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Borel Extension of Tensor Products of Vector Measures with Applications

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ABSTRACT. The purpose of the paper is to report the existence and uniqueness of the Borel injective tensor product of two Banach space-valued vector measures and the validity of a Fubini-type theorem. Thanks to this result, the convolution of vector measures on a topological semigroup is defined as the measure induced by their Borel injective tensor product and the semigroup operation. The joint weak continuity of Borel injective tensor products or convolutions of vector measures is also reported.

1. Introduction

In the usual set-theoretical theory, the notion of a product measure is of central importance. In the case of Borel measures on topological spaces S and T, we are immediately confronted with the problem that the Borel σ-field $\mathcal{B}(S \times T)$ is in general larger than the product σ-field $\mathcal{B}(S) \times \mathcal{B}(T)$; see [**30**, Corollary to Lemma I.1.1]. Our main purpose of the paper is to report the existence and uniqueness of the Borel injective tensor product of two Banach space-valued vector measures and the validity of a Fubini-type theorem.

Let X and Y be Banach spaces. Denote by $X\hat{\otimes}_{\varepsilon}Y$ the injective tensor product of X and Y. We say that vector measures $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(T) \to Y$ have their *Borel injective tensor product* if there is a unique vector measure $\lambda : \mathcal{B}(S \times T) \to X\hat{\otimes}_{\varepsilon}Y$ such that $\lambda(A \times B) = \mu(A) \otimes \nu(B)$ for all $A \in \mathcal{B}(S)$ and $B \in \mathcal{B}(T)$. In Section 2 we shall formulate some notation and results which are needed in the sequel, and it is reported in Section 3 that if two Borel vector measures are τ-smooth, one of them with some separability condition with respect to the other, then they have their τ-smooth Borel injective tensor product; see Theorem 3.3. This type of problems has been already investigated in [**11, 12, 16**] in more general settings. However, it should be remarked that the vector measures in those papers are always supposed to be of bounded variation and be defined on compact or locally compact spaces.

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In Section 4 we shall report the joint weak continuity of Borel injective tensor products of Banach lattice-valued positive vector measures. This result has already been proved in [**21**, Theorem 5.4] with the additional assumption that the spaces S and T, on which the vector measures are defined, satisfy $\mathcal{B}(S \times T) = \mathcal{B}(S) \times \mathcal{B}(T)$.

In Section 5, the convolution of Banach algebra-valued vector measures on a topological semigroup is defined as the measure induced by their Borel injective tensor product and the semigroup operation. The paper concludes with reporting the joint weak continuity of convolutions of positive vector measures with values in special Banach algebras. The proofs of the results in this paper will be appeared in [**22**].

2. Preliminaries

All the topological spaces in this paper are Hausdorff and the scalar fields of Banach spaces are taken to be the field $\mathbb{R}$ of all real numbers. Denote by $\mathbb{N}$ the set of all natural numbers.

Let T be a topological space and $\mathcal{B}(T)$ the σ-field of all Borel subsets of T, that is, the σ-field generated by the open subsets of T. Let X be a Banach space and X^* a topological dual of X. Denote by $\boldsymbol{B}_X$ and $\boldsymbol{B}_{X^*}$ the closed unit balls of X and X^*, respectively. A countably additive set function $\nu : \mathcal{B}(T) \to X$ is called a *vector measure.* The *semivariation* of ν is the set function $\|\nu\|(B) := \sup\{|x^*\nu|(B) : x^* \in \boldsymbol{B}_{X^*}\}$, where $B \in \mathcal{B}(T)$ and $|x^*\nu|(\cdot)$ is the total variation of the real measure $x^*\nu$. Then $\|\nu\|(T) < \infty$ [**2**, Lemma 2.2].

We define several notions of regularity for vector measures. A vector measure $\nu : \mathcal{B}(T) \to X$ is said to be *Radon* if for each $\eta > 0$ and $B \in \mathcal{B}(T)$, there exists a compact subset K of T with $K \subset B$ such that $\|\nu\|(B - K) < \eta$, and it is said to be *tight* if the condition is satisfied for $B = T$. We say that ν is *τ-smooth* if for every increasing net $\{G_\alpha\}_{\alpha\in\Gamma}$ of open subsets of T with $G = \bigcup_{\alpha\in\Gamma} G_\alpha$, we have $\lim_{\alpha\in\Gamma} \|\nu\|(G - G_\alpha) = 0$. It follows that ν is Radon (respectively, tight, τ-smooth) if and only if for each $x^* \in X^*$, $x^*\nu$ is Radon (respectively, tight, τ-smooth). In fact, this is a consequence of the Rybakov theorem [**9**, Theorem IX.2.2], which ensures that there exists $x_0^* \in X^*$ for which $x_0^*\nu$ and ν are mutually absolutely continuous. Thus, all the regularity properties which are valid for positive finite measures remain true of vector measures.

In this paper, we shall employ two types of integration which will be briefly explained below. Let $\nu : \mathcal{B}(T) \to X$ be a vector measure. Denote by χ_B the indicator function of a set B. A $\|\nu\|$-null set is a set $B \in \mathcal{B}(T)$ for which $\|\nu\|(B) = 0$; the term $\|\nu\|$*-almost everywhere* (for short, $\|\nu\|$-a.e.) refers to the complement of a $\|\nu\|$-null set. Given a $\mathcal{B}(T)$-simple function of the form $f = \sum_{i=1}^k a_i\chi_{B_i}$ with $a_1, \ldots, a_k \in \mathbb{R}, B_1, \ldots, B_k$ are pairwise disjoint sets in $\mathcal{B}(T)$, and $k \in \mathbb{N}$, define its integral $\int_B f\,d\nu$ over a set $B \in \mathcal{B}(T)$ by $\int_B f\,d\nu := \sum_{i=1}^k a_i\nu(B_i \cap B)$. We say that a $\mathcal{B}(T)$-measurable function $f : T \to \mathbb{R}$ is *ν-integrable* if there is a sequence $\{f_n\}$ of $\mathcal{B}(T)$-simple functions converging $\|\nu\|$-almost everywhere to f such that the sequence $\{\int_B f_n d\nu\}$ converges in the norm of X for each $B \in \mathcal{B}(T)$. This limit $\int_B f\,d\nu$ does not depend on the choice of such $\mathcal{B}(T)$-simple functions $f_n, n \in \mathbb{N}$. By the Orlicz-Pettis theorem [**9**, Corollary I.4.4], the indefinite integral $B \mapsto \int_B f\,d\nu$ is countably additive. For further properties of this integral see [**2, 14**].

Next we define the Bartle bilinear integration in our setting. Let X, Y and Z be Banach spaces with a continuous bilinear mapping $b : X \times Y \to Z$. Let $\nu : \mathcal{B}(T) \to Y$ be a vector measure. The *semivariation of ν with respect to b* is the set function $\|\nu\|_b(B) := \sup \|\sum_{i=1}^k b(x_i, \nu(B_i))\|$, where B is a subset of T and the supremum is taken for all finite families $\{B_i\}_{i=1}^k$ of pairwise disjoint sets in $\mathcal{B}(T)$ contained in B and all families $\{x_i\}_{i=1}^k$ of elements in $\boldsymbol{B}_X$. In what follows, we shall assume that ν has the (*)*-property with respect to b*, that is, there is a positive finite measure q on $\mathcal{B}(T)$ such that $q(B) \to 0$ if and only if $\|\nu\|_b(B) \to 0$ [**3**, Definition 2]. Then $\|\nu\|_b(T) < \infty$ [**28**, Lemma 1]. It will be verified below that this assumption is automatically satisfied whenever we use this integration in this paper.

A $\|\nu\|_b$-null set is a subset B of T for which $\|\nu\|_b(B) = 0$; the term $\|\nu\|_b$-*almost everywhere* (for short, $\|\nu\|_b$-a.e.) refers to the complement of a $\|\nu\|_b$-null set. Given an X-valued $\mathcal{B}(T)$-simple function $\varphi = \sum_{i=1}^k x_i\chi_{B_i}$ with $x_1, \dots, x_k \in X$, $B_1, \dots, B_k$ are pairwise disjoint sets in $\mathcal{B}(T)$, and $k \in \mathbb{N}$, define its integral $\int_B b(\varphi, d\nu)$ over a set $B \in \mathcal{B}(T)$ by $\int_B b(\varphi, d\nu) := \sum_{i=1}^k b(x_i, \nu(B_i \cap B))$. We say that a vector function $\varphi : T \to X$ is (ν, b)-*measurable* if there is a sequence $\{\varphi_n\}$ of X-valued $\mathcal{B}(T)$-simple functions converging $\|\nu\|_b$-almost everywhere to φ. The function φ is said to be (ν, b)-*integrable* if there is a sequence $\{\varphi_n\}$ of X-valued $\mathcal{B}(T)$-simple functions converging $\|\nu\|_b$-almost everywhere to φ such that the sequence $\{\int_B b(\varphi_n, d\nu)\}$ converges in the norm of Z for each $B \in \mathcal{B}(T)$. This limit $\int_B b(\varphi, d\nu)$ does not depend on the choice of such X-valued $\mathcal{B}(T)$-simple functions $\varphi_n, n \in \mathbb{N}$, and the indefinite integral $B \mapsto \int_B b(\varphi, d\nu)$ is a Z-valued vector measure on $\mathcal{B}(T)$.

Since we always assume that ν has the (*)-property with respect to b, we make free use of the (*)-theorems of [**3**, Section 4]. Among others we shall frequently use the following theorem and refer to it as the *Bounded convergence theorem*; see [**3**, Theorem 7] and the statement after [**3**, (*) Theorem 10]. For further properties of this integral see [**3, 15**].

PROPOSITION 2.1 (Bounded Convergence Theorem). *Let $\{\varphi_n\}$ be a sequence of (ν, b)-integrable X-valued functions on T which converges $\|\nu\|_b$-almost everywhere to an X-valued function φ on T. If there is a $M > 0$ such that $\|\varphi_n(t)\| \le M$ for $\|\nu\|_b$-almost everywhere $t \in T$, then φ is (ν, b)-integrable over any set $B \in \mathcal{B}(T)$ and $\int_B b(\varphi, d\nu) = \lim_{n\to\infty} \int_B b(\varphi_n, d\nu)$.*

3. Borel injective tensor product of vector measures

Throughout this section, we assume that S and T are topological spaces and X and Y are Banach spaces. Denote by U_{X^*} and U_{Y^*} the closed unit balls $\boldsymbol{B}_{X^*}$ and $\boldsymbol{B}_{Y^*}$ equipped with the relative topologies induced by the weak* topologies on X^* and Y^*, respectively. Then, they are compact spaces by the Alaoglu theorem [**14**, Theorem V.4.2]. Denote by $X\hat{\otimes}_\varepsilon Y$ the injective tensor product of X and Y with injective norm

$$\|z\|_\varepsilon := \sup\left\{\left|\sum_{i=1}^k x^*(x_i)y^*(y_i)\right| : x^* \in \boldsymbol{B}_{X^*}, y^* \in \boldsymbol{B}_{Y^*}\right\},$$

where $\sum_{i=1}^k x_i \otimes y_i$ is any representation of $z \in X \otimes Y$. If $x^* \in X^*$ and $y^* \in Y^*$, then there is a unique bounded linear functional on $X\hat{\otimes}_\varepsilon Y$, which is called the *tensor product* of x^* and y^* and denoted by $x^* \otimes y^*$, such that $(x^* \otimes y^*)(x \otimes y) = x^*(x)y^*(y)$ for every $x \in X$ and $y \in Y$. Then it is easily verified that the equation $\|x^* \otimes y^*\| = \|x^*\|\|y^*\|$ holds. For the above results and further information on the injective tensor product of Banach spaces see [**9**, Chapter VIII].

Let $\varepsilon : X \times Y \to X\hat{\otimes}_\varepsilon Y$ be the bilinear mapping defined by $\varepsilon(x, y) = x \otimes y$ for all $x \in X$ and $y \in Y$. Let $\nu : \mathcal{B}(T) \to Y$ be a vector measure. Then the semivariation $\|\nu\|_\varepsilon$ with respect to ε coincides with the usual semivariation $\|\nu\|$ on $\mathcal{B}(T)$, so that ν has the (*)-property with respect to ε [**31**, Remark 2]. In this case we shall write $\int_B \varphi \otimes d\nu := \int_B \varepsilon(\varphi, d\nu)$ for every (ν, ε)-integrable function $\varphi : T \to X$ and $B \in \mathcal{B}(T)$.

We first give a useful criterion for the τ-smoothness of vector measures with values in $X\hat{\otimes}_\varepsilon Y$.

PROPOSITION 3.1. *Let $\lambda : \mathcal{B}(S \times T) \to X\hat{\otimes}_\varepsilon Y$ be a vector measure. Then λ is τ-smooth if $(x^* \otimes y^*)\lambda$ is τ-smooth for every $x^* \in X^*$ and $y^* \in Y^*$.*

Our goal in this section is to report the existence and uniqueness of the Borel injective tensor product of two Banach space-valued vector measures and the validity of a Fubini-type theorem; see Theorems 3.3 and 3.4 and Proposition 3.6. This has been already accomplished in [**6**, Lemma 2] and [**30**, Theorem I.4.1 and its Corollary] for τ-smooth or Radon probability measures, and the result can be extended to τ-smooth or Radon positive finite measures verbatim (see [**1**, Corollary 2.1.11 and Theorem 2.1.12]).

In what follows, denote by $B(S)$ the set of all Borel measurable (that is, $\mathcal{B}(S)$-measurable), bounded real functions on S. Similar definitions are made for $B(T)$ and $B(S \times T)$. For each $E \in \mathcal{B}(S \times T)$ and $t \in T$ put $E^t := \{s \in S : (s,t) \in E\}$. Then $E^t \in \mathcal{B}(S)$.

Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(T) \to Y$ be vector measures. We say that μ has the *separability condition with respect to* ν if for each $E \in \mathcal{B}(S \times T)$ the vector function $t \in T \mapsto \mu(E^t)$ is $\|\nu\|$-essentially separably valued, that is, there exists a set $N \in \mathcal{B}(T)$ with $\|\nu\|(N) = 0$ such that the set $\{\mu(E^t) : t \in T - N\}$ is a separable subset of X.

REMARK 3.2. Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(T) \to Y$ be vector measures. Then μ has the separability condition with respect to ν if any of the following assumptions is valid; see [**16**, page 125].

(i) μ has separable range.
(ii) μ has relatively compact range.
(iii) μ is given by a σ-finite positive measure p on $\mathcal{B}(S)$ and a p-measurable, Pettis integrable vector function $\varphi : S \to X$ as the Pettis integral $\mu(A) = \int_A \varphi dp$ for all $A \in \mathcal{B}(S)$ [**27**, page 363].
(iv) μ is of bounded variation and X has the Radon-Nikodým property [**9**, Definition III.1.3].
(v) There is a countable subset $\mathcal{D}$ of $\mathcal{B}(S)$ with the property that for any $\eta > 0$ and $A \in \mathcal{B}(S)$ there exists $D \in \mathcal{D}$ such that $\|\mu\|(A \triangle D) < \eta$, where $A \triangle D := (A - D) \cup (D - A)$.
(vi) The spaces S and T satisfy $\mathcal{B}(S \times T) = \mathcal{B}(S) \times \mathcal{B}(T)$ [**5**, Theorem 2].
(vii) One of the spaces S or T has a countable base of open sets [**18**, Theorem 8.1].
(viii) The spaces S and T are locally compact and μ and ν are τ-smooth [**29**, P16, page xiii] and [**31**, Theorem 2.2].

The following theorem is our first goal in this paper, and it extends [**31**, Theorem 2.2].

THEOREM 3.3. *Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(T) \to Y$ be τ-smooth vector measures. If one of them has the separability condition with respect to the other, then there is a unique τ-smooth vector measure $\lambda : \mathcal{B}(S \times T) \to X \hat{\otimes}_\varepsilon Y$ such that*

$$\lambda(A \times B) = \mu(A) \otimes \nu(B) \quad \text{for all } A \in \mathcal{B}(S) \text{ and } B \in \mathcal{B}(T). \tag{3.1}$$

Further we have the following properties.

(i) $(x^* \otimes y^*)\lambda = x^*\mu \times y^*\nu$ *and* $|(x^* \otimes y^*)\lambda| = |x^*\mu| \times |y^*\nu|$ *for every* $x^* \in X^*$ *and* $y^* \in Y^*$.
(ii) $\|\lambda\|(A \times B) = \|\mu\|(A) \cdot \|\nu\|(B)$ *for every* $A \in \mathcal{B}(S)$ *and* $B \in \mathcal{B}(T)$.

From Theorem 3.3 we have

THEOREM 3.4. *Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(T) \to Y$ be Radon vector measures. If one of them has the separability condition with respect to the other, then there is a unique Radon vector measure $\lambda : \mathcal{B}(S \times T) \to X \hat{\otimes}_\varepsilon Y$ such that $\lambda(A \times B) = \mu(A) \otimes \nu(B)$ for all $A \in \mathcal{B}(S)$ and $B \in \mathcal{B}(T)$. Further properties (i) and (ii) in Theorem 3.3 are valid.*

Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(T) \to Y$ be vector measures. We say that μ has a *proper Borel injective tensor product with respect to* ν if there is a unique vector measure $\lambda : \mathcal{B}(S\times T) \to X\hat{\otimes}_\varepsilon Y$ such that $\lambda(A\times B) = \mu(A)\otimes\nu(B)$ for all $A \in \mathcal{B}(S)$ and $B \in \mathcal{B}(T)$, and the following two conditions are valid;

(i) for each $E \in \mathcal{B}(S \times T)$, the function $t \in T \mapsto \mu(E^t)$ is (ν, ε)-integrable and

(ii) the primitive Fubini theorem holds:

$$\lambda(E) = \int_T \mu(E^t) \otimes \nu(dt), \quad E \in \mathcal{B}(S \times T). \tag{3.2}$$

Then λ is called a *proper Borel injective tensor product of the ordered pair* μ, ν. It should be remarked that this is not a symmetrical definition for μ and ν, contrary to the case of the Borel injective tensor product. From theorems 3.3 and 3.4 we have

COROLLARY 3.5. *Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(T) \to Y$ be τ-smooth (respectively, Radon) vector measures. Assume that μ has the separability condition with respect to ν. Then μ has a τ-smooth (respectively, Radon) proper Borel injective tensor product with respect to ν.*

The following result shows that a Fubini-type theorem for bounded Borel measurable functions is valid for proper Borel injective tensor products. The Fubini theorem for real measures involves two pairs of iterated integrals; however, in our present context, only one of the pairs makes sense.

PROPOSITION 3.6. *Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(T) \to Y$ be vector measures. Assume that μ has a proper Borel injective tensor product λ with respect to ν. Let $h \in B(S \times T)$. Then the following assertions are valid:*

(i) *For each $t \in T$, the function $s \in S \mapsto h(s,t)$ is μ-integrable,*

(ii) *the vector function $t \in T \mapsto \int_S h(s,t)\mu(ds)$ is (ν, ε)-integrable, and*

(iii) $\int_{S\times T} h d\lambda = \int_T \left\{\int_S h(s,t)\mu(ds)\right\} \otimes \nu(dt)$.

The notion that ν has a proper Borel injective tensor product with respect to μ can be introduced in a similar fashion and the corresponding results are valid by obvious modifications.

4. Weak convergence of injective tensor products of vector measures

In this section, we shall consider the weak convergence of proper Borel injective tensor products of Banach lattice-valued positive vector measures. The importance of the facts in section 3 will be apparent here. Otherwise the weak convergence of vector measures in the product space would not make any sense.

Throughout this section, we assume that X and Y are Banach lattices such that their injective tensor product $X\hat{\otimes}_\varepsilon Y$ is also a Banach lattice and satisfies the positivity condition (P_ε): $x \otimes y \geq 0$ for every $x \in X$ with $x \geq 0$ and $y \in Y$ with $y \geq 0$. We refer to [**24**] for the basic theory of Banach lattices. In general, the injective tensor product $X\hat{\otimes}_\varepsilon Y$ may not be a vector lattice for the natural ordering. However, for instance, the following Banach lattices satisfy the above assumption; see examples in [**24**, pages 274–276] and [**17**, page 90].

EXAMPLE 4.1. (1) Let Ω be a compact space and Y be any Banach lattice. Denote by $C(\Omega, Y)$ the Banach lattice with its canonical ordering of all (bounded) continuous functions $\varphi : \Omega \to Y$. We write $C(\Omega) := C(\Omega, \mathbb{R})$. Then $C(\Omega)\hat{\otimes}_\varepsilon Y$ is isometrically lattice isomorphic to the Banach lattice $C(\Omega, Y)$. Especially, when $Y = C(\Omega')$ for some compact space Ω', $C(\Omega)\hat{\otimes}_\varepsilon C(\Omega')$ is isometrically lattice isomorphic to $C(\Omega \times \Omega')$.

(2) Let Λ be a locally compact space and Y be any Banach lattice. Denote by $C_0(\Lambda, Y)$ the Banach lattice with its canonical ordering of all (bounded) continuous

functions $\varphi : \Lambda \to Y$ such that for every $\eta > 0$, the set $\{\omega \in \Lambda : \|\varphi(\omega)\| \geq \eta\}$ is compact. We write $C_0(\Lambda) := C_0(\Lambda, \mathbb{R})$. Then $C_0(\Lambda)\hat{\otimes}_\varepsilon Y$ is isometrically lattice isomorphic to $C_0(\Lambda, Y)$. Especially, when $Y = C_0(\Lambda')$ for some locally compact space Λ', $C_0(\Lambda)\hat{\otimes}_\varepsilon C_0(\Lambda')$ is isometrically lattice isomorphic to $C_0(\Lambda \times \Lambda')$.

(3) Let $(\Phi, \mathcal{A}, p)$ be a measure space and Y be any Banach lattice. Denote by $L^\infty(\Phi, Y)$ the Banach lattice of all (equivalent classes of) p-essentially bounded measurable functions $\varphi : \Phi \to Y$ with its canonical ordering. We write $L^\infty(\Phi) := L^\infty(\Phi, \mathbb{R})$. Then $L^\infty(\Phi)\hat{\otimes}_\varepsilon Y$ is a Banach lattice. However, in general, $L^\infty(\Phi)\hat{\otimes}_\varepsilon Y$ is a proper closed subset of $L^\infty(\Phi, Y)$.

Let S be a topological space. We say that a vector measure $\mu : \mathcal{B}(S) \to X$ is *positive* if $\mu(A) \geq 0$ for every $A \in \mathcal{B}(S)$. Then for any positive vector measure μ, the equation $\|\mu\|(A) = \|\mu(A)\|$ holds for all $A \in \mathcal{B}(S)$ [**26**, Lemma 1.1]. Then we have

PROPOSITION 4.2. *Let S and T be topological spaces. Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(T) \to Y$ be vector measures. Assume that μ has a proper Borel injective tensor product λ with respect to ν. If μ and ν are positive, then so is λ.*

Let S and T be topological spaces. Let $C(S)$ be the Banach space of all bounded continuous real functions on S with norm $\|f\|_\infty := \sup_{s\in S} |f(s)|$. Denote by $\mathcal{M}^+(S, X)$ the set of all positive vector measures $\mu : \mathcal{B}(S) \to X$, and denote by $\mathcal{M}_t^+(S, X)$ the set of all $\mu \in \mathcal{M}^+(S, X)$ which are tight. Similar definitions are made for $\mathcal{M}^+(T, Y)$ and $\mathcal{M}_t^+(T, Y)$.

We recall the definition of weak convergence of vector measures. Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a net in $\mathcal{M}^+(S, X)$ and $\mu \in \mathcal{M}^+(S, X)$. We say that $\{\mu_\alpha\}_{\alpha\in\Gamma}$ *converges weakly* to μ, and write $\mu_\alpha \xrightarrow{w} \mu$, if for each $f \in C(S)$ we have $\int_S f\,d\mu = \lim_{\alpha\in\Gamma} \int_S f\,d\mu_\alpha$ in the norm of X; see [**8, 19, 20, 21, 26**]. The following result shows the joint weak continuity of proper Borel injective tensor products of positive vector measures. Except for assuming the tightness of vector measures it extends previous results for probability measures; see [**4**, Theorem 3.2] and [**30**, Proposition I.4.1].

THEOREM 4.3. *Let S and T be completely regular spaces. Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a net in $\mathcal{M}^+(S, X)$ and $\mu \in \mathcal{M}_t^+(S, X)$. Let $\{\nu_\alpha\}_{\alpha\in\Gamma}$ be a net in $\mathcal{M}_t^+(T, Y)$ and $\nu \in \mathcal{M}_t^+(T, Y)$. Assume that μ_α and μ have τ-smooth proper Borel injective tensor products λ_α and λ with respect to ν_α and ν, respectively. Further assume that ν is τ-smooth. If $\mu_\alpha \xrightarrow{w} \mu$ and $\nu_\alpha \xrightarrow{w} \nu$ then $\lambda_\alpha \xrightarrow{w} \lambda$.*

REMARK 4.4. Theorem 4.3 has already been proved in [**21**] with the additional assumption $\mathcal{B}(S \times T) = \mathcal{B}(S) \times \mathcal{B}(T)$. However, the result obtained here applies to many cases which are not covered by Theorem 5.4 of [**21**]; see Remark 3.2 and Corollary 3.5. Similar results are given in [**19**] for not necessarily positive vector measures with values in certain nuclear spaces.

5. Weak convergence of convolutions of vector measures

In this section, we shall define the convolution of Banach algebra-valued vector measures on a topological semigroup and consider their weak convergence.

Let X be a Banach algebra with the multiplication $(x, y) \in X \times X \mapsto xy$. We write $m(x, y) = xy$. Then $m : X \times X \to X$ is a continuous bilinear mapping. Denote by $\tilde{m}$ the *linearization* of m, that is, $\tilde{m}$ is a mapping from the algebraic tensor product $X \otimes X$ into X defined by $\tilde{m}(u) = \sum_{i=1}^k m(x_i, y_i)$ for $u = \sum_{i=1}^k x_i \otimes y_i$ in $X \otimes X$. This is well-defined since the value of $\tilde{m}(u)$ does not depend on the representation of u. We suppose throughout this section that the mapping $\tilde{m}$ satisfies the condition (M_ε): $\|\tilde{m}(u)\| \leq \|u\|_\varepsilon$ for all $u \in X \otimes X$, which was also supposed in [**31**, Section 3]. In this case, $\tilde{m}$ has a unique continuous extension, which we also denote by $\tilde{m}$; thus $\tilde{m} : X\hat{\otimes}_\varepsilon X \to X$ is a continuous linear mapping with $\|\tilde{m}\| \leq 1$. Although the condition (M_ε) is a fairly strong assumption, it is satisfied, for instance, X is the Banach algebra $C(\Omega)$ with a compact space Ω or the Banach algebra $C_0(\Lambda)$ with a locally compact space Λ; see Example 4.1.

Let T be a topological space. Let $\nu : \mathcal{B}(T) \to X$ be a vector measure. Then the inequality $\|\nu\|_m(B) \leq \|\nu\|_\varepsilon(B)$ holds for all subsets of T [**31**, Theorem 3.1]. Since every vector measure $\nu : \mathcal{B}(T) \to X$ has the (*)-property with respect to the tensor mapping $\varepsilon(x,y) = x \otimes y$ [**31**, Remark 2], it follows from the above inequality and [**13**, Lemma 2] that ν also has the (*)-property with respect to m. In what follows, we shall write $\int_B \varphi d\nu := \int_B m(\varphi, d\nu)$ for every (ν, m)-integrable function $\varphi : T \to X$ and $B \in \mathcal{B}(T)$. Then we have

PROPOSITION 5.1. *Let S and T be topological spaces. Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(T) \to X$ be vector measures. Assume that μ has a proper Borel injective tensor product λ with respect to ν. Let $h \in B(S \times T)$. Then the following assertions are valid:*

(i) *The vector function $t \in T \mapsto \int_S h(s,t)\mu(ds)$ is (ν, m)-integrable, and*

(ii) $\tilde{m}\left(\int_{S\times T} h d\lambda\right) = \int_T \int_S h(s,t)\mu(ds)\nu(dt)$.

From now throughout this paper, assume that S is a topological semigroup with a completely regular topology, that is, S is a completely regular space with a jointly continuous, associative, semigroup operation $(s,t) \in S \times S \mapsto st \in S$. We write $\theta(s,t) := st$. Then θ is measurable with respect to $\mathcal{B}(S \times S)$ and $\mathcal{B}(S)$. The following theorem extends [**31**, Theorem 3.2] and shows that the convolution of X-valued vector measures can be defined as the measure induced by their proper Borel injective tensor product and the semigroup operation.

THEOREM 5.2. *Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(S) \to X$ be vector measures. Assume that μ has a τ-smooth proper Borel injective tensor product with respect to ν. Then there is a unique τ-smooth vector measure $\gamma : \mathcal{B}(S) \to X$ such that*

$$\int_S f d\gamma = \int_S \int_S f(st)\mu(ds)\nu(dt) \quad \text{for all } f \in C(S). \tag{5.1}$$

Then γ is called the convolution of the ordered pair μ, ν and denoted by $\mu * \nu$. Further, (5.1) *continues to hold for every $f \in B(S)$.*

In the following, we shall assume that the Banach algebra X is one of the spaces $C(\Omega)$ with a compact space Ω or $C_0(\Lambda)$ with a locally compact space Λ. Then X satisfies the condition (M_ε). Further, X is a Banach lattice such that $X\hat{\otimes}_\varepsilon X$ is also a Banach lattice and satisfies the positivity condition (P_ε) in Section 4. We recall that $C(\Omega)$ and $C_0(\Lambda)$ are separable if Ω is metrizable and Λ is separable and metrizable; see [**14**, Exercise IV.13.16] and [**25**, Theorem II.6 in Part I, page 111].

PROPOSITION 5.3. *Let X be one of the spaces $C(\Omega)$ with a compact space Ω or $C_0(\Lambda)$ with a locally compact space Λ. Let $\mu : \mathcal{B}(S) \to X$ and $\nu : \mathcal{B}(S) \to X$ be vector measures. Assume that μ has a τ-smooth proper Borel injective tensor product with respect to ν (this condition is satisfied if, for instance, μ and ν are τ-smooth, Ω is metrizable, and Λ is separable and metrizable by Corollary 3.5). If μ and ν are positive then so is the convolution $\mu * \nu$.*

The following theorem shows the joint weak convergence of convolutions of vector measures with values in special Banach algebras $C(\Omega)$ or $C_0(\Lambda)$. Except for assuming the tightness of vector measures, it extends previous results for probability measures; see [**7**, Corollary] and [**30**, Proposition I.4.5].

THEOREM 5.4. *Let X be one of the spaces $C(\Omega)$ with a compact space Ω or $C_0(\Lambda)$ with a locally compact space Λ. Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a net in $\mathcal{M}^+(S,X)$ and $\mu \in \mathcal{M}_t^+(S,X)$. Let $\{\nu_\alpha\}_{\alpha\in\Gamma}$ be a net in $\mathcal{M}_t^+(S,X)$ and $\nu \in \mathcal{M}_t^+(S,X)$. Assume that μ_α and μ have τ-smooth proper Borel injective tensor products with respect to ν_α and ν, respectively. Further assume that ν is τ-smooth (these conditions are satisfied if, for instance, μ_α, ν_α, μ and ν are all Radon, Ω is metrizable, and Λ is separable and metrizable by Corollary 3.5). If $\mu_\alpha \xrightarrow{w} \mu$ and $\nu_\alpha \xrightarrow{w} \nu$ then $\mu_\alpha * \nu_\alpha \xrightarrow{w} \mu * \nu$.*

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