数理解析研究所講究録 1341 巻 2003 年 77-84

ARGUMENT ESTIMATES FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. Let $p(z)$ be analytic in the open unit disk $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. S.S.Miller and P.T.Mocanu (J. Math. Anal. Appl. 276(2002)) have shown some interesting subordination theorems for such functions $p(z)$. The object of the present paper is to discuss some sufficient conditions for arguments of $p(z)$ to be $|\arg p(z)| < \frac{\pi}{2}\rho$ for $z \in \mathbb{U}$.

1. INTRODUCTION

Let $p(z)$ be analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with $p(0) = 1$ and $p'(0) = 0$. For such functions $p(z)$, Miller and Mocanu [3] have showm some interesting subordination theorems.

Theorem A. ([3]) *For $\frac{1}{2} < \rho \leqq 1$ define the function $q(z)$ by*

$$q(z) = q_\rho(z) = \left(\frac{1+z}{1-z}\right)^\rho,$$

and let $t_0 \in (0,1)$ be the unique solution of

$$t^\rho \left\{(1-\rho)t^2 \cos\left(\frac{\pi}{2}\rho\right) + t\sin\left(\frac{\pi}{2}\rho\right) - (1-\rho)\cos\left(\frac{\pi}{2}\rho\right)\right\} + t^2 - 1 = 0.$$

If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = 1, p'(0) = 0$ and

$$\left|\arg\left(zp'(z) + p(z)^2 + p(z)\right)\right| < \frac{\pi}{2}(\rho+1) - \mathrm{Tan}^{-1}\left(\frac{t_0}{1+\rho-(1-\rho)t_0^2}\right),$$

then $p(z) \prec q_\rho(z)$, where the simbol "$\prec$" means the subordinations.

To discuss our problems for functions $p(z)$, we need the following lemma due to Hallenbeck and Ruscheweyh [2] which is the same as one by Fukui and Sakaguchi [1].

2000 Mathematics Subject Classification. Primary 30C45.
Key Words and Phrases. analytic function, argument estimate, subordination.

Lemma 1.1. *Let $p(z)$ be analytic in $|z| < R$ and $p^{(k)}(0) = 0 (0 \leqq k \leqq n)$. Then if $|p(z)|$ attains its maximum value on the circle $|z| = r < R$ at a point z_0, we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} \geqq n+1. \tag{1.1}$$

Applying the above lemma, we derive

Lemma 1.2. *Let $p(z)$ be analytic in $\mathbb{U}, p(0) = 1, p'(0) = 0$, and let $p(z) \neq 0 (z \in \mathbb{U})$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \qquad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

for some $\alpha > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k, \tag{1.2}$$

where

$$k \geqq \left(a + \frac{1}{a}\right) \geqq 2 \qquad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leqq -\left(a + \frac{1}{a}\right) \leqq -2 \qquad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where $p(z_0)^{1/\alpha} = \pm ia$ and $a > 0$.

Proof. We use the same manner which was used by Nunokawa [4] for the proof of the lemma. Let us put

$$q(z) = p(z)^{1/\alpha}. \tag{1.3}$$

Then we see that $\text{Re}\, q(z) > 0 \ (|z| < |z_0|), \text{Re}\, q(z_0) = 0, q(0) = 1$ and $q'(0) = 0$. Defining the function $\phi(z)$ by

$$\phi(z) = \frac{1 - q(z)}{1 + q(z)}, \tag{1.4}$$

we have that $\phi(0) = 0, |\phi(z)| < 1 \ (|z| < |z_0|)$, and $|\phi(z_0)| = 1$.
In view of Lemma 1.1, we know that

$$\frac{z_0 \phi'(z_0)}{\phi(z_0)} = \frac{-2z_0 q'(z_0)}{1 - q(z_0)^2} \tag{1.5}$$

$$= \frac{-2z_0 q'(z_0)}{1+|q(z_0)|^2} \geqq 2\,.$$

It follows from (1.5) that

$$-z_0 q'(z_0) \geqq (1+|q(z_0)|^2) \tag{1.6}$$

and $z_0 q'(z_0)$ is a negative real number. Since $q(z_0)$ is a non-vanishing pure imaginary number, we can put $q(z_0) = ia$, where a is a non-vanishing real number.
We have, for $a > 0$,

$$\operatorname{Im}\left(\frac{z_0 q'(z_0)}{q(z_0)}\right) = \operatorname{Im}\left(-\frac{iz_0 q'(z_0)}{|q(z_0)|}\right) \geqq \left(\frac{1+a^2}{a}\right) \geqq 2 \tag{1.7}$$

and, for $a < 0$,

$$\operatorname{Im}\left(\frac{z_0 q'(z_0)}{q(z_0)}\right) = \operatorname{Im}\left(\frac{iz_0 q'(z_0)}{|q(z_0)|}\right) \leqq -\left(\frac{1+a^2}{a}\right) \leqq -2 \tag{1.8}$$

On the other hand, it follows that

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{1}{\alpha}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right). \tag{1.9}$$

This completes the proof of Lemma 1.2. □

2. Argument estimates

Our first property for argument estimates of analytic function $p(z)$ is contained in

Theorem 2.1. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies*

$$\left|\arg\left(zp'(z) + p(z)^2 + \alpha p(z)\right)\right| < \pi\rho \qquad (z \in \mathbb{U}) \tag{2.1}$$

for some $\alpha\,(\alpha > 0), \rho\,(0 < \rho \leqq \rho_0)$, where $\rho_0\,(0 < \rho_0 < 1)$ is given by

$$\tan\left(\frac{\pi}{2}\rho_0\right) = \frac{2}{\alpha}\rho_0\,,$$

then

$$|\arg p(z)| < \frac{\pi}{2}\rho \qquad (z \in \mathbb{U})\,. \tag{2.2}$$

Proof. Let a function $p(z)$ satisfy the conditions of the theorem. If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\rho \qquad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\rho\,,$$

then applying Lemma 1.2, we have that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\rho k, \tag{2.3}$$

where

$$k \geqq a + \frac{1}{a} \geqq 2 \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\rho$$

and

$$k \leqq -\left(a + \frac{1}{a}\right) \leqq -2 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\rho$$

with $p(z_0)^{1/\rho} = \pm ia$ $(a > 0)$. It follows that, for $\arg p(z_0) = \frac{\pi}{2}\rho$ and $k \geqq a + \frac{1}{a} \geqq 2$,

$$\arg\left(z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0)\right) = \arg p(z_0)\left(\frac{z_0 p'(z_0)}{p(z_0)} + p(z_0) + \alpha\right) \tag{2.4}$$

$$= \frac{\pi}{2}\rho + \arg\left(i\rho k + a^\rho e^{i\frac{\pi}{2}\rho} + \alpha\right) = \frac{\pi}{2}\rho + \mathrm{Tan}^{-1}\left(\frac{\rho k + a^\rho \sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)}\right)$$

Since, by $0 < \rho \leqq \rho_0 < 1$ and $k \geqq 2$,

$$\mathrm{Tan}^{-1}\left(\frac{\rho k + a^\rho \sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)}\right) \geqq \mathrm{Tan}^{-1}\left(\frac{2\rho + a^\rho \sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)}\right) > 0, \tag{2.5}$$

we define $g(a)$ by

$$g(a) = \frac{2\rho + a^\rho \sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)} \qquad (a > 0). \tag{2.6}$$

Noting that

$$g'(a) = \frac{\alpha\rho a^{\rho-1}\cos\left(\frac{\pi}{2}\rho\right)\left(\tan\left(\frac{\pi}{2}\rho\right) - \frac{2\rho}{\alpha}\right)}{\left(\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)\right)^2}, \tag{2.7}$$

we define $h(\rho)$ by

$$h(\rho) = \tan\left(\frac{\pi}{2}\rho\right) - \frac{2\rho}{\alpha} \qquad (0 < \rho \leqq \rho_0 < 1). \tag{2.8}$$

Then $h(0) = 0, h(\rho_0) = 0$, and

$$h''(\rho) = \frac{\pi^2}{2}\sec^2\left(\frac{\pi}{2}\rho\right)\tan\left(\frac{\pi}{2}\rho\right) > 0. \tag{2.9}$$

This shows that $g'(a) \leqq 0$ for $a > 0$, that is, that

$$\mathrm{Tan}^{-1}\left(\frac{\rho k + a^\rho \sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)}\right) \geqq \mathrm{Tan}^{-1}\left(\tan\left(\frac{\pi}{2}\rho\right)\right) = \frac{\pi}{2}\rho. \tag{2.10}$$

Therefore, we conclude that

$$\arg\left(z_0p'(z_0)+p(z_0)^2+\alpha p(z_0)\right) \geqq \pi\rho \tag{2.11}$$

when $\arg p(z_0)=\frac{\pi}{2}\rho$.

Similarly, for $\arg p(z_0)=-\frac{\pi}{2}\rho$ and $k \leqq -\left(a+\frac{1}{a}\right) \leqq -2$, we have that

$$\begin{aligned}\arg\left(z_0p'(z_0)+p(z_0)^2+\alpha p(z_0)\right) &= -\frac{\pi}{2}\rho+\arg\left(i\rho k+a^{\rho}e^{-i\frac{\pi}{2}\rho}+\alpha\right) \\ &= -\frac{\pi}{2}\rho+\mathrm{Tan}^{-1}\left(\frac{\rho k-a^{\rho}\sin\left(\frac{\pi}{2}\rho\right)}{\alpha+a^{\rho}\cos\left(\frac{\pi}{2}\rho\right)}\right) \\ &\leqq -\frac{\pi}{2}\rho+\mathrm{Tan}^{-1}\left(\frac{-2\rho-a^{\rho}\sin\left(\frac{\pi}{2}\rho\right)}{\alpha+a^{\rho}\cos\left(\frac{\pi}{2}\rho\right)}\right) \\ &= -\frac{\pi}{2}\rho-\mathrm{Tan}^{-1}\left(\frac{2\rho+a^{\rho}\sin\left(\frac{\pi}{2}\rho\right)}{\alpha+a^{\rho}\cos\left(\frac{\pi}{2}\rho\right)}\right) \\ &\leqq -\frac{\pi}{2}\rho-\frac{\pi}{2}\rho=-\pi\rho.\end{aligned} \tag{2.12}$$

Thus, for such a point $z_0\in\mathbb{U}$, we see that

$$\left|\arg\left(z_0p'(z_0)+p(z_0)^2+\alpha p(z_0)\right)\right| \geqq \pi\rho, \tag{2.13}$$

which contradicts our condition for $p(z)$.

Consequently, we conclude that

$$|\arg p(z)| < \frac{\pi}{2}\rho \qquad (z\in\mathbb{U}).$$

□

Example 2.1. *Let us consider the function $p(z)$ defined by*

$$p(z)=1+\frac{1}{5}z^2.$$

Then we see that

$$zp'(z)+p(z)^2+\frac{1}{2}p(z)=\frac{3}{2}+\frac{9}{10}z^2+\frac{1}{25}z^4.$$

Letting $\alpha=\frac{1}{2}$ and

$$\rho=\frac{1}{\pi}\mathrm{Sin}^{-1}\left(\frac{19}{30}\right)$$

in Theorem 2.1, we have that

$$\left|\arg\left(zp'(z)+p(z)^2+\frac{1}{2}p(z)\right)\right| < \pi\rho=\mathrm{Sin}^{-1}\left(\frac{19}{30}\right)$$

and

$$|\arg p(z)| < \mathrm{Sin}^{-1}\left(\frac{1}{5}\right) < \frac{\pi}{2}\rho.$$

If we take $\alpha = 1$ in Theorem 2.1, then

Corollary 2.1. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies*

$$\text{(2.14)} \qquad |\arg\left(zp'(z) + p(z)^2 + p(z)\right)| < \pi\rho \qquad (z \in \mathbb{U})$$

for some ρ $\left(0 < \rho \leqq \frac{1}{2}\right)$, then

$$\text{(2.15)} \qquad |\arg p(z)| < \frac{\pi}{2}\rho \qquad (z \in \mathbb{U}).$$

Remark 2.1. (1) *If $\alpha = \frac{4}{5}$, then $0 < \rho \leqq \rho_0$ and $0.647873 < \rho_0 < 0.647874$.*

(2) *If $\alpha = \frac{1}{2}$, then $0 < \rho \leqq \rho_0$ and $0.809251 < \rho_0 < 0.809252$.*

(3) *If $\alpha = \frac{1}{3}$, then $0 < \rho \leqq \rho_0$ and $0.880966 < \rho_0 < 0.880967$.*

(4) *If $\alpha = \frac{1}{4}$, then $0 < \rho \leqq \rho_0$ and $0.913417 < \rho_0 < 0.913418$.*

(5) *If $\alpha = 1.1$, then $0 < \rho \leqq \rho_0$ and $0.401247 < \rho_0 < 0.491248$.*

(6) *If $\alpha = 1.2$, then $0 < \rho \leqq \rho_0$ and $0.262943 < \rho_0 < 0.262944$.*

(7) *If $\alpha = 1.3$, then there is no $\rho_0 > 0$ such that $\tan\left(\frac{\pi}{2}\rho_0\right) = \frac{2}{\alpha}\rho$. Thus we see that $0 < \alpha < 1.3$ in Theorem 2.1.*

Next, we derive

Theorem 2.2. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies*

$$\text{(2.16)} \qquad \left|\arg\left(zp'(z) + p(z)^2 + \alpha p(z)\right)\right| < \frac{\pi}{2}\rho + \mathrm{Tan}^{-1}\left(\frac{2\rho}{\alpha}\right) \qquad (z \in \mathbb{U})$$

for some α $(\alpha > 0)$, ρ $(\rho_0 \leqq \rho < 1)$, where ρ_0 $(0 < \rho_0 < 1)$ is given by $\tan\left(\frac{\pi}{2}\rho_0\right) = \frac{2}{\alpha}\rho_0$, then

$$\text{(1.7)} \qquad |\arg p(z)| < \frac{\pi}{2}\rho \qquad (z \in \mathbb{U}).$$

Proof. Using the same technique as in the proof of Theorem 2.1, we know that

$$\mathrm{Tan}^{-1}\left(\frac{2\rho + a^{\rho}\sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^{\rho}\cos\left(\frac{\pi}{2}\rho\right)}\right)$$

is increasing for $a > 0$. Thus, we obtain

$$\left|\arg\left(z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0)\right)\right| \geqq \frac{\pi}{2}\rho + \mathrm{Tan}^{-1}\left(\frac{2\rho}{\alpha}\right) \tag{2.18}$$

for $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\rho \qquad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\rho.$$

This contradicts our condition of the theorem. Therefore,

$$|\arg p(z)| < \frac{\pi}{2}\rho \qquad (z \in \mathbb{U}).$$

□

Letting $\alpha = 1$ in Theorem 2.2, we obtain

Corollary 2.2. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies*

$$\left|\arg\left(zp'(z) + p(z)^2 + p(z)\right)\right| < \frac{\pi}{2}\rho + \mathrm{Tan}^{-1}(2\rho) \qquad (z \in \mathbb{U}) \tag{2.19}$$

for some ρ $\left(\frac{1}{2} \leqq \rho < 1\right)$, then

$$|\arg p(z)| < \frac{\pi}{2}\rho \qquad (z \in \mathbb{U}). \tag{2.20}$$

Finally, we note that

Remark 2.2. (1) *If $\alpha = \frac{4}{5}$, then $0 < \rho \leqq \rho_0$ and $0.647873 < \rho_0 < 0.647874$.*

(2) *If $\alpha = \frac{1}{2}$, then $0 < \rho \leqq \rho_0$ and $0.809251 < \rho_0 < 0.809252$.*

(3) *If $\alpha = \frac{1}{3}$, then $0 < \rho \leqq \rho_0$ and $0.880966 < \rho_0 < 0.880967$.*

(4) *If $\alpha = \frac{1}{4}$, then $0 < \rho \leqq \rho_0$ and $0.913417 < \rho_0 < 0.913418$.*

(5) *If $\alpha = 1.1$, then $0 < \rho \leqq \rho_0$ and $0.401247 < \rho_0 < 0.491248$.*

(6) *If $\alpha = 1.2$, then $0 < \rho \leqq \rho_0$ and $0.262943 < \rho_0 < 0.262944$.*

(7) *If $\alpha = 1.3$, then there is no $\rho_0 > 0$ such that $\tan\left(\frac{\pi}{2}\rho_0\right) = \frac{2}{\alpha}\rho$. Thus we see that $0 < \alpha < 1.3$ in Theorem 2.2.*

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