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Inequalities for Saigo's Fractional Calculus Operator

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Abstract

Let A be the class of functions $f(z)$ of the form $f(z)=z+a_2z^2+a_3z^3+\cdots$ which are analytic in the open unit disk U. For $f(z)\in A$, the subclass $A(n,\delta)$ of A satisfying the coefficient inequalities $|a_k|\leq k^{n+\delta}$ $(k\geq 2)$ is introduced. The object of the present paper is to derive some inequalities for Saigo's fractional calculus operator $I_{0,z}^{\alpha,\beta,\eta}f(z)$ of $f(z)\in A(n,\delta)$.

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1. Introduction

Let A be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

which are analytic in the unit disk $U=\{z\in C:|z|<1\}$. Saigo's fractional calculus operator $I_{0,z}^{\alpha,\beta,\eta}f(z)$ of $f(z)\in A$ is defined in Srivastava, Saigo and Owa [7] (see also Saigo [5]) as follows.

Definition 1.1. For real numbers $\alpha>0,\beta$ and η, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}f(z)$ of $f(z)$ is defined by

$$I_{0,z}^{\alpha,\beta,\eta}f(z)=\frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}\,{}_2F_1\left(\begin{matrix}\alpha+\beta,-\eta\\ \alpha\end{matrix};1-\frac{\zeta}{z}\right)f(\zeta)d\zeta, \tag{1.2}$$

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where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\epsilon}) \qquad (z \to 0),$$

where

$$\varepsilon > \max\{0, \beta - \eta\} - 1,$$

and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Remark 1. It follows from Definition 1.1 that

$$I_{0,z}^{\alpha,-\alpha,\eta} f(z) = D_z^{-\alpha} f(z) = \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\alpha}} d\zeta, \tag{1.3}$$

when $\beta = -\alpha$, where $D_z^{-\alpha}$ is the fractional integral of order α defined by Owa [3].

Definition 1.2. For real numbers $0 \leq \alpha < 1, \beta$ and η, the fractional derivative operator $J_{0,z}^{\alpha,\beta,\eta} f(z)$ of $f(z)$ is defined by

$$J_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left\{ z^{\alpha-\beta} \int_0^z (z-\zeta)^{-\alpha} {}_2F_1 \left(\begin{matrix} \beta-\alpha, 1-\eta \\ 1-\alpha \end{matrix} ; 1 - \frac{\zeta}{z} \right) f(\zeta) d\zeta \right\} \tag{1.3}$$

and

$$J_{0,z}^{m+\alpha,\beta,\eta} f(z) = \frac{d^m}{dz^m} \left(J_{0,z}^{\alpha,\beta,\eta} f(z) \right) \qquad (m = 0, 1, 2, \ldots), \tag{1.5}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\epsilon}) \qquad (z \to 0),$$

where

$$\varepsilon > \max\{0, \beta - \eta\} - 1,$$

and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed as in Definition 1.1 above.

Remark 2. We also note that, when $\beta = \alpha$,

$$J_{0,z}^{\alpha,\alpha,\eta} f(z) = D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left\{ \int_0^z \frac{f(z) d\zeta}{(z-\zeta)^{\alpha}} \right\} \tag{1.6}$$

and

$$J_{0,z}^{m+\alpha,\alpha,\eta} f(z) = D_z^{m+\alpha} f(z) = \frac{d^m}{dz^m} \left(D_z^{\alpha} f(z) \right), \tag{1.7}$$

where $D_z^{\alpha} f(z)$ and $D_z^{m+\alpha} f(z)$ are fractional derivatives of $f(z)$ defined by Owa [3].

A function $f(z) \in A$ is said to be starlike in U if it satisfies

$$\Re \left(\frac{z f'(z)}{f(z)} \right) > 0 \qquad (z \in U). \tag{1.8}$$

We denote by S^* the subclass of A consisting of all starlike functions in U. A function $f(z) \in A$ is said to be convex in U if it satisfies

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \qquad (z\in U). \tag{1.9}$$

We also denote by K the subclass of A consisting of functions $f(z)$ which are convex in U. Note that $f(z) \in K$ if and only if $zf'(z) \in S^*$.

It is well-known that:

(i) if $f(z) \in S^*$, then $|a_k| \leq k$ $(k = 2,3,4,\ldots)$, see e.g. [1],

and

(ii) if $f(z) \in K$, then $|a_k| \leq 1$ $(k = 2,3,4,\ldots)$, see e.g. [4].

Now, let $A(n,\delta)$ be the subclass of A consisting of all functions $f(z)$ which satisfy

$$|a_k| \leq k^{n+\delta} \tag{1.10}$$

for some $n = 0,1,2,\ldots$, and for some $0 \leq \delta \leq 1$. Then we see that $S^* \subset A(1,0)$ and $K \subset A(0,0)$.

2. Inequalities

Let ${}_pF_q(z)$ be the generalized hypergeometric function defined by (for all details, see [2])

$${}_pF_q(z) \equiv {}_pF_q\left(\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p\\ \beta_1,\beta_2,\ldots,\beta_q\end{matrix};z\right) = \sum_{k=0}^{\infty}\left(\frac{\prod_{j=1}^{p}(\alpha_j)_k}{\prod_{j=1}^{q}(\beta_j)_k}\right)\frac{z^k}{(1)_k}, \tag{2.1}$$

where $(\alpha_j)_k$ means the Pochhammer symbol defined by

$$(\alpha_j)_k = \begin{cases} 1 & (k=0) \\ \alpha_j(\alpha_j+1)\ldots(\alpha_j+k-1) & (k=1,2,3,\ldots). \end{cases} \tag{2.2}$$

In order to derive our inequalities for Saigo's fractional calculus operators, we need the following lemma due to Srivastava, Saigo and Owa [7].

Lemma 2.1. *Let $\alpha > 0, \beta$ and η be real. Then, for $k > \max\{0, \beta-\eta\} - 1$,*

$$I_{0,z}^{\alpha,\beta,\eta} z^{\beta} = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} z^{k-\beta}. \tag{2.3}$$

We also have

Lemma 2.2. *Let $0 \leq \alpha < 1, \beta$ and η be real. Then, for $k > \max\{0, \beta-\eta\} - 1$,*

$$J_{0,z}^{\alpha,\beta,\eta} z^{k} = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k-\alpha+\eta+1)} z^{k-\beta} \tag{2.4}$$

$$J_{0,z}^{m+\alpha,\beta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta-m+1)\Gamma(k-\alpha+\eta+1)} z^{k-\beta-m} \quad (m=0,1,2,\ldots). \tag{2.5}$$

Our first inequality for $I_{0,z}^{\alpha,\beta,\eta} f(z)$ is contained in the following theorem.

Theorem 2.1. *Let $\alpha > 0, \beta$ and η be real, and let $2-\beta > 0, 2-\beta+\eta > 0$, and $2+\alpha+\eta > 0$. If $f(z) \in A(n,\delta)$, then*

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} |z|^{1-\beta} {}_{n+3}F_{n+2}\left(\begin{matrix} 2,\ldots,2,2-\beta+\eta \\ 1,\ldots,1,2-\beta,2+\alpha+\eta \end{matrix} ; |z| \right) \tag{2.6}$$

for $0 < |z| < 1$.

P r o o f. Applying Lemma 2.1 for $f(z) \in A(n,\delta)$, we have

$$\begin{aligned} |I_{0,z}^{\alpha,\beta,\eta} f(z)| &= \left|\sum_{k=1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} a_k z^{k-\beta}\right| \qquad (a_1 = 1) \qquad (2.7) \\ &\leq \sum_{k=1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} k^{n+1} |z|^{k-\beta} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k+2)\Gamma(k-\beta+\eta+2)}{\Gamma(k-\beta+2)\Gamma(k+\alpha+\eta+2)} (k+1)^{n+1} |z|^{k+1-\beta}. \end{aligned}$$

Since

$$\Gamma(k+\gamma) = \Gamma(\gamma)(\gamma)_k \quad (\gamma > 0) \qquad \text{and} \qquad k+1 = \frac{(2)_k}{(1)_k},$$

we obtain that

$$\begin{aligned} |I_{0,z}^{\alpha,\beta,\eta} f(z)| &\leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} |z|^{1-\beta} \left(\sum_{k=0}^{\infty} \frac{((2)_k)^{n+2}(2-\beta+\eta)_k}{((1)_k)^n (2-\beta)_k (2+\alpha+\eta)_k} \frac{|z|^k}{(1)_k} \right) \qquad (2.8) \\ &= \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} |z|^{1-\beta} {}_{n+3}F_{n+2}\left(\begin{matrix} 2,\ldots,2,2-\beta+\eta \\ 1,\ldots,1,2-\beta,2+\alpha+\eta \end{matrix} ; |z| \right) \end{aligned}$$

which completes the proof of the theorem. ∎

If we take $\beta = -\alpha$ in Theorem 2.1, then we have

Corollary 2.1. *If $f(z) \in A(n,\delta)$, then*

$$|D_z^{-\alpha} f(z)| \leq \frac{1}{\Gamma(2+\alpha)} |z|^{1+\alpha} {}_{n+2}F_{n+1}\left(\begin{matrix} 2,\ldots,2 \\ 1,\ldots,1,2+\alpha \end{matrix} ; |z| \right) \tag{2.9}$$

for $0 < |z| < 1$ *and* $\alpha > 0$.

Taking special values of n and δ in Theorem 2.1, we derive the following corollary.

Corollary 2.2. *Let $\alpha > 0, \beta$ and η be real, and let $2-\beta > 0, 2-\beta+\eta > 0$, and $2+\alpha+\eta > 0$. If $f(z) \in A(1,0)$, then*

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} |z|^{1-\beta} {}_4F_3\left(\begin{matrix} 2,2,2,2-\beta+\eta \\ 1,2-\beta,2+\alpha+\eta \end{matrix} ; |z| \right) \tag{2.10}$$

for $0 < |z| < 1$. If $f(z) \in A(0,0)$, then

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} |z|^{1-\beta} {}_3F_2\left(\begin{matrix} 2,2,2-\beta+\eta \\ 2-\beta,2+\alpha+\eta \end{matrix} ; |z| \right). \tag{2.11}$$

Similarly, for Saigo's fractional derivative operator $J_{0,z}^{\alpha,\beta,\eta} f(z)$ of $f(z)$, we have

Theorem 2.2. *Let $0 \leq \alpha < 1, \beta$ and η be real, and let $2-\beta-m > 0$, $2-\beta+\eta > 0$, $2-\alpha+\eta > 0$, and $m = 0,1,2,\dots$. If $f(z) \in A(n,\delta)$, then*

$$|J_{0,z}^{m+\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta-m)\Gamma(2-\alpha+\eta)} \times |z|^{1-\beta-m} {}_{n+3}F_{n+2}\left(\begin{matrix} 2,\dots,2,2-\beta+\eta \\ 1,\dots,1,2-\beta-m,2-\alpha+\eta \end{matrix} ; |z| \right) \tag{2.12}$$

for $0 < |z| < 1$.

P r o o f. Since Lemma 2.2 implies that

$$J_{0,z}^{m+\alpha,\beta,\eta} f(z) = \sum_{k=1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta-m+1)\Gamma(k-\alpha+\eta+1)} a_k z^{k-\beta-m}, \tag{2.13}$$

we easily see the inequality (2.12). ■

If we put $\beta = \alpha$ in Theorem 2.2, then we have

Corollary 2.3. *If $f(z) \in A(n,\delta)$, then*

$$|D_z^{\alpha} f(z)| \leq \frac{1}{\Gamma(2-\alpha-m)} |z|^{1-\alpha-m} {}_{n+2}F_{n+1}\left(\begin{matrix} 2,\dots,2 \\ 1,\dots,1,2-\alpha-m \end{matrix} ; |z| \right) \tag{2.14}$$

for $0 < |z| < 1$, $0 \leq \alpha < 1$ and $m = 0,1$.

Taking special values for n and δ, we have

Corollary 2.4. *Let $0 \leq \alpha < 1, \beta$ and η be real, and let $2-\beta-m > 0, 2-\beta+\eta > 0$, $2-\alpha+\eta > 0$ and $m = 0,1,2,\dots$. If $f(z) \in A(1,0)$, then*

$$|J_{0,z}^{m+\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta-m)\Gamma(2-\alpha+\eta)} |z|^{1-\beta-m} {}_4F_3\left(\begin{matrix} 2,2,2,2-\beta+\eta \\ 1,2-\beta-m,2-\alpha+\eta \end{matrix} ; |z| \right) \tag{2.15}$$

for $0<|z|<1$. *If* $f(z)\in A(0,0)$, *then*

$$|J_{0,z}^{m+\alpha,\beta,\eta}f(z)|\leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta-m)\Gamma(2-\alpha+\eta)}|z|^{1-\beta-m}\,{}_3F_2\left(\begin{matrix}2,2,2-\beta+\eta\\ 2-\beta-m,2-\alpha+\eta\end{matrix};|z|\right). \tag{2.16}$$

3. Appendix

We introduce the following formula for the generalized hypergeometric functions ([2]).

Lemma 3.1. (Srivastava [6]) *If* m *is a positive integer, then*

$$ {}_pF_q\left(\begin{matrix}\beta_1+m,\alpha_2,\dots,\alpha_p\\ \beta_1,\beta_2,\dots,\beta_q\end{matrix};z\right)=\sum_{j=0}^{m}\binom{m}{j}\left(\frac{\prod_{s=2}^{p}(\alpha_s)_j}{\prod_{s=1}^{q}(\beta_s)_j}\right)z^j\,{}_{p-1}F_{q-1}\left(\begin{matrix}\alpha_2+j,\dots,\alpha_p+j\\ \beta_2+j,\dots,\beta_q+j\end{matrix};z\right) \tag{3.1}$$

In view of Lemma 3.1, for ${}_{n+3}F_{n+2}$ in Theorem 2.1, we see that

$$\begin{aligned}
{}_{n+3}F_{n+2}&\left(\begin{matrix}2,\dots,2,2-\beta+\eta\\ 1,\dots,1,2-\beta,2+\alpha+\eta\end{matrix};|z|\right)=\sum_{j=0}^{1}\binom{1}{j}\frac{((2)_j)^{n+1}(2-\beta+\eta)_j}{((1)_j)^n(2-\beta)_j(2+\alpha+\eta)_j} \qquad (3.2)\\
&\times|z|^j\,{}_{n+2}F_{n+1}\left(\begin{matrix}2+j,\dots,2+j,2-\beta+\eta+j\\ 1+j,\dots,1+j,2-\beta+j,2+\alpha+\eta+j\end{matrix};|z|\right)\\
&={}_{n+2}F_{n+1}\left(\begin{matrix}2,\dots,2,2-\beta+\eta\\ 1,\dots,1,2-\beta,2+\alpha+\eta\end{matrix};|z|\right)\\
&\quad+\frac{2^{n+1}(2-\beta+\eta)}{(2-\beta)(2+\alpha+\eta)}|z|\,{}_{n+2}F_{n+1}\left(\begin{matrix}3,\dots,3,3-\beta+\eta\\ 2,\dots,2,3-\beta,3+\alpha+\eta\end{matrix};|z|\right)\\
&={}_{n+1}F_n\left(\begin{matrix}2,\dots,2,2-\beta+\eta\\ 1,\dots,1,2-\beta,2+\alpha+\eta\end{matrix};|z|\right)\\
&\quad+\frac{3\cdot2^{n}(2-\beta+\eta)}{(2-\beta)(2+\alpha+\eta)}|z|\,{}_{n+1}F_{n}\left(\begin{matrix}3,\dots,3,3-\beta+\eta\\ 2,\dots,2,3-\beta,3+\alpha+\eta\end{matrix};|z|\right)\\
&+\frac{4\cdot3^{n}(2-\beta+\eta)(3-\beta+\eta)}{(2-\beta)(3-\beta)(2+\alpha+\eta)(3+\alpha+\eta)}|z|^2\,{}_{n+1}F_{n}\left(\begin{matrix}4,\dots,4,4-\beta+\eta\\ 3,\dots,3,4-\beta,4+\alpha+\eta\end{matrix};|z|\right).\\
&=\dots
\end{aligned}$$

Therefore, Corollary 2.2 can be written as:

Corollary 3.1. *Let $\alpha > 0, \beta$ and η be real, and let $2-\beta > 0, 2-\beta+\eta > 0$, and $2+\alpha+\eta > 0$. If $f(z) \in A(1,0)$, then*

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}\, {}_3F_2\left(\begin{matrix} 2,2,2-\beta+\eta \\ 2-\beta, 2+\alpha+\eta \end{matrix} ; |z| \right) + \frac{4\Gamma(2-\beta+\eta)}{\Gamma(3-\beta)\Gamma(3+\alpha+\eta)} |z| {}_3F_2\left(\begin{matrix} 3,3,3-\beta+\eta \\ 3-\beta, 3+\alpha+\eta \end{matrix} ; |z| \right) \tag{3.3}$$

for $0 < |z| < 1$. Furthermore, let $\alpha > 0, \beta < 2$, and η be a positive integer. If $f(z) \in A(1,0)$, then

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} |z|^{1-\beta} \times \left\{ \sum_{j=0}^{\eta} \binom{\eta}{j} \left\{ \frac{((2)_j)^2}{(2-\beta)_j (2+\alpha+\eta)_j} {}_2F_1\left(\begin{matrix} 2+\eta, 2+\eta \\ 2+\alpha+2\eta \end{matrix} ; |z| \right) + \frac{((2)_{j+1})^2 (2-\beta+\eta)}{(2-\beta)_{j+1}(2+\alpha+\eta)_{j+1}} |z| {}_2F_1\left(\begin{matrix} 3+\eta, 3+\eta \\ 3+\alpha+2\eta \end{matrix} ; |z| \right) \right\} |z|^j \right\} \tag{3.4}$$

for $0 < |z| < 1$. If $f(z) \in A(0,0)$, then

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} |z|^{1-\beta} \times \left\{ \sum_{j=0}^{\eta} \binom{\eta}{j} \frac{((2)_j)^2}{(2+\alpha+\eta)_j} |z|^j {}_2F_1\left(\begin{matrix} 2+\eta, 2+\eta \\ 2+\alpha+2\eta \end{matrix} ; |z| \right) \right\} \tag{3.5}$$

for $0 < |z| < 1$.

We also see from Corollary 2.4 that

Corollary 3.2. *Let $0 \leq \alpha < 1, \beta$ and η be real, and let $2-\beta-m > 0, 2-\beta+\eta > 0$, $2-\alpha+\eta > 0$ and $m = 0,1,2,\ldots$. If $f(z) \in A(1,0)$, then*

$$|J_{0,z}^{m+\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta-m)\Gamma(2-\alpha+\eta)} |z|^{1-\beta-m} {}_3F_2\left(\begin{matrix} 2,2,2-\beta+\eta \\ 2-\beta-m, 2-\alpha+\eta \end{matrix} ; |z| \right) + \frac{4\Gamma(3-\beta+\eta)}{\Gamma(3-\beta-m)\Gamma(3-\alpha+\eta)} |z|^{2-\beta-m} {}_3F_2\left(\begin{matrix} 3,3,3-\beta+\eta \\ 3-\beta-m, 3-\alpha+\eta \end{matrix} ; |z| \right) \tag{3.6}$$

for $0 < |z| < 1$. Furthermore, let $0 \leq \alpha < 1$ and $\beta < 2$ be real, and η be a positive integer, and let $2-\beta-m > 0$ and $m = 0,1,2,\ldots$. If $f(z) \in A(1,0)$, then

$$|J_{0,z}^{m+\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta-m)\Gamma(2-\alpha+\eta)} |z|^{1-\beta-m}$$

(3.7)

$$\times \left\{ \sum_{j=0}^{\eta+m} \binom{\eta+m}{j} \left\{ \frac{((2)_j)^2}{(2-\beta-m)_j(2-\alpha+\eta)_j} {}_2F_1 \left(\begin{matrix} 2+\eta+m, 2+\eta+m \\ 2-\alpha+2\eta+m \end{matrix} ; |z| \right) \right.\right.$$

$$\left.\left. + \frac{((2)_{j+1})^2(2-\beta+\eta)}{(2-\beta-m)_{j+1}(2-\alpha+\eta)_{j+1}} |z| {}_2F_1 \left(\begin{matrix} 3+\eta+m, 3+\eta+m \\ 3-\alpha+2\eta+m \end{matrix} ; |z| \right) \right\} |z|^j \right\}$$

for $0 < |z| < 1$. *If* $f(z) \in A(0,0)$, *then*

$$|J_{0,z}^{m+\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta-m)\Gamma(2-\alpha+\eta)} |z|^{1-\beta-m} \tag{3.8}$$

$$\times \left\{ \sum_{j=0}^{\eta+m} \binom{\eta+m}{j} \frac{((2)_j)^2}{(2-\beta-m)_j(2-\alpha+\eta)_j} |z|^j {}_2F_1 \left(\begin{matrix} 2+\eta+m, 2+\eta+m \\ 2-\alpha+2\eta+m \end{matrix} ; |z| \right) \right\}$$

for $0 < |z| < 1$.

Further, we consider the case of $\beta = -\alpha$ in Corollary 2.2.

Corollary 3.3. *Let* $\alpha > 0$. *If* $f(z) \in A(1,0)$, *then*

$$|D_z^{-\alpha} f(z)| \leq \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)(1-|z|)^{3-\alpha}}$$

$$\times \left\{ (1-|z|) {}_2F_1 \left(\begin{matrix} \alpha, \alpha \\ 2+\alpha \end{matrix} ; |z| \right) + \frac{4|z|}{2+\alpha} {}_2F_1 \left(\begin{matrix} \alpha, \alpha \\ 3+\alpha \end{matrix} ; |z| \right) \right\} \tag{3.9}$$

for $z \in U$. *If* $f(z) \in A(0,0)$, *then*

$$|D_z^{-\alpha} f(z)| \leq \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)} {}_2F_1 \left(\begin{matrix} \alpha, \alpha \\ 2+\alpha \end{matrix} ; |z| \right) \tag{3.10}$$

for $z \in U$.

P r o o f. Applying Lemma 3.1 and the formula (see [2])

$${}_2F_1 \left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} ; z \right) = (1-z)^{\gamma-\alpha-\beta} {}_2F_1 \left(\begin{matrix} \gamma-\alpha, \gamma-\beta \\ \gamma \end{matrix} ; z \right) \tag{3.11}$$

we see that

$$|D_z^{-\alpha} f(z)| \leq \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)} {}_3F_2 \left(\begin{matrix} 2,2,2 \\ 1, 2+\alpha \end{matrix} ; |z| \right) \tag{3.12}$$

$$= \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)} \left\{ \sum_{j=0}^{1} \binom{1}{j} \frac{(2)_j (2)_j}{(1)_j (2+\alpha)_j} |z|^j {}_2F_1 \left(\begin{matrix} 2+j, 2+j \\ 2+\alpha+j \end{matrix} ; |z| \right) \right\}$$

$$= \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)} \left\{ \sum_{j=0}^{1} \binom{1}{j} \frac{(2)_j (2)_j}{(1)_j (2+\alpha)_j} \frac{|z|^j}{(1-|z|)^{2+j-\alpha}} {}_2F_1\left(\begin{matrix} \alpha, \alpha \\ 2+\alpha+j \end{matrix} ; |z| \right) \right\}$$

$$= \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)(1-|z|^{3-\alpha})} \left\{ (1-|z|)\, {}_2F_1\left(\begin{matrix} \alpha, \alpha \\ 2+\alpha \end{matrix} ; |z| \right) + \frac{4|z|}{2+\alpha} {}_2F_1\left(\begin{matrix} \alpha, \alpha \\ 3+\alpha \end{matrix} ; |z| \right) \right\}$$

for $z \in U$. ∎

Letting $\beta = \alpha$ in Corollary 2.4, we have

Corollary 3.4. *let* $0 \leq \alpha < 1$, $m = 0, 1$ *and* $\alpha + m < 2$. *If* $f(z) \in A(1,0)$, *then*

$$|D_z^{m+\alpha} f(z)| \leq \frac{|z|^{1-\alpha-m}}{\Gamma(2-\alpha-m)(1-|z|)^{3+\alpha+m}}$$

$$\times \left\{ (1-|z|)\, {}_2F_1\left(\begin{matrix} -\alpha-m, -\alpha-m \\ 2-\alpha-m \end{matrix} ; |z| \right) + \frac{4|z|}{2-\alpha-m} {}_2F_1\left(\begin{matrix} -\alpha-m, -\alpha-m \\ 3-\alpha-m \end{matrix} ; |z| \right) \right\} \tag{3.13}$$

for $0 < |z| < 1$. *If* $f(z) \in A(0,0)$, *then*

$$|D_z^{m+\alpha} f(z)| \leq \frac{|z|^{1-\alpha-m}}{\Gamma(2-\alpha-m)} {}_2F_1\left(\begin{matrix} 2, 2 \\ 2-\alpha-m \end{matrix} ; |z| \right) \tag{3.14}$$

for $0 < |z| < 1$.

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