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INDUCTION THEORY OF EQUIVARIANT-SURGERY-OBSTRUCTION GROUPS

(同変手術障害類群の誘導理論)

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Abstract

In the present article, we recall the definitions of the Hermitian-representation ring $\mathrm{G}_1(R,G)$, the Grothendieck-Witt rings $\mathrm{GW}(G,R)$ and $\mathrm{GW}_0(R,G)$, the Wall groups $\mathrm{L}_n^h(R[G],w)$, and the Bak groups $\mathrm{L}_n^h(R[G],\Lambda,w)$ of a finite group G, and we discuss induction theory concerned with these rings and groups using the notion of w-Mackey functor.

1. INTRODUCTION

Throughout this article, let G be a finite group.

After works on surgery by J. Milnor, S. P. Novikov, W. Browder, and etc., C. T. C. Wall [18], [19] formulated the surgery-obstruction groups $\mathrm{L}_n^h(\mathbb{Z}[G],w)$ using quadratic modules and automorphisms. In the case where the orientation homomorphism w is trivial, C. B. Thomas [17, Theorems 1, 3] in 1971 proved that $\mathrm{L}_n^h(\mathbb{Z}[G],w)$ is a module

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over the Hermitian-representation ring $G_1(\mathbb{Z}, G)$, and moreover the pairing of functors

$$G_1(\mathbb{Z}, -) \times L_n^h(\mathbb{Z}[-], w|_-) \to L_n^h(\mathbb{Z}[-], w|_-)$$

is a Frobenius pairing (see Section 3). The Grothendieck-Witt ring $GW_0(\mathbb{Z}, G)$ defined in [7], [15] is the quotient ring of $G_1(\mathbb{Z}, G)$ with respect to the Quillen relation. We note that another Grothendieck-Witt ring $GW(G, \mathbb{Z})$ is defined in [8] and the canonical homomorphism $GW(G, \mathbb{Z}) \to GW_0(\mathbb{Z}, G)$ is an isomorphism. It is a folklore since 1970's, perhaps regarded as a corollary to [17, Theorems 1, 3], that if w is trivial, then $L_n^h(\mathbb{Z}[G], w)$ is a module over the ring $GW_0(\mathbb{Z}, G)$ and

$$GW_0(\mathbb{Z}, -) \times L_n^h(\mathbb{Z}[-], w|_-) \to L_n^h(\mathbb{Z}[-], w|_-)$$

is a Frobenius pairing. This was a main motivation of the study of $GW_0(\mathbb{Z}, G)$ and $GW(G, \mathbb{Z})$ by A. Dress [6], [7], [8] in the respect of induction and restriction. By using the Frobenius structure above and the induction theory of $GW_0(\mathbb{Z}, -)$, various authors computed $L_n(\mathbb{Z}[G], w)$ for many finite groups G (cf. [9]). In addition, A. Bak [1] introduced the notion of form parameter Λ and defined various K-theoretic groups for the category of quadratic modules with form parameter (see Section 5). We [11], [12] and [13] showed that certain Bak groups $W_n(\mathbb{Z}[G], \Lambda; w)$ are equivariant-surgery-obstruction groups, as the groups $L_n^h(\mathbb{Z}[G], w)$ are surgery-obstruction groups. The groups $W_n(\mathbb{Z}[G], \Lambda; w)$ are denoted by $L_n^h(\mathbb{Z}[G], \Lambda, w)$ in the current paper. In the case where Λ is the minimal form parameter *min*, the group $L_n^h(\mathbb{Z}[G], \Lambda, w)$ coincides with the Wall group $L_n^h(\mathbb{Z}[G], w)$. It is important to ask whether the Bak-group functor $L_n^h(\mathbb{Z}[-], \Lambda_-; w|_-)$ is a Frobenius module over the Grothendieck-Witt-ring functor $GW_0(\mathbb{Z}, -)$. We have an affirmative answer as in the theorem below. Particularly if n is an even integer, the answer was obtained in [15].

Let $\mathcal{S}(G)$ denote the set of all subgroups of G and let $G(2)$ denote the set consisting of all elements g in G of order 2. Let $w : G \to \{1, -1\}$ be a homomorphism. For each $H \in \mathcal{S}(G)$, let $w_H : H \to \{1, -1\}$ denote the restriction of w. The group ring $\mathbb{Z}[H]$ has the involution $- : \mathbb{Z}[H] \to \mathbb{Z}[H]$ associated with w_H. Let n be an integer and set $\lambda = (-1)^k$ and regard it as the symmetry of $\mathbb{Z}[H]$, where k is the integer such that $n = 2k$ or $2k+1$. Let Q be a conjugation-invariant subset of $G(2)$ satisfying $w(g) = (-1)^{k+1}$ and set $Q_H = H \cap Q$. The form parameter Λ_H of $\mathbb{Z}[H]$ is defined by

$$\Lambda_H = \{x - \lambda\overline{x} \mid x \in \mathbb{Z}[H]\} + \langle Q_H \rangle.$$

Similarly to the Wall-group functor, the bifunctor $\mathrm{L}_n^h(\mathbb{Z}[-], \Lambda_-, w_-)$ on $\mathcal{S}(G)$ with canonical correspondence of morphisms is not a Mackey functor if w is nontrivial. However, we have

Theorem 1.1. *The bifunctor* $\mathrm{L}_n^h(\mathbb{Z}[-], \Lambda_-, w_-)$ *on* $\mathcal{S}(G)$ *with canonical correspondence of morphisms is a* w*-Mackey functor* (see Section 3) *and furthermore a module over the Grothendieck-Witt ring functor* $\mathrm{GW}_0(\mathbb{Z}, -)$ *on* $\mathcal{S}(G)$ *with canonical correspondence of morphisms.*

Let $\mathcal{H}_2(G)$ denote the set of all 2-hyperelementary subgroups and elementary subgroups of G. By [8, Theorem 1] and [1, Theorem 12.13 (a)], the Green functor $\mathrm{GW}_0(\mathbb{Z}, -)$ on $\mathcal{S}(G)$ is $\mathcal{H}_2(G)$-computable. By replacing the correspondence of morphisms as in [15, Proposition 2.3], the w-Mackey functor $\mathrm{L}_n^h(R[-], \Lambda_-, w_-)$ on $\mathcal{S}(G)$ is modified to a Mackey functor on $\mathcal{S}(G)$.

Corollary 1.2. *The modified Mackey functor* $\mathrm{L}_n^h(\mathbb{Z}[-], \Lambda_-, w_-)$ *is* $\mathcal{H}_2(G)$*-computable* (see Section 3). *In particular, the restriction homomorphism*

$$\mathrm{Res} : \mathrm{L}_n^h(\mathbb{Z}[G], \Lambda_G, w) \longrightarrow \bigoplus_{H \in \mathcal{H}_2(G)} \mathrm{L}_n^h(\mathbb{Z}[H], \Lambda_H, w_H)$$

is injective, and the induction homomorphism

$$\mathrm{Ind} : \bigoplus_{H \in \mathcal{H}_2(G)} \mathrm{L}_n^h(\mathbb{Z}[H], \Lambda_H, w_H) \longrightarrow \mathrm{L}_n^h(\mathbb{Z}[G], \Lambda_G, w)$$

is surjective.

Further results are discussed in Section 6. The other sections are organized as follows. In Section 2, we describe the definitions of the rings $G_1(R, G)$, $GW(G, R)$, and $GW_0(R, G)$. In Section 3, we give the definition of a Frobenius pairing and recall results obtained by C. B. Thomas, A. Dress and A. Bak. In Section 4, we describe the definitions of the category $\mathcal{G}$ $(= \mathcal{G}(G))$ and a w-Mackey functor given in [15] and recall relevant results. Section 5 is devoted to recalling the definitions of groups $L_n^h(R[G], \Lambda, w)$.

2. The Grothendieck-Witt rings

Let R be a commutative ring with 1. Let $\mathfrak{B}(G)$ denote the category of all pairs (M, B) consisting of a finitely generated R-projective $R[G]$-module M and a symmetric, G-invariant, nonsingular R-bilinear form $B : M \times M \to R$, namely

$$B(ax + a'x', by) = abB(x, y) + a'bB(x', y),$$

$$B(x, y) = B(y, x),$$

$$B(gx, gy) = B(x, y),$$

for any a, a', $b \in R$, x, x', $y \in M$, $g \in G$, and

$$M \longrightarrow \mathrm{Hom}_R(M, R);\ x \longmapsto B(x, -)$$

is a bijection. The set $\mathrm{Morph}_{\mathfrak{B}(G)}((M, B), (M', B'))$ of morphisms $(M, B) \to (M', B')$ in $\mathfrak{B}(G)$ consists of all R-linear maps $f : M \to M'$ compatible with forms, namely

$$B'(f(x), f(y)) = B(x, y)$$

for all x, $y \in M$. For an $R[G]$-submodule U of M, we define the $R[G]$-submodule $U^\perp$ of M by

$$U^\perp = \{x \in M \mid B(x, y) = 0\ (\forall\, y \in U)\}.$$

If U is R-projective and $U = U^\perp$ then we say that U is a *Lagrangian*. More generally, if an $R[G]$-submodule U of M is an R-direct summand of M and satisfies $U \subseteq U^\perp$, then we refer to U as a *Quillen submodule* of (M, B) (or simply, M). In the case where U is a Quillen submodule of (M, B), the pair $(U^{\perp}/U, B^{\perp})$ defined by

$$B^{\perp}(x + U, y + U) = B(x, y)$$

for x, $y \in U^{\perp}$ is an object in $\mathfrak{B}(G)$. For a finitely generated R-projective $R[G]$-module N, the *associated hyperbolic module* (in $\mathfrak{B}(G)$) $\mathrm{H}(N) = (N \oplus N^*, B_N)$ is defined so that $B_N(N, N) = 0 = B_N(N^*, N^*)$, $B_N(n, v) = v(n)$ for $n \in N$ and $v \in N^*$, where $N^* = \mathrm{Hom}_R(N, R)$ with $(g \cdot v)(n) = v(g^{-1}n)$.

C. B. Thomas [17] defined the group

$$\mathrm{G}_1(R, G)$$

to be the Grothendieck Group of the category $\mathfrak{B}(G)$ with respect to orthogonal sum:

$$[M_1, B_1] + [M_2, B_2] = [M_1 \oplus M_2, B_1 \perp B_2].$$

This set also has a product operation

$$([M_1, B_1], [M_2, B_2]) \mapsto [M_1, B_1] \cdot [M_2, B_2] = [M_1 \otimes_R M_2, B_1 \otimes_R B_2],$$

and is a commutative ring with 1, actually

$$1 = [R, B_0]$$

such that R has the trivial G-action and $B_0(a, b) = ab$ for a, $b \in R$. The ring $\mathrm{G}_1(R, G)$ is called the *Hermitian-representation ring.* A. Dress [8] defined a Grothendieck-Witt ring

$$\mathrm{GW}(G, R)$$

to be the quotient $\mathrm{G}_1(R, G)/\langle[(M, B)]\rangle$, where (M, B) ranges over all objects in $\mathfrak{B}(G)$ having Lagrangians. In addition, A. Dress [7, p.472] defined the ring

$$\mathrm{GU}_0(R, G)$$

as the quotient

$$\mathrm{G}_1(R, G)/\langle[(M, B)] - [(U^{\perp}/U, B^{\perp})] - [\mathrm{H}(U)]\rangle$$

and another Grothendieck-Witt ring

$$\mathrm{GW}_0(R, G)$$

as the quotient

$$\mathrm{G}_1(R, G)/\langle [(M, B)] - [(U^{\perp}/U, B^{\perp})] \rangle,$$

where (M, B) and U range over all objects (M, B) of $\mathfrak{B}(G)$ with Quillen submodule U. We remark that A. Bak [1] used the same notation $\mathrm{GW}_0(R, G)$ to denote the group $\mathrm{GW}(G, R)$ by it. Clearly, we have the canonical ring-epimorphisms

$$\mathrm{G}_1(R, G) \longrightarrow \mathrm{GW}(G, R) \longrightarrow \mathrm{GW}_0(R, G).$$

By [8, Theorem 5], the last arrow is an isomorphism if R is a Dedekind domain and $|G|$ is invertible in its field of fractions.

3. Frobenius pairing

Let $\mathfrak{F}$ be a category such that $\mathrm{Obj}(\mathfrak{F}) = \mathcal{S}(G)$ the set of all subgroups of G, let $\mathfrak{A}$ denote the category of abelian groups, and let L, M, $N : \mathfrak{F} \to \mathfrak{A}$ be bifunctors. Namely $L = (L^*, L_*)$ consists of a contravariant functor $L^* : \mathfrak{F} \to \mathfrak{A}$ and a covariant functor $L_* : \mathfrak{F} \to \mathfrak{A}$ such that $L^*(H) = L_*(H)$ for all $H \in \mathcal{S}(G)$. So, we usually write $L(H)$ instead of $L^*(H)$, $L_*(H)$.

We mean by a *pairing* $L \times M \to N$ a family of biadditive maps

$$L(H) \times M(H) \to N(H); (x, y) \mapsto x \cdot y,$$

where H runs over $\mathcal{S}(G)$. We mean by a *Frobenius pairing* a paring satisfying the conditions:

(1) $N^*(f)(x \cdot y) = L^*(f)(x) \cdot M^*(f)(y)$ for $x \in L(H), y \in M(H), f \in \mathrm{Morph}_{\mathfrak{F}}(H, K)$,

(2) $x \cdot M^*(f)(y) = N_*(f)(L^*(f)(x) \cdot y)$ for $x \in L(K), y \in M(H), f \in \mathrm{Morph}_{\mathfrak{F}}(H, K)$,

(3) $L_*(f)(x) \cdot y = N_*(f)(x \cdot M^*(f)(y))$ for $x \in L(H), y \in M(K), f \in \mathrm{Morph}_{\mathfrak{F}}(H, K)$.

Let us note the following.

(1) C. B. Thomas [17] showed that in the case where $\mathrm{Morph}_{\mathfrak{F}}(H, K)$ consists of inclusions $H \to K$ and w is the trivial homomorphism $G \to \{1\}$,

$$\mathrm{G}_1(\mathbb{Z}, -) \times \mathrm{L}_n^h(\mathbb{Z}[-], w_-) \to \mathrm{L}_n^h(\mathbb{Z}[-], w_-)$$

is a Frobenius pairing.

(2) In the case where $\mathrm{Morph}_{\mathfrak{F}}(H, K)$ consists of all monomorphisms $H \to K$, A. Dress [8, p. 292, ℓ. 3] claimed that

$$\mathrm{GW}(-, \mathbb{Z}) \times \mathrm{L}_n^h(\mathbb{Z}[-], w_-) \to \mathrm{L}_n^h(\mathbb{Z}[-], w_-)$$

is a Frobenius pairing. A similar version of quadratic forms with form parameter is given by A. Bak [1, Theorems 12.6, 12.7] where proof of the odd-dimensional case is omitted.

(3) In the case where $\mathrm{Morph}_{\mathfrak{F}}(H, K)$ consists of inclusions $H \to K$, conjugations $H \to gHg^{-1}$ and their compositions and w is trivial, one has perhaps regarded that

$$\mathrm{GW}_0(\mathbb{Z}, -) \times \mathrm{L}_n^h(\mathbb{Z}[-], w_-) \to \mathrm{L}_n^h(\mathbb{Z}[-], w_-)$$

is a Frobenius pairing, as a corollary to [17, Theorems 1, 3]. In fact, A. Dress [8, p. 742, $\ell\ell$. -6–-5] claimed without showing a detailed and precise proof that $\mathrm{GU}_0(\mathbb{Z}, -)$ acts on $\mathrm{L}_n^h(\mathbb{Z}[-], w_-)$ as a Frobenius functor.

Thus, it would serve our convenience to describe a detailed and precise proof of the fact that

$$\mathrm{GW}_0(\mathbb{Z}, -) \times \mathrm{L}_n^h(\mathbb{Z}[-], \Lambda_-, w_-) \to \mathrm{L}_n^h(\mathbb{Z}[-], \Lambda_-, w_-)$$

is a Frobenius pairing for certain form parameters Λ_- and general w. For the case $n = 2k$, one can find a proof with details in [15] (cf. [15, Theorem 12.10]).

4. w-Mackey functor

We begin this section with recalling the category $\mathcal{G} = \mathcal{G}(G)$: The set $\mathrm{Obj}(\mathcal{G})$ is same as $\mathcal{S}(G)$. For H, $K \in \mathcal{S}(G)$, $\mathrm{Morph}_{\mathcal{G}}(H, K)$ is the set of all homomorphisms

$$\varphi_{(H,g,K)} : H \to K;\ \varphi_{(H,g,K)}(h) = ghg^{-1}\quad (h \in H)$$

for $g \in G$ such that $gHg^{-1} \subseteq K$. The composition of morphisms is given by the composition of maps. Adopting the notation in [15], we also use $j_{H,K}$ and $c_{(H,g)}$ for $\varphi_{(H,e,K)}$ and $\varphi_{(H,g,gHg^{-1})}$, respectively.

We mean by a bifunctor $M = (M^*, M_*) : \mathcal{G} \to \mathfrak{A}$ a pair consisting of a contravariant functor $M^* : \mathcal{G} \to \mathfrak{A}$ and covariant functor $M_* : \mathcal{G} \to \mathfrak{A}$ such that $M^*(H) = M_*(H)$, which will be denoted by $M(H)$, for all $H \in \mathcal{S}(G)$. By [15, Proposition 2.1], we obtain

Proposition 4.1. *Let $M : \mathcal{G} \to \mathfrak{A}$ be a bifunctor satisfying $M_*(c_{(gHg^{-1},g^{-1})}) = M^*(c_{(H,g)})$ for all $H \in \mathcal{S}(G)$ and $g \in G$. The Burnside ring $\Omega(G)$ canonically acts on $M(G)$ if and only if*

$$M^*(c_{(G,g)})M_*(j_{H,G})M^*(j_{H,G}) = M_*(j_{H,G})M^*(j_{H,G})M^*(c_{(G,g)}) \tag{1}$$

for all $H \in \mathcal{S}(G)$ and $g \in G$.

Let $w : G \to \{1, -1\}$ be a homomorphism.

Definition 4.2. A bifunctor $M : \mathcal{G} \to \mathfrak{A}$ is called a *w-Mackey functor* if the following conditions are fulfilled:

(1) $M_*(c_{(H,g)}) = M^*(c_{(gHg^{-1},g^{-1})})$ for all $H \in \mathcal{S}(G)$ and $g \in G$,

(2) $M^*(c_{(H,h)}) = w(h)id_{M(H)}$ (hence $M_*(c_{(H,h)}) = w(h)id_{M(H)}$) for all $H \in \mathcal{S}(G)$ and $h \in H$,

(3) $M^*(j_{K,G}) \circ M_*(j_{H,G})$ coincides with

$$\bigoplus_{KgH \in K\backslash G/H} M_*(j_{K\cap gHg^{-1},K}) \circ (w(g)M_*(c_{(H\cap g^{-1}Kg,g)}) \circ M^*(j_{H\cap g^{-1}Kg,H})$$

for any H, $K \in \mathcal{S}(G)$.

We note that a w-Mackey functor for trivial w is a *Mackey functor.*

Recall the next proposition.

Proposition 4.3 ([15, Proposition 2.3]). *Let $M : \mathcal{G} \to \mathfrak{A}$ be a w-Mackey functor. Then bifunctor $M^w : \mathcal{G} \to \mathfrak{A}$ given by*

$$M^w(H) = M(H),$$

$$M^w_*(\varphi_{(H,g,K)}) = w(g)M_*(\varphi_{(H,g,K)}) \quad \textit{and}$$

$$M^{w*}(\varphi_{(H,g,K)}) = w(g)M^*(\varphi_{(H,g,K)})$$

for H, $K \in \mathcal{S}(G)$, $\varphi_{(H,g,K)} \in \mathrm{Morph}_{\mathcal{G}}(H,K)$ with $g \in G$ is a Mackey functor.

For a w-Mackey functor M, we say that M^w is the *Mackey functor associated with* M.

The next proposition is fundamental in geometric applications of the notion of w-Mackey functor.

Proposition 4.4 ([15, Proposition 2.6]). *A w-Mackey functor $M : \mathcal{G} \to \mathfrak{A}$ is a module over the Burnside-ring functor $\Omega : \mathcal{G} \to \mathfrak{A}$.*

Proof. Since $M^*(c_{(G,g)}) = \pm id_{M(G)}$, the equality (1) in Proposition 4.1 obviously holds. Thus $M(G)$ is a module over $\Omega(G)$. Similarly, $M(H)$ is a module over $\Omega(H)$. The naturalities (1)–(3) required for a Frobenius pairing in Section 3 can be checked in a straightforward way. ☐

Let $\mathcal{F}$ be a conjugation-invariant lower-closed subset of $\mathcal{S}(G)$, namely $gHg^{-1} \in \mathcal{F}$ and $K \in \mathcal{F}$ both hold whenever $H \in \mathcal{F}$, $g \in G$ and $K \subset H$. A Mackey functor $L : \mathcal{G} \to \mathfrak{A}$ is said to be *$\mathcal{F}$-computable* if

$$L(G) = \lim_{\longleftarrow \mathcal{G}|_{\mathcal{F}}} L(-) \text{ and } L(G) = \lim_{\longrightarrow \mathcal{G}|_{\mathcal{F}}} L(-).$$

5. Equivariant-surgery-obstruction groups

Let $A = (A, -, \lambda, \Lambda)$ be a form ring: A is a ring with 1, $-$ is an involution on A such that $\overline{ab} = \overline{b}\overline{a}$, λ is a symmetry, namely an element of Center(A) such that $\overline{\lambda}\lambda = 1$, and Λ is a form parameter, namely an additive subgroup satisfying

(1) $\{a - \lambda\overline{a} \mid a \in A\} \subseteq \Lambda \subseteq \{a \in A \mid a = -\lambda\overline{a}\}$ and

(2) $a\Lambda\overline{a} \subseteq \Lambda$ for all $a \in A$.

Let M be a finitely generated A-module. A biadditive map $B : M \times M \to A$ is called a λ-*Hermitian form* if

(1) $B(ax, by) = bB(x, y)\overline{a}$ and

(2) $B(x, y) = \lambda\overline{B(y, x)}$

for all a, $b \in A$, x, $y \in M$. A map $q : M \to A/\Lambda$ is called a *quadratic 'form'* with respect to B if

(1) $q(x + y) - q(x) - q(y) = B(x, y)$ in A/Λ,

(2) $q(ax) = aq(x)\overline{a}$ in A/Λ and

(3) $B(x, x) = \widetilde{q(x)} + \lambda\overline{\widetilde{q(x)}}$ in A

for all $a \in A$, x, $y \in M$, where $\widetilde{q(x)} \in A$ is a lifting of $q(x) \in A/\Lambda$. Such (M, B, q) is referred to as an A-*quadratic module.*

Let $\mathbf{H}(A)$ denote the *standard hyperbolic plane.* That is, $\mathbf{H}(A)$ is the A-quadratic module (M, B, q) consisting of an A-free module M with basis $\{e, f\}$, a λ-Hermitian form $B : M \times M \to A$ such that

$$B(e, e) = B(f, f) = 0, B(e, f) = 1,$$

and a quadratic 'form' $q : M \to A/\Lambda$ such that

$$q(e) = q(f) = 0.$$

A *hyperbolic module* is an A-quadratic module isomorphic to

$$\mathbf{H}(A^n) = \mathbf{H}(A) \perp \cdots \perp \mathbf{H}(A)$$

the orthogonal sum of n copies of the standard hyperbolic plane. Let $\mathcal{Q}(A)$ denote the category of A-quadratic modules (M, B, q) such that M is a free A-module and B is a nonsingular form, namely

$$M \longrightarrow \mathrm{Hom}_A(M, A);\ x \longmapsto B(x, -)$$

is a bijection. The set $\mathrm{Morph}_{\mathcal{Q}(A)}((M, B, q), (M', B', q'))$ of morphisms $(M, B, q) \to (M', B', q')$ in $\mathcal{Q}(A)$ consists of A-linear maps $f : M \to M'$ satisfying $B' \circ (f \times f) = B$ and $q' \circ f = q$.

We define $\mathrm{KQ}_0(A)_{\mathrm{free}}$ to be the Grothendieck Group of the category $\mathcal{Q}(A)$ with respect to orthogonal sum. Let $\mathrm{WQ}_0(A)_{\mathrm{free}}$ denote the quotient group $\mathrm{KQ}_0(A)_{\mathrm{free}}/\langle \mathbf{H}(A)\rangle$.

Let R be a commutative ring with 1, let $w : G \to \{1, -1\}$ be a homomorphism, let $-$ denote the involution on $R[G]$ associated to w, let n be an integer, and set $\lambda = (-1)^k$, where $k \in \mathbb{Z}$ with $n = 2k$ or $2k + 1$. The involution $-$ on $R[G]$ associated with w is the map

$$\sum_{g \in G} r_g g \longmapsto \sum_{g \in G} w(g) r_g g^{-1},$$

where $r_g \in R$.

First, consider the case where $n = 2k$ is an even integer. Given a form parameter Λ of $(R[G], -, \lambda)$, we define the group $\mathrm{L}_n^h(R[G], \Lambda, w)$ by

$$\mathrm{L}_n^h(R[G], \Lambda, w) = \mathrm{WQ}_0(A)_{\mathrm{free}}.$$

Thus in particular, Wall's group $\mathrm{L}_n^h(R[G], w)$ is $\mathrm{L}_n^h(R[G], min, w)$, where

$$min = \{x - \lambda\overline{x} \mid x \in R[G]\}.$$

For defining $\mathrm{L}_n^h(R[G], min, w)$ with n odd, we use notation below. Let $\mathrm{SU}_m(A, \Lambda)$ denote the subgroup of $\mathrm{GL}_{2m}(A)$ corresponding to $\mathrm{Aut}(\mathbf{H}(A^m))$, let $\mathrm{EU}_m(A, \Lambda)$ denote the subgroup of $\mathrm{SU}_m(A,\Lambda)$ consisting of elementary Λ-quadratic matrices, and let $\mathrm{TU}_m(A,\Lambda)$ denote the subgroup of $\mathrm{SU}_m(A,\Lambda)$ corresponding to the group consisting of $\alpha \in \mathrm{Aut}(\mathbf{H}(A^m))$ such that

$$\alpha(\langle e_1,\dots,e_m\rangle) = \langle e_1,\dots,e_m\rangle,$$

where $\langle e_1,\dots,e_m\rangle$ is the canonical Lagrangian of $\mathbf{H}(A^m)$. Let

$$\sigma \in \mathrm{SU}_1(A,\Lambda)$$

denote the matrix corresponding to $\alpha \in \mathrm{Aut}(\mathbf{H}(A))$ such that $\alpha(e) = f$ and $\alpha(f) = \overline{\lambda}e$. We set

$$\mathrm{RU}_m(A,\Lambda) = \langle \mathrm{TU}_m(A,\Lambda), \sigma\rangle.$$

Then, $\mathrm{SU}(A,\Lambda)$ is defined to be the direct limit $\varinjlim \mathrm{SU}_m(A,\Lambda)$ in a canonical way; moreover $\mathrm{EU}(A,\Lambda)$, $\mathrm{TU}(A,\Lambda)$, and $\mathrm{RU}(A,\Lambda)$ are similarly defined.

We obtain the next lemma by using 3.5 (the Whitehead Lemma) and Corollary 3.9 of [1].

Lemma 5.1. *If a subgroup K of $\mathrm{SU}(A,\Lambda)$ contains $\mathrm{EU}(A,\Lambda)$, then $[K,K] = \mathrm{EU}(A,\Lambda)$.*

Define

$$\mathrm{KQ}_1(A,\Lambda) = \mathrm{SU}(A,\Lambda)/\mathrm{EU}(A,\Lambda)$$

and

$$\mathrm{WQ}_1(A,\Lambda) = \mathrm{KQ}_1(A,\Lambda)/\langle \text{hyperbolic matrices}\rangle,$$

where we mean by a hyperbolic matrix a matrix in $\mathrm{SU}_m(A,\Lambda)$, for some m, of the form

$$\mathbf{H}(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}$$

with $\alpha \in \mathrm{GL}_m(A)$. It follows from arguments in [1, p. 27] that $\mathrm{WQ}_1(A,\Lambda)$ coincides with

$$\mathrm{KQ}_1(A,\Lambda)/[\mathrm{TU}(A,\Lambda)].$$

Now we consider the case where $n = 2k+1$ is an odd integer. Since $\mathrm{RU}(A,\Lambda) \supseteq \mathrm{EU}(A,\Lambda)$ (cf. [13, Propostion 2.7]), the quotient

$$\mathrm{L}_n^h(\mathbb{Z}[G],\Lambda,w) = \mathrm{SU}(A,\Lambda)/\mathrm{RU}(A,\Lambda)$$

is an abelian group and coincides with

$$\mathrm{WQ}_1(A,\Lambda)/\langle\sigma\rangle.$$

In particular, the Wall group $\mathrm{L}_n^h(R[G],w)$ is $\mathrm{L}_n^h(R[G],min,w)$.

6. Results

Let G be a finite group, $w : G \to \{1,-1\}$ a homomorphism, n an integer, Q an involution invariant subset of $G(2)$ satisfying $w(g) = -(-1)^k$ for all $g \in Q$, where k is an integer with $n = 2k$ or $2k+1$. For $H \leq G$, we set $Q_H = Q \cap H$, $w_H = w|_H$, and

$$\Lambda_H = \{x - (-1)^k \overline{x} \mid x \in R[H]\} + \langle Q_H \rangle_R.$$

Then, our main result is

Theorem 6.1. *The bifunctor* $\mathrm{L}_n^h(R[-],\Lambda_-,w_-) : \mathcal{G}(G) \to \mathfrak{A}$ *is a* w*-Mackey functor and moreover a module over the Grothendieck- Witt-ring functor* $\mathrm{GW}_0(\mathbb{Z},-) : \mathcal{G}(G) \to \mathfrak{A}$.

The assertion for the case $n = 2k$ follows from arguments in [15]. A detailed proof for the case $n = 2k+1$ will be given in a forthcoming paper.

Let $\mathcal{H}_2(G)$ denote the set of all 2-hyperelementary subgroups and elementary subgroups of G.

Corollary 6.2. *With respect to the associated-Mackey-functor structure, the bifunctor* $\mathrm{L}_n^h(R[-],\Lambda_-,w_-) : \mathcal{G}(G) \to \mathfrak{A}$ *is* $\mathcal{H}_2(G)$*-computable. In particular, the restriction homomorphism*

$$\mathrm{Res} : \mathrm{L}_n^h(R[G],\Lambda_G,w) \longrightarrow \bigoplus_{H\in\mathcal{H}_2(G)} \mathrm{L}_n^h(R[H],\Lambda_H,w_H)$$

is injective, and the induction homomorphism

$$\mathrm{Ind} : \bigoplus_{H \in \mathcal{H}_2(G)} \mathrm{L}_n^h(R[H], \Lambda_H, w_H) \longrightarrow \mathrm{L}_n^h(R[G], \Lambda_G, w)$$

is surjective.

This follows from [8, Theorem 1] and [1, Theorem 12.13 (a)].

Corollary 6.3. *Let β be an element in the Burnside ring $\Omega(G)$ such that $\chi_H(\beta) = 0$ for all $H \in \mathcal{H}_2(G)$ (resp. cyclic subgroup H of G). Then one has*

$$\beta \mathrm{L}_n^h(R[G], \Lambda_G, w) = 0 \quad \left(\textit{resp.}\ \ \beta^{2(a+1)} \mathrm{L}_n^h(R[G], \Lambda_G, w) = 0\right),$$

where a is the integer such that $|G| = 2^a m$ with odd integer m.

This follows from [7, Theorems 1, 3 (iii)] and [10, Proposition 6.3].

Finally we remark that the construction of smooth actions on spheres of finite groups in [16] is a geometric application of the induction theory above.

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