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AUTOMORPHIC GREEN FUNCTIONS ON ARITHMETIC QUOTIENTS OF TYPE IV SYMMETRIC DOMAIN

MASAO TSUZUKI
(都築正男, 上智大学理工学部)

1. Introduction

This article is a short summary of the forthcoming paper:

'Automorphic Green functions associated with the secondary spherical functions'
(Takayuki Oda and Masao Tsuzuki)

Let $G := O_0(n,2)$ be the identity component of the orthogonal group with signature $(n+,2-)$ and $K := G \cap \operatorname{diag}(O(n), O(2))$ a maximal compact subgroup of G. The Lie algebra $\mathfrak{g} := \operatorname{Lie}(G)$ is identified with the space of matrices $X \in \operatorname{Mat}_{n+2}(\mathbb{R})$ satisfying ${}^tXI_{n,2}+I_{n,2}X = O$ with the bracket product $[X,Y] = XY-YX$. Let E_{ij} $(1 \leqslant i,j \leqslant n+2)$ be the usual matrix unit of $\operatorname{Mat}_{n+2}(\mathbb{R})$.

The homogenous manifold G/K is a symmetric space of type IV, which is a Hermitian symmetric domain with the G-invariant complex structure coming from the adjoint action $J := \operatorname{ad}(\tilde{Z}_0)|\mathfrak{p}$ with $\tilde{Z}_0 := E_{n+1,n+2} - E_{n+2,n+1} \in \mathfrak{k} := \operatorname{Lie}(K)$ on $\mathfrak{p}$, the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$. The K-invarinat alternating form $\tilde{\omega}(X,Y) := (8n)^{-1}B(X, J(Y))$ on $\mathfrak{p}$ is uniquely extended to a G-invariant C^∞ differential from ω of $(1,1)$ type on G/K, by which G/K is a Kähler manifold.

Any arithmetic subgroup Γ of G acts discontinuously on G/K through bi-holomrphic automorphisms of G/K. When Γ is neat, taking the quotient by Γ we have a Kähler manifold $\Gamma\backslash G/K$ with Kähler form $\omega_{\Gamma\backslash G/K}$ such that the quotient map $\pi : G/K \to \Gamma\backslash G/K$ is holomorphic and $\pi^*\omega_{\Gamma\backslash G/K} = \omega$.

Consider the symmetric subgroup $H = O_0(n-1,2)$ consisting of fixed points of the involution σ of G defined by $\sigma(g) = SgS$ with $S := \operatorname{diag}(E_{n-1}, -1, E_2)$. We assume that H is 'Γ-rational' in a proper sence. In particular the invarinat volume of $\Gamma_H\backslash H/K_H$ is finite, where $\Gamma_H := \Gamma \cap H$ and $K_H := H \cap K$. Let D be the image of the natural holomorphic map $\Gamma_H\backslash H/K_H \to \Gamma\backslash G/K$. Then D is a closed complex analytic subset of $\Gamma\backslash G/K$ with complex codimension 1, which defines a closed current δ_D by integration

$$\langle \delta_D, \alpha \rangle = \int_{D_{\mathrm{ns}}} j^*\alpha, \quad \alpha \in A_{\mathrm{c}}(\Gamma\backslash G/K).$$

Here D_{ns} denotes the smooth locus of D and $A_{\mathrm{c}}(M)$ denotes the space of compactly supported smooth forms on a complex manifold M.

Then our aim here is to explain an explicit construction of the Green current for D following [1]. Though a similar construction for the 'unitary case' (i.e., for the modular divisors in an arithmetic quotient of a complex-hyperball) is proved to be possible, we focus only on the 'orthogonal case' setting aside the 'unitary case' for simplicity of presentation.

2. Secondary spherical functions

Let $\mathfrak{a}$ be the maximal abelian subspace $\mathbb{R}Y_0$ of $\mathfrak{p}\cap\mathfrak{q}$ with the basis $Y_0 = E_{n,n+1} + E_{n+1,n}$. Here $\mathfrak{q}$ is the orthogonal complement of $\mathfrak{h} := \mathrm{Lie}(H)$. Then the group G is a union of the double cosets Ha_tK $(t \geqslant 0)$ with $a_t := \exp(tY_0)$. We introduce two functions $\phi_s^{(2)}$ and ψ_s with singularities on G.

2.0.1. *The function* $\phi_s^{(2)}$. There exists a unique family of functions $\phi_s^{(2)}$ $(\mathrm{Re}(s) > n/2)$ such that

- $\phi_s^{(2)}$ is a C^∞-function on $G - HK$ and (H,K)-invarinat, i.e.,

$$\phi_s^{(2)}(hgk) = \phi_s^{(2)}(g) \quad \forall h \in H, \forall g \in G - HK, \forall k \in K.$$

- $\phi_s^{(2)}$ satisfies the differential equation

$$\Omega\phi_s^{(2)}(g) = (s^2 - (n/2)^2)\,\phi_s^{(2)}(g), \quad g \in G - HK.$$

- There exists a poistive δ such that $\phi_s^{(2)}(\exp(tY_0)) - \log(t)$ is bounded on the interval $(0,\delta)$.
- $\phi_s^{(2)}(a_t)$ decays exponentially as t getting large:

$$\phi_s^{(2)}(a_t) = O(e^{-(\mathrm{Re}(s)+n/2)t}) \quad (t \to +\infty).$$

([1, Proposition 2.4.2]).

We have the explicit formula:

$$\phi_s^{(2)}(a_t) = -\frac{1}{2}\frac{\Gamma((s+n/2)/2)\,\Gamma((s-n/2)/2+1)}{\Gamma(s+1)} \times (\cosh t)^{-(s+n/2)}{}_2F_1\left(\frac{s+n/2}{2}, \frac{s-n/2}{2}+1\,; s+1\,; \frac{1}{\cosh^2}\right), \quad (t>0).$$

([1, 2.5]).

2.0.2. *The function* ψ_s. Let $\mathfrak{p}_\pm$ be the $\pm\sqrt{-1}$-eigen space of the complex linear extension of J to $\mathfrak{p}_\mathbb{C}$. Then $\mathfrak{p}_+ = \sum_{i=0}^{n-1}\mathbb{C}X_i$ and $\mathfrak{p}_- = \sum_{i=0}^{n-1}\mathbb{C}\bar{X}_i$ with

$$X_0 = E_{n,n+1} + E_{n+1,n} + \sqrt{-1}(E_{n,n+2} + E_{n+2,n}),$$
$$X_i = E_{i,n+1} + E_{n+1,i} + \sqrt{-1}(E_{i,n+2} + E_{n+2,i}), \quad 1 \leqslant i \leqslant n-1.$$

Let $\{\omega_i\}$ and $\{\bar\omega_i\}$ be the dual basis of $\{X_i\}$ and $\{\bar X_i\}$ respectively. Put

$$\mathsf{v}_{11} := \frac{1}{4}\left(\sum_{i=1}^{n-1}\omega_i\wedge\bar\omega_i - (n-1)\omega_0\wedge\bar\omega_0\right) \; (\in \mathfrak{p}_+^*\wedge\mathfrak{p}_-^*)$$

Then $(\mathfrak{p}_+^*\wedge\mathfrak{p}_-^*)^M$ is a two dimensional space generated by v_{11} and the Kähler form $\tilde\omega = \frac{\sqrt{-1}}{2}\sum_{i=0}^{n-1}\omega_i\wedge\bar\omega_i$. For $\mathrm{Re}(s) > n/s$, put

$$\psi_s(g) = \frac{1}{4}\sum_{i,j=0}^{n-1} R_{X_i\bar X_j}\phi_s^{(2)}(g)\,\omega_i\wedge\bar\omega_j \quad g \in G - HK.$$

Here are some properties of the function ψ_s.

- ψ_s is a C^∞-function on $G - HK$ such that

$$\psi_s(hgk) = (\mathrm{Ad}^*_{\mathfrak{p}_+} \wedge \mathrm{Ad}^*_{\mathfrak{p}_-})(k)^{-1}\psi_s(g), \quad \forall h \in H, \forall g \in G - HK, \forall k \in K.$$

Here $\mathrm{Ad}^*_{\mathfrak{p}+\pm}$ be the coadjoint representation of K on $\mathfrak{p}^*_\pm$.

- We have $\psi_s(a_t) = f_s(t)\,\mathrm{v}_{11}$ with

$$f_s(t) = \left(\mathrm{tanh}t\frac{d}{dt} - \frac{s^2-(n/2)^2}{n}\right)\phi_s^{(2)}(a_t), \quad t > 0.$$

- There exists a positive δ such that $f_s(t) + \frac{s^2-(n/2)^2}{2n}\log t$ is bounded on the interval $(0,\delta)$.
- We have the estimation:

$$f_s(t) \prec e^{-(\mathrm{Re}(s)+n/2)t}, \quad t \in [1,\infty).$$

3. Currents defined by Poincare series

Let Γ be as in the introduction. For $\alpha \in A(\Gamma\backslash G/K)$, we have a unique C^∞-function $\tilde{\alpha} : G \to \bigwedge \mathfrak{p}^*_{\mathbb{C}}$ such that $\tilde{\alpha}(\gamma g k) = \tau(k)^{-1}\tilde{\alpha}(g)$, $(\gamma \in \Gamma,\ k \in K)$ and such that

$$(1) \qquad \langle(\pi^*\alpha)(gK), dL_g(\xi_o)\rangle = \langle\tilde{\alpha}(g), \xi_o\rangle, \quad g \in G,\ \xi_o \in \bigwedge \mathfrak{p} = \bigwedge \mathrm{T}_o(G/K)$$

holds. Here L_g denotes the left translation on G/K by the element g and we identify $\mathfrak{p}$ with $\mathrm{T}_o(G/K)$, the tangent space of G/K at $o = eK$. Let dk (resp. dk_0) be the normalized Haar measure of K (resp. K_H) with total volume 1. Then there exists a Haar measure dg (resp. dh) of G (resp. H) such that $\frac{dg}{dk}$ (resp. $\frac{dh}{dk_0}$) corresponds to the measure of the symmetric space G/K (resp. H/K_H) determined by the invarinat volume form associated to the Kähler form.

For any left Γ-invariant function f on G, put

$$\mathfrak{I}_H(f\,;g) = \int_{\Gamma_H\backslash H} f(hg)\,d\dot{h}, \quad g \in G.$$

Let $\varphi_s = \phi_s^{(2)}$ $(\mathrm{Re}(s) > n/2)$ or ψ_s $(\mathrm{Re}(s) > n/2)$. Then the integral

$$\int_{\Gamma\backslash G}\left(\sum_{\gamma\in\Gamma_H\backslash\Gamma} \|\varphi_s(\gamma g)\|\right)d\dot{g}$$

is locally bounded in $\mathrm{Re}(s) > n/2$ ([1, Proposition3.1.1]), and there exists a unique current $P(\varphi_s)$ on $\Gamma\backslash G/K$ such that

$$\begin{aligned}\langle P(\varphi_s), *\bar{\alpha}\rangle &= \int_{\Gamma\backslash G}\Big(\sum_{\gamma\in\Gamma_H\backslash\Gamma}\varphi_s(\gamma g)|\tilde{\alpha}(g)\Big)\,d\dot{g}\\ &= \frac{\pi}{2}\int_0^\infty (\varphi_s(a_t)|\mathfrak{I}_H(\tilde{\alpha}\,;a_t))\,\mathrm{sinh}t\,(\mathrm{cosh}t)^{n-1}\,dt, \quad \forall\alpha \in A_c(\Gamma\backslash G/K)\end{aligned}$$

Here $(\cdot|\cdot)$ is the Hermitian inner product of $\mathfrak{p}^*_{\mathbb{C}}$ canonically induced by the inner product $(8n)^{-1}B(X,Y)$ on $\mathfrak{p}$.

We have the current $G_s := P(\phi_s^{(2)})$ of $(0,0)$-type and the one $\Psi_s := P(\psi_s)$ of $(1,1)$-type on $\Gamma\backslash G/K$ which depends holomorphically on $\mathrm{Re}(s) > n/2$.

4. Differential equations

Let $\mathrm{Re}(s) > n/2$. Then the currents G_s and Ψ_s satisfy the differential equations:

$$\triangle G_s = -((2s)^2 - n^2)\, G_s - 2\pi\, \Lambda\delta_D,$$

$$\triangle \Psi_s = -((2s)^2 - n^2)\Big(\Psi_s - \frac{\pi\sqrt{-1}}{4}\delta_D - \frac{\pi\sqrt{-1}}{4n}L\Lambda\,\delta_D\Big),$$

$$\partial\bar{\partial}G_s + \pi\sqrt{-1}\delta_D = \frac{\sqrt{-1}}{2n}((2s)^2 - n^2)L\,G_s + 4\Psi_s.$$

Here Λ is the adjoint of the Lefschets operator $L\alpha = \omega_{\Gamma\backslash G/K} \wedge \alpha$ ([1, Theorem 7.6.1]).

5. Meromorphicity

Suppose $\Gamma\backslash G$ is compact. Let $\{\lambda_m\}_{m\in\mathbb{N}}$ be the increasing sequence of the eigenvalues of the negative of the Casimir operator $-R_\Omega$ acting on $L^2(\Gamma\backslash G/K)$ such that each eigenvalue occurs with its multiplicity. We fix an orthonormal basis $\{\varphi_m\}_{m\in\mathbb{N}}$ of $L^2(\Gamma\backslash G/K)$ consisiting of automorphic forms on $\Gamma\backslash G/K$ such that $-R_\Omega\varphi_m = \lambda_m\,\varphi_m\;(\forall m \in \mathbb{N})$. Then we have the spectral expansion of G_s ($\mathrm{Re}(s) > n/2$):

$$\langle G_s, *\bar{\alpha}\rangle = \sum_{m=0}^{\infty} \frac{\mathfrak{I}_H(\bar{\varphi}_m\,; e)}{(n/2)^2 - \lambda_m - s^2}\,\langle\varphi_m|\bar{\alpha}\rangle_{L^2}, \quad \alpha \in A_c(\Gamma\backslash G/K). \tag{2}$$

Here $\langle\cdot|\cdot\rangle_{L^2}$ is the L^2-inner product of $L^2(\Gamma\backslash G/K)$. The corresponding result for the 'unitary case' is proved in [1, Proposition 6.2.2]. The proof for the present case is pararell since we assume $\Gamma\backslash G$ is compact. Then by an estimation similar to that in [1, Theorem 6.2.1 (1)], the series (2) is absolutely convergent for an arbitrary $s \in \{s \in \mathbb{C}|\, s^2 \neq (n/2)^2 - \lambda_m\,(\forall m)\}$ locally uniformly. Hence the current $s \mapsto G_s$, which is originally holomorphic only on $\mathrm{Re}(s) > n/2$, has a meromorphic continuation to the whole s-plane with possible simple poles at the points $s \in \mathbb{C}$ such that $s^2 = (n/2)^2 - \lambda_m\,(\exists m)$.

6. Green current

The point $s = n/2$ is a simple pole of G_s with the residue

$$\mathrm{Res}_{s=n/2}G_s = -\frac{1}{n}\frac{\mathrm{vol}(\Gamma_H\backslash H)}{\mathrm{vol}(\Gamma\backslash G)},$$

a constant function on $\Gamma\backslash G/K$.

Definition

We put $\mathcal{G}$ to be $(-2\pi)^{-1}$ times the constant term of the Laurent expansion of G_s at $s = n/2$, i.e.,

$$\mathcal{G}(x) = \frac{-1}{2\pi}\lim_{s\to n/2}\Big(G_s(x) - \frac{\kappa}{s - n/2}\Big)$$

with $\kappa = -\frac{1}{n}\frac{\mathrm{vol}(\Gamma_H\backslash H)}{\mathrm{vol}(\Gamma\backslash G)}$.

Theorem

- The current-valued function $s \mapsto \Psi_s$ on $\mathrm{Re}(s) > n/2$ has a meromorphic continuation to the whole s-plane. The point $s = n/2$ is a regular point of the meromorphic function Ψ_s and the value $\Psi_{n/2}$ is harmonic, i.e.,

$$\triangle \Psi_{n/2} = 0.$$

- The current $\mathcal{G}$ satisfies Green's equation:

$$\mathrm{dd^c}\mathcal{G} + \delta_D = \frac{1}{\pi}(\kappa\, \omega_{\Gamma\backslash G/K} + 4\Psi_{n/2}).$$

REFERENCES

[1] Oda, T., Tsuzuki, M.,*Automorphic Green functions associated with the secondary spherical functions*, to appear in Publication RIMS.

Masao TSUZUKI
Department of Mathematics
Sophia University, Kioi-cho 7-1 Chiyoda-ku Tokyo, 102-8554, Japan
E-mail : `tsuzuki@mm.sophia.ac.jp`