数理解析研究所講究録 1340 巻 2003 年 18-26

On 'Monotonic' Binomial Distribution

Naofumi Muraki

Mathematics Laboratory, Iwate Prefectural University
Takizawa, Iwate 020-0193, Japan

Abstract. The 'monotonic' analogue of binomial distribution is discussed. Its probability distribution is determined in a recursive way. We also give a graphical simulation of monotonic central limit theorem and of monotonic Poisson limit theorem (= monotonic law of small numbers), through this monotonic binomial distribution.

1. The notion of *monotonic independence* was introduced by the author [6] as an example of universal notions of independence in non-commutative probability theory. It is well-known that a non-commutative analogue of classical probability theory, that is *free probability theory*, can be developed based on the notion of *freeness* (= free independence) of D. V. Voiculescu [2][11]. After trying to find other possibilities of such non-commutative notions of independence, the author found a new example (= monotonic independence) [6]. It was introduced as the algebraic abstraction of a structure which have been hidden in the discussion of a ceratin central limit type argument in monotone Fock space [4][5] (or in chronological Fock space discussed independently by Y. G. Lu [3]). In the way parallel to the free probability theory of Voiculescu, we can develope the monotonic analogue of several probabilistic notions, for example, the analogue of central limit theorem, law of small numbers, Brownian motion, convolution of probability measures, Lévy-Hinčin formula, Lévy processes, and stochastic calculus [5][6][7][1]. Also interesting is the monotone product construction for non-commutative probability spaces [7], which can be compared with the tensor product construction in classical probability theory and the free product construction in free probability theory.

In this note, as a continuation of my program of developing 'monotone probability,' we consider about the probability distribution of monotonically independent sum of identically distributed Bernoulli random variables (= 'monotonic' binomial distribution). We give a recursive description for the monotonic binomial distribution. Plotting the graph of monotonic binomial distribution, we can certify in a visual way the monotonic central limit theorem (which asserts, in its special case, the convergence of 'monotonic' binomial distribution to 'monotonic' Gaussian distribution) and the monotoni Poisson limit theorem (which asserts, in its special case, the convergence of 'monotonic' binomial distribution to 'monotonic' Poisson distribution) although these limit theorems have been already established in [6] for possibly non-binomial random variables.

2. Let $(\mathcal{A}, \phi)$ be a C^*-probability space consisting of a unital C^*-algebra $\mathcal{A}$ and a state ϕ over $\mathcal{A}$. Let us be given a linearly ordered family $\{\mathcal{A}_i\}_{i\in I}$ of C^*-subalgebras of $\mathcal{A}$, where the index set I is linearly ordered. Here we do not assume that the unit $1_{\mathcal{A}}$ of $\mathcal{A}$ is contained in each $\mathcal{A}_i$. The family of subalgebras $\{\mathcal{A}_i\}_{i\in I}$ is said to be *monotonically independent* if the following conditions are satisfied.

(M1) The factorizaion

$$\phi(YX_iX_jX_kZ) = \phi(X_j)\phi(YX_iX_kZ)$$

holds whenever $i < j > k$ and $X_i \in \mathcal{A}_i$, $X_j \in \mathcal{A}_j$, $X_k \in \mathcal{A}_k$, $Y, Z \in \mathcal{A}$.

(M2) The factorization

$$\begin{aligned}&\phi(X_{i_m}\cdots X_{i_2}X_{i_1}X_jX_{k_1}X_{k_2}\cdots X_{k_n})\\ &\quad= \phi(X_{i_m})\cdots\phi(X_{i_2})\phi(X_{i_1})\phi(X_j)\phi(X_{k_1})\phi(X_{k_2})\cdots\phi(X_{k_n})\end{aligned}$$

holds whenever $i_m > \cdots > i_2 > i_1 > j < k_1 < k_2 \cdots < k_n$ and $X_{i_1} \in \mathcal{A}_{i_1}$, $X_{i_2} \in \mathcal{A}_{i_2}$, $\cdots$, $X_{i_m} \in \mathcal{A}_{i_m}$, $X_j \in \mathcal{A}_j$, $X_{k_1} \in \mathcal{A}_{k_1}$, $X_{i_2} \in \mathcal{A}_{i_2}$, $\cdots$, $X_{k_n} \in \mathcal{A}_{k_n}$.

For any family of C^*-probability spaces $(\mathcal{A}_i, \phi_i)_{i\in I}$ with linearly ordered index set I, there exists a C^*-probability space $(\tilde{\mathcal{A}}, \tilde{\phi})$ so that every $(\mathcal{A}_i, \phi_i)$'s are embeded as monotonically independent subalgebras of $(\tilde{\mathcal{A}}, \tilde{\phi})$. This construction (= monotone product construction) can be characterized by some universal property in the category of non-commutative probabity spaces.

3. In the usual probability theory, the notion of convolution of probability measures is useful for the description of probability distribution of the sum of independent random variables. Also in the setting of 'monotone probability', we can introduce a certain kind of convolution for probability measures, which is associated to the notion of monotonic independence [7].

For any probability measure μ on the real line $\mathbf{R}$, its *Cauchy transform* $G_\mu(z)$ is defined by

$$G_\mu(z) := \int_{-\infty}^{+\infty} \frac{1}{z-x} d\mu(x), \quad z \in \mathbf{C}^+,$$

where $\mathbf{C}^+$ denotes the complex upper half plane. Its reciprocal

$$H_\mu(z) = \frac{1}{G_\mu(z)}, \quad z \in \mathbf{C}^+$$

is called the *reciprocal Cauchy transform* of μ. For any self-adjoint random variable $X = X^* \in \mathcal{A}$ in a C^*-probability space $(\mathcal{A}, \phi)$, we define its Cauchy transform (resp. reprocal Cauchy transform) by $G_X(z) := G_{\mu_X}(z)$ (resp. $H_X(z) := H_{\mu_X}(z)$) where μ_X is the probabity distribution of X under the state ϕ. A family of random variables is said to be monotonically independent if the family of subalgebras generated by each random variavles is monotonically independent. Then we have the following.

Theorem [7] *Let $X_1, X_2, \cdots, X_n \in \mathcal{A}$ be monotonically independent self-adjoint random variables, in the natural order, over a C^*-probability space $(\mathcal{A}, \phi)$. Then*

$$H_{X_1+X_2+\cdots+X_n}(z) = H_{X_1}(H_{X_2}(\cdots H_{X_n}(z)\cdots)).$$

This theorem tells us that the role of the reciprocal Cauchy transform in monotone probability is analogous to that of the Fourier transform in classical probability and to that of the R transform of Voiculescu in free probability. Based on the reciprocal Cauchy transform, the *monotonic convolution* $\lambda = \mu \rhd \nu$ of two probability measures μ, ν on the real line $\mathbf{R}$, which are possibly unbounded, is defined by $H_\lambda(z) = H_\mu(H_\nu(z))$. This notion is well-defined [7].

4. Let X_1, X_2, $\cdots$, X_n, $\cdots$ be monotonically independent and identically distributed Bernoulli random variables. So the same distribution $\mu := \mu_{X_i}$ of each X_i is given by

$$\mu = p \cdot \delta_a + q \cdot \delta_b,$$

where $p \geq 0,\ q \geq 0,\ p + q = 1$ and $a < b$. Here δ_{x_0} denotes the Dirac measure at a point x_0. Let us investigate the probability distribution μ_n of the monotonically independent sum $Y_n := X_1 + X_2 + \cdots + X_n$. The distribution μ_n should be called the *monotonic binomial distribution.*

Using the reciprocal Cauchy transform, we can determine in the recursive way the probability distribution μ_n of the random variable Y_n $(= Y_{n-1} + X_n)$ as

$$\mu_n = \sum_{\sigma \in \{-,+\}^n} p(\sigma) \cdot \delta_{a(\sigma)},$$

where the coefficients $a(\sigma)$, $p(\sigma)$ $(\sigma = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \{-,+\}^n)$ satisfy the initial conditions

$$a(-) := a, \quad a(+) := b; \quad p(-) := p, \quad p(+) := q$$

and the recursive relations

$$\begin{aligned} a(*,\varepsilon) &= \frac{a(*) + (a+b) + \varepsilon\sqrt{(a(*) + (a-b))^2 + 4(b-a)q\,a(*)}}{2}, \\ p(*,\varepsilon) &= p(*) \times \varepsilon \times \frac{a(*\varepsilon) - (a\,q + b\,p)}{a(*+) - a(*-)}, \end{aligned}$$

where $*$ is an arbitrary element in $\{-,+\}^{n-1}$ and ε is an element in $\{-,+\}$.

Specifying the scaling of the parameter of the distribution μ_n, let us visualize the behaviour of the monotonic binomial distribution μ_n with the number of trials $n \to \infty$. We plot the graph of μ_n with use of Mathematica.

A. *Scaling of the central limit type.* Let each X_i be the symmetric Bernoulli random variables with values $a = -1$, $b = +1$ and the respective probabilities $1/2$. In this case, the coefficients $\alpha(*)$, $p(*)$ satisfy the recursive relations

$$\begin{aligned} a(*,\varepsilon) &= \frac{a(*) + \varepsilon\sqrt{a(*)^2 + 4}}{2}, \\ p(*,\varepsilon) &= p(*) \times \varepsilon \times \frac{a(*\varepsilon)}{\sqrt{a(*)^2 + 4}}. \end{aligned}$$

We note that the coefficients $a(\sigma)$, $p(\sigma)$ describing the monotonic binomial distribution μ_n have the following properties.

1) The correspondence $a(\sigma) \mapsto a(\sigma,+)$ $(\sigma \in \{-,+\}^{n-1})$ preserves the order relation. So for any $\sigma_1, \sigma_2 \in \{-,+\}^{n-1}$,

$$a(\sigma_1) < a(\sigma_2) \implies a(\sigma_1,+) < a(\sigma_2,+).$$

2) Under the inversion $\sigma \mapsto \sigma'$ defined for $\sigma = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ by $\sigma' = (\varepsilon_1', \varepsilon_2', \cdots, \varepsilon_n')$, $+' = -$, $-' = +$, we have $a(\sigma') = -a(\sigma)$, $p(\sigma') = p(\sigma)$. Of course μ_n is the symmetric probability distribution.
3) The correspondence $\sigma \mapsto a(\sigma)$ preserves the lexicographic order among σ's. So we have

$$\sigma_1 \prec \sigma_2 \implies a(\sigma_1) < a(\sigma_2) \qquad (\sigma_1, \sigma_2 \in \{-,+\}^n)$$

Here the lexicographic ordering among σ's is defined in the way that, in the evaluation for the ordering, the letter in the right hand side is more dominant than the letter in the left hand side. For example, we have

- $(-) \prec (+)$,
- $(-,-) \prec (+,-) \prec (-,+) \prec (+,+)$,
- $(-,-,-) \prec (+,-,-) \prec (-,+,-) \prec (+,+,-)$
$\prec (-,-,+) \prec (+,-,+) \prec (-,+,+) \prec (+,+,+)$.

We plot, in the figures G[1], $\cdots$, G[7] and mG, the graphs of the symmetric monotonic binomial distributions with the number of trials $n = 1, 2, \cdots, 7$ and its limit ($n = \infty$). The vertical axis in G[1], $\cdots$, G[7] (resp. mG) express the weight $p(\sigma)$ (resp. the probability density). As shown in [4][6], the limit distribution of the scaled sum $\frac{1}{\sqrt{n}} \cdot Y_n$ is just the standard arcsine law with mean 0 and variance 1 given by

$$\frac{1}{\pi\sqrt{2-x^2}}dx, \qquad -\sqrt{2} < x < \sqrt{2}$$

(see Figure mG). So the arcsine law plays the role of 'monotonic' Gaussian law. Furthermore we recognize from figures G[1], $\cdots$, G[7] a cetain kind of fractal property of monotonic binomial distribution.

B. *Scaling of Poisson type.* Let us treat the scaling of Poisson type with the parameter $\lambda > 0$. In this case, we put $Y_n = X_1^{(n)} + \cdots + X_n^{(n)}$ and assume that, for any fixed n, the random variables $X_1^{(n)}, \cdots, X_n^{(n)}$ are monotonically independent and identically distributed. The distribution μ of 1 trial (in the total n trials) is already in the dependency on n as $\mu = \mu^{(n)}$. To be more concrete, for each fixed n, every variables $X_i^{(n)}$ takes the values $a = 0$, $b = 1$ with the respective probability $p = 1 - \lambda/n$, $q = \lambda/n$ ($\lambda > 0$). That is, we have

$$\mu^{(n)} = \left(1 - \frac{\lambda}{n}\right) \cdot \delta_0 + \frac{\lambda}{n} \cdot \delta_1.$$

Now we put $Y_k^{(n)} = X_1^{(n)} + \cdots + X_k^{(n)}$ ($k \leq n$). Also we denote by $\nu_k^{(n)}$ the distribution of $Y_k^{(n)}$. Then $\nu_k^{(n)}$ is given by

$$\nu_k^{(n)} = \sum_{\sigma \in \{-,+\}^k} p^{(n)}(\sigma) \cdot \delta_{a^{(n)}(\sigma)},$$

where the finite sequence of families of coefficients $\{a^{(n)}(\sigma), p^{(n)}(\sigma)\}_{\sigma\in\{-,+\}^k}$ ($k = 1, 2, \cdots, n$) is determined in the recursive way by

$$a^{(n)}(-) := 0, \quad a^{(n)}(+) := 1, \quad p^{(n)}(-) := p^{(n)} = 1 - \frac{\lambda}{n}, \quad p^{(n)}(+) := q^{(n)} = \frac{\lambda}{n},$$

$$a^{(n)}(*, \varepsilon) = \frac{a^{(n)}(*) + 1 + \varepsilon\sqrt{(a^{(n)}(*) - 1)^2 + 4\,q^{(n)}\,a^{(n)}(*)}}{2},$$

$$p^{(n)}(*, \varepsilon) = p^{(n)}(*) \times \varepsilon \times \frac{a^{(n)}(*, \varepsilon) - p^{(n)}}{\sqrt{(a^{(n)}(*) - 1)^2 + 4\,q^{(n)}\,a^{(n)}(*)}},$$

with $* \in \cup_{k=1}^{n-1}\{-,+\}^k$. Note that, in the case of Poisson type scaling, the propability p (resp. q) of tail (resp. head) in a coin toss is in the dependency on n as $p = p^{(n)}$, $q = q^{(n)}$.

The coefficients $a^{(n)}(\sigma)$, $p^{(n)}(\sigma)$ describing the binomial distribution $\nu_k^{(n)}$ have the following properties.

1) The correspondence $a^{(n)}(\sigma) \mapsto a^{(n)}(\sigma,+) \quad (\sigma \in \{-,+\}^{k-1},\ k \leq n)$ preserves the order relation.
2) The relation $a^{(n)}(-,\sigma) = a^{(n)}(\sigma)$ holds. By the mapping $\{-,+\}^{k-1} \ni \sigma \mapsto (-,\sigma) \in \{-,+\}^k$, the family $\{a^{(n)}(\sigma) | \sigma \in \{-,+\}^{k-1}\}$ is extended to the family $\{a^{(n)}(\sigma) | \sigma \in \{-,+\}^k\}$.
3) The correspondence $\sigma \mapsto a^{(n)}(\sigma)$ preserves the lexicographic ordering of $\sigma \in \{-,+\}^k$.

We plot, in the figures P[1,1/2], $\cdots$, P[7,1/2] and mP[1/2], the graphs of the monotonic binomial distributions $\nu_n^{(n)}$ with the number of trials $n = 1, 2, \cdots, 7$ and its limit ($n = \infty$). In these figures, the parameter λ of Poisson distribution is fixed to be $\lambda := 1/2$. The vertical axis in P[1,1/2], $\cdots$, P[7,1/2] (resp. mP[1/2]) express the weight $p(\sigma)$ (resp. the probability density). We remark that the value of weight $p^{(n)}(-,-,\cdots,-)$ is out of the frame of each graph. By the result in [6], the limit distribution of the binomial dsitribution $\nu_n^{(n)}$ is just the 'monotonic' Posson law (see mP[1/2]). The monotonic Poisson distribution ν with parameter λ consisits of the absolutely continuous part ν_1 and the atomic part ν_2. The absolutely continuous part ν_1 is given by

$$\frac{1}{\pi}\,\mathrm{Im}\frac{1}{W_{-1}(-x\,e^{\lambda-x})}\,dx, \qquad a < x < b,$$

and the atomic part is given by $\nu_2 = c\,\delta_0$ with the Dirac measure δ_0 at the origin $x = 0$, where the constants a, b, c are defined by

$$a = -W_0\left(-\frac{1}{e^{1+\lambda}}\right), \quad b = -W_{-1}\left(-\frac{1}{e^{1+\lambda}}\right), \quad c = \frac{1}{e^{\lambda}}.$$

Here $W_n(z)$ is the nth branch of the the Lambert W function (a special function). Also in the Poisson case, we recognize from figures P[1,1/2], $\cdots$, G[7,1/2] a certain kind of fractal property of monotonic binomial distribution.

References

[1] Franz, U., Unification of boolean, monotone, anti-monotone, and tensor independence and Lévy processes, Math. Z., Vol. 243, No. 4 (2003), 779-816.

[2] Hiai, F., and Petz, D., The Semicircle Law, Free Random Variables and Entropy, Mathematical Surveys and Monographs **77**, Amer. Math. Soc., Providence, RI, 2000.

[3] Lu, Y. G., An interacting free Fock space and the arcsine law, Probability and Mathematical Statistics, **17**(1997), 149-166.

[4] Muraki, N., A new example of noncommutative "de Moivre-Laplace theorem", Probability Theory and Mathematical Statistics, Proc. Seventh Japan-Russia Symp., Tokyo 1995 (edited by Watanabe, S., Fukushima, M., Prohorov, Yu. V., and Shiryaev, A. N.), World Scientific, Singapore, 1996, 353-362.

[5] Muraki, N., Noncommutative Brownian motion in monotone Fock space, Commun. Math. Phys., **183**(1997), 557-570.

[6] Muraki, N., Monotonic independence, monotonic central limit theorem and monotonic law of small numbers, Infinite Dim. Anal. Quantum Probab., **4**(2001), 39-58.

[7] Muraki, N., Monotonic convolution and monotonic Lévy-Hinčin formula, preprint, Iwate Pref. Univ., 2000. Submitted to Probab. Theory Relat. Fields.

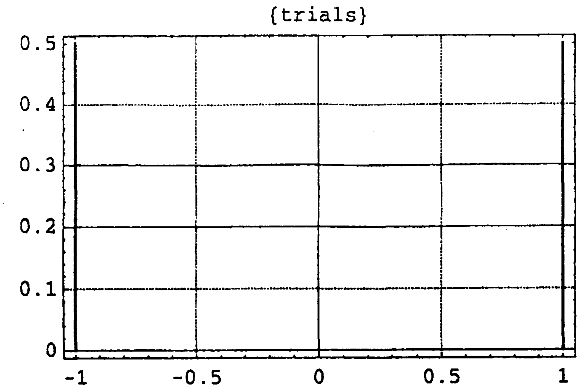
[8] Muraki, N., The five independences as quasi-universal products, Infinite Dim. Anal. Quantum Probab., **5**(2002), 113-134.

[9] Muraki, N., The five independences as natural products, preprint, EMAU Greifswald, 2002. Submitted to Infinite Dim. Anal. Quantum Probab.

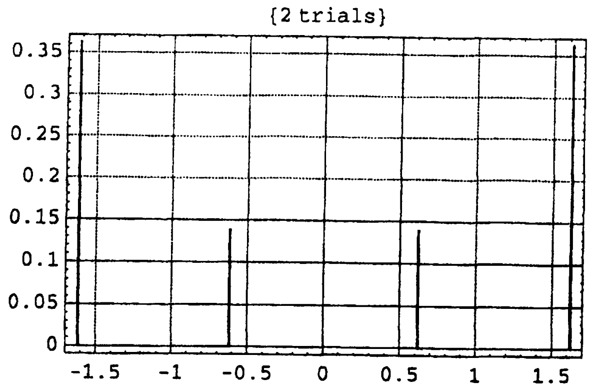
[10] Muraki, N., Notions of 'independence' in noncommutative probability theory, Bull. JSIAM, Vol.13 No2 (2003) 137-149. (in Japanese)

[11] Voiculescu, D. V., Dykema, K. J., and Nica, A., Free Random Variables, CRM Monograph Series, Amer. Math. Soc., Providence, RI, 1992.

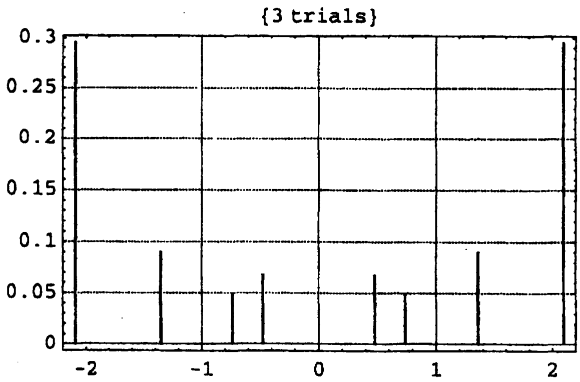
G[1]



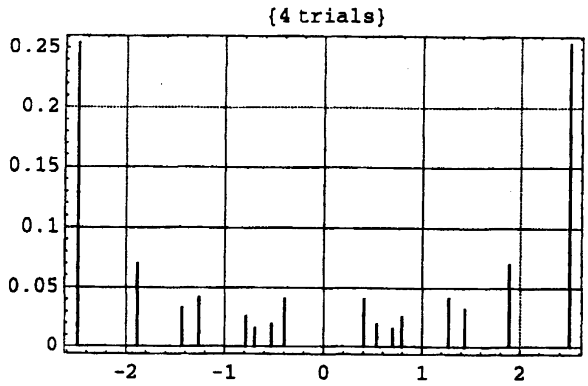
G[2]



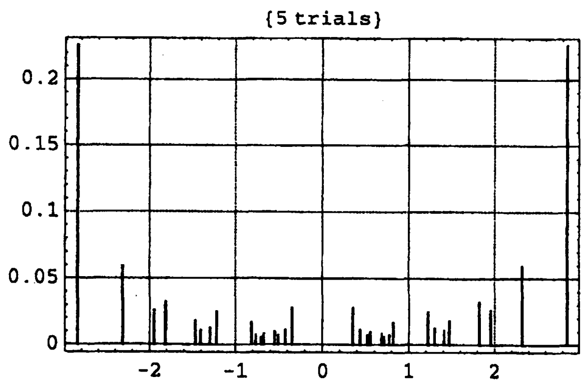
G[3]



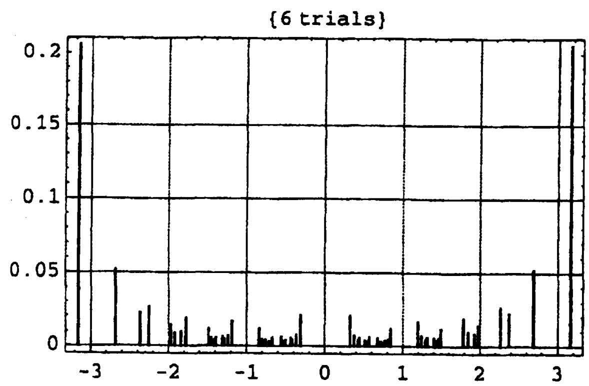
G[4]



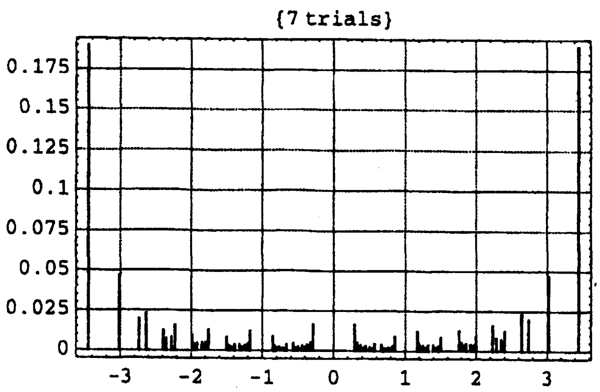
G[5]



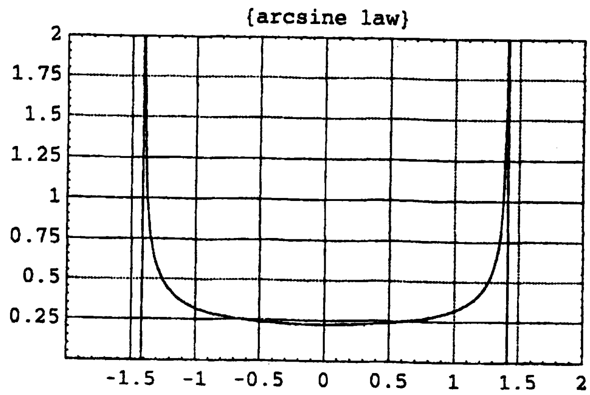
G[6]



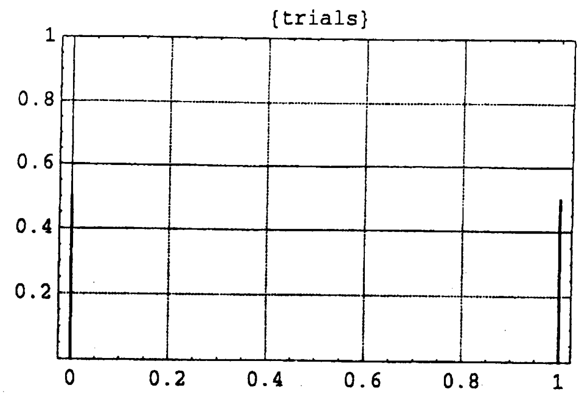
G[7]



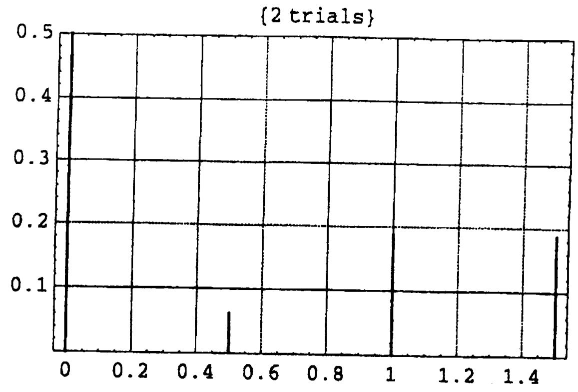
mG



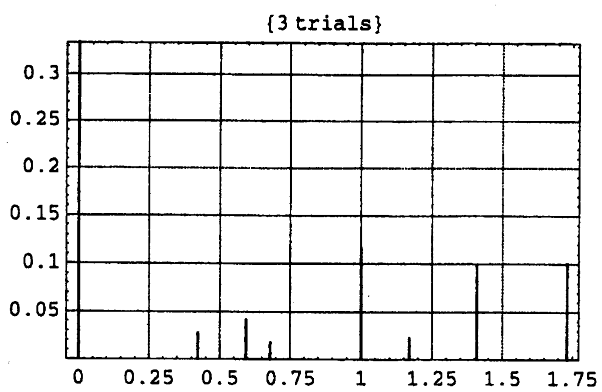
P[1, 1/2]



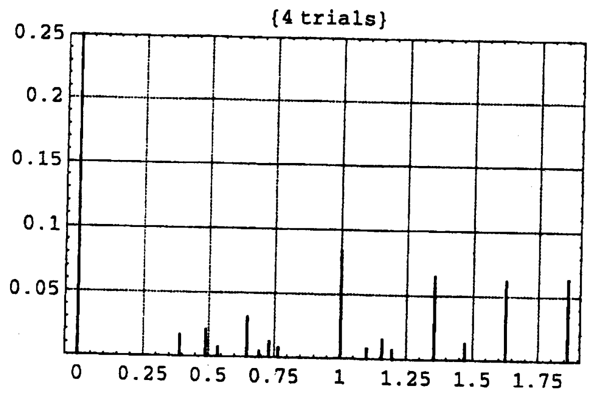
P[2, 1/2]



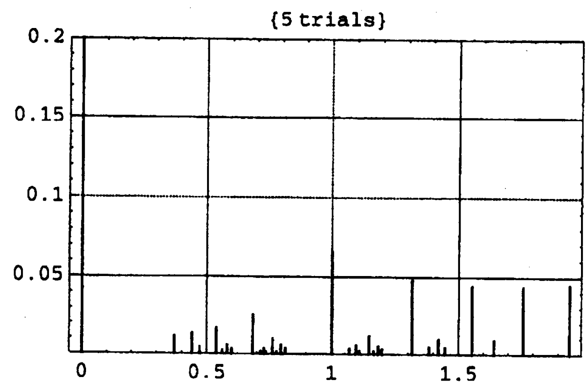
P[3, 1/2]



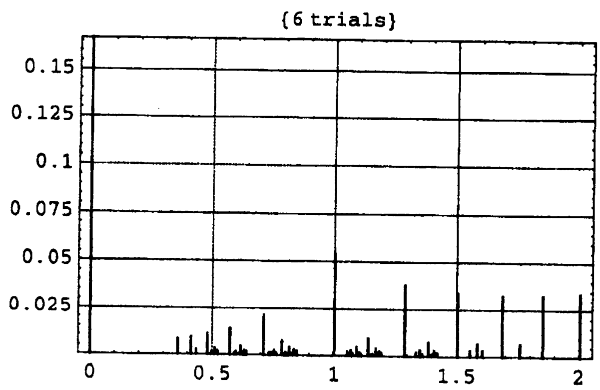
P[4, 1/2]



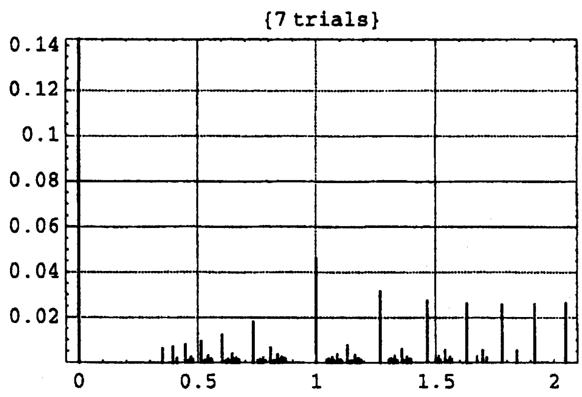
P[5, 1/2]



P[6, 1/2]



P[7, 1/2]



mP[1/2]