stagnation points are not invariant to Galilean transformations and neither is their number density as the argument given in section 2 based on comparing the magnitudes of $Vk$ and of $\sqrt{k^3E(k)}$ can be applied to the Galilean transformation and lead to the same conclusions. This is because the number of stagnation points in instantaneous velocity fields is affected by $\mathbf{u}' = \mathbf{u} + \mathbf{V}$ but is not affected by $\mathbf{x}' = \mathbf{x} + \mathbf{V}t$. How can the statistics of straining stagnation points, which are not Galilean invariant, determine the Galilean invariant statistics of pair separations?

One way to resolve this question is to consider that the relevant straining stagnation points are not those of $\mathbf{u}$ but those of $\mathbf{u} - \langle \mathbf{u} \rangle$, where the brackets signify an average over realisations or time or space. The statistics of stagnation points of $\mathbf{u} - \langle \mathbf{u} \rangle$ are clearly Galilean invariant. However, the transformation of Davila & Vassilicos (2003) should be generalised to mean a superposition of a uniform velocity $\mathbf{V}$ on $\mathbf{u} - \langle \mathbf{u} \rangle$ but not on $\mathbf{u}$ and $\langle \mathbf{u} \rangle$ separately. Defining the fluctuation trajectory $\mathbf{x}' \equiv \mathbf{x} - \langle \mathbf{x} \rangle$, where $\frac{d}{dt}\mathbf{x} = \mathbf{u}$ and $\frac{d}{dt} \langle \mathbf{x} \rangle = \langle \mathbf{u} \rangle$, the transformation of Davila & Vassilicos (2003) amounts to $\frac{d}{dt}\mathbf{x}' = \mathbf{u} - \langle \mathbf{u} \rangle + \mathbf{V}$. This transformation is of course meaningless from the physical point of view but can serve as a numerical experiment implementable in the computer to test the effects of flow structure on diffusion (as Davila & Vassilicos (2003) did).

We consider the issue of the Galilean invariance of flow structural mechanisms responsible for pair separation in turbulent flows to remain open at this stage and are currently working on it. An important concern is the nature of the average defining $\langle \mathbf{u} \rangle$ as well as the potential time and space dependencies of $\langle \mathbf{u} \rangle$ and their own effects on turbulent pair diffusion. In unravelling these effects, another variant of the Davila & Vassilicos (2003) transformation might prove useful: namely, $\frac{d}{dt}\mathbf{x} = \mathbf{u} + \mathbf{V}$, $\frac{d}{dt} \langle \mathbf{x} \rangle = \langle \mathbf{u} \rangle + \mathbf{V}$, $\frac{d}{dt}\mathbf{x}' = \mathbf{u} - \langle \mathbf{u} \rangle$.

## 4. Final comments

The effects of persistent straining stagnation points on turbulent diffusion manifest themselves in more than one way. Goto & Kida (2003) identified persistent hyperbolic points created by antiparallel vortex pairs as being responsible for enhanced stretching of material lines in turbulence and as the cause for the breakdown of Batchelor's (1952) relation between local and global stretching rates. Furthermore, Nicolleau & Vassilicos (2003) pointed out that the dependence of pair diffusion statistics on the pairs' initial separation $\Delta_0 \leq \eta$ that is observed in Kinematic Simulations (Nicolleau & Vassilicos 2003) and Direct Numerical Simulations (Boffetta & Sokolov 2002) over times comparable to the integral time-scale is consistent with a diffusion mechanism dominated by persistent straining stagnation points, in which case it is not a finite size effect and should be expected to persist at arbitrarily high Reynolds number.

**Acknowledgements**

SG and JCV gratefully acknowledge the support of the Japanese Ministry of Education, Culture, Sports, Science and Technology and of the Royal Society of London respectively. This paper is the result of questions raised by Shigeo Kida at the lecture that JCV gave at the RIMS Symposium on Turbulence transport, diffusion and mixing, January 15-17, 2003, RIMS, Kyoto, Japan.

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