数理解析研究所講究録 1343 巻 2003 年 10-24

Classification of compact transformation groups on complex quadric with codimension one orbits

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Abstract

We classify compact connected Lie transformation groups on cohomology complex quadrics with codimension one orbits.

1 Introduction

1.1 Motivation

In 1960, H.C. Wang([8]) investigated compact transformation groups on spheres with codimension one orbits, after (in 1979) the classification of compact connected Lie groups on rational cohomology projective spaces with codimension one orbits was done completely by F. Uchida([6]). Similar problems were studied by T. Asoh([1] on Z_2-cohomology spheres) and K.Iwata([4] on rational cohomology Cayley projective planes).

In this paper we shall study the similar classification problem of rational cohomology complex quadrics. The author is grateful to F. Uchda, M. Masuda and S. Kikuchi for their hertful help.

1.2 Problem setting, Method and Result

Let G be a compact connected Lie group and let M be a compact connected manifold with the rational cohomology ring of a complex quadric.

Definition (complex quadric Q_{2n} ($n \neq 1$))

$$\begin{aligned} Q_{2n} &= \{z \in P_{2n+1}(C) \mid z_0^2 + z_1^2 + \cdots + z_{2n+1}^2 = 0\} \\ &\simeq SO(2n+2)/SO(2n) \times SO(2). \end{aligned}$$

It is well known that the rational cohomology ring of complex quadric. That is

$$H^*(Q_{2n}; \mathbf{Q}) = \mathbf{Q}[c, x]/(c^{n+1} - cx, x^2, c^{2n+1})$$

where $deg(x) = 2n, deg(c) = 2$.

G acts on M smoothly with codimension one orbits. The purpose of this paper is to classify such pairs denoted by (G, M) up to essentially isomorphic. Here we say that (G, M) is essentially isomorphic to (G', M') if their induced effective actions are isomorphic. This notion is defined precisely.

To classify such pairs we use the similar method of Uchida([6]). First we calculate the Poincaré polynominals of the singular orbits. Second we determine the transformation groups G from the Poincaré polynominals using well known fact of Lie theory([5]). Finally we classify (G, M) by making use of the differentiable slice theorem.

Theorem 1.1 *(G,M) is essentially isomorphic to one of the pairs in following list*

n	(G,M)	action
$n \geq 2$	$(SO(2n+1), Q_{2n})$	*canonical*
$n \geq 2$	$(U(n+1), Q_{2n})$	$U(n+1) \to SO(2n+2)$
$n \geq 2$	$(SU(n+1), Q_{2n})$	$SU(n+1) \to SO(2n+2)$
3	(G_2, Q_6)	$G_2 \to SO(7)$
2	$(Sp(2), S^7 \times_{Sp(1)} P_2(\mathbf{C}))$	*canonical*

2 Preliminary

Let us first recall some basic facts about the structure of (G, M).

Theorem 2.1 (Uchida[6]) *Let G be a compact connected Lie group. Let M be a compact connected manifold without boundary and assume*

$$H^1(M; Z_2) = 0.$$

Assume that G acts smoothly on M with an orbit $G(x)$ of codimension one. Then $G(x) = G/K$ is a principal orbit and (G,M) has just two singular orbits $G(x_1) = G/K_1$ and $G(x_2) = G/K_2$. Moreover there exists a closed invariant tubular neighborhood X_s of $G(x_s)$ such that

$$M = X_1 \cup X_2 \quad \textit{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

3 Poincaré polynominal

Let M be a compact connected manifold with the same cohomology ring as Q_{2n}, and G be a compact connected Lie group which acts on M with codimension one orbits. Then the pair (G, M) satisfies Theorem 2.1.

Hence we can show the following theorem.

Theorem 3.1 *If the two orbits are both orientable,*

(1) $G/K_s \sim P_n(\mathbf{C}); k_1 = 2n = k_2, n_1 = n = n_2.$

(2) $G/K_1 \sim P_{2n-1}(\mathbf{C}), G/K_2 \sim S^{2n},$
$k_1 = 2, k_2 = 2n, n_1 = 2n-1, n_2 = 0.$

(3) $P(G/K_s : t) = (1 + t^{k_r - 1})a(n),$
$k_1 + k_2 = 2n+1, n_1 = n = n_2, s + r = 3.$

(4) $P(G/K_1 : t) = (1 + t^{2n+1})(1 + t^{n-1})(1 + t^2 + \cdots + t^{n-1}),$
$P(G/K_2 : t) = (1 + t^{2n})(1 + t^n),$
$k_1 = 2, k_2 = n(odd), n_1 = 2n-1, n_2 = 0.$

(5) $P(G/K_1 : t) = (1 + t^{2n})(1 + t^{n-1})(1 + t^2 + \cdots + t^{n-1}),$
$P(G/K_2 : t) = (1 + t)(1 + t^n + t^{2n})(1 + t^2 + \cdots + t^{n-1}),$
$n = 2n_1 + 1, n_2 = 3n_1 + 1.$

If G/K_1 is orientable and G/K_2 non-orientable,

- $G/K_1 \sim P_{2n-1}(\mathbf{C}),$
  $P(G/K_2 : t) = (1 + t^{2n}), P(G/K_2^o : t) = (1 + t^n)(1 + t^{2n}),$
  $G/K^o \sim S^{4n-1}, n_1 = 2n-1, n_2 = 0, k_1 = 2, k_2 = n.$

If the two orbits are both non-orientable,

- $P(G/K_s : t) = 1 + t^2 + t^4, P(G/K_s^o : t) = (1 + t^2)(1 + t^2 + t^4),$
  $P(G/K : t) = P(G/K^o : t) = (1 + t^3)(1 + t^2 + t^4),$
  $n = n_1 = n_2 = k_1 = k_2 = 2.$

4 Examples

4.1 $G = SO(2n+1)$

$M = Q_{2n}$. $SO(2n+1)$ acts through the canonical representation to $SO(2n+2)$. Then there are two singular orbits, S^{2n} and $P_{2n-1}(\mathbf{C})$. The principal orbit type is $SO(2n+1)/SO(2n-1)$.

4.2 $G = SU(n+1)$

$M = Q_{2n}$. $SU(n+1)$ acts through the representation to $SO(2n+2)$;

$$SU(n+1) \ni A + B\mathbf{i} \to \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in SO(2n+2).$$

Then there two singular orbits, both orbit types are $P_n(\mathbf{C})$. The principal orbit type is $SU(n+1)/(SO(2) \times SU(n-1))$.

For $G = U(n+1)$ we get the same result.

4.3 $G = G_2$

$M = Q_6$. The exceptional Lie group G_2 acts through the canonical representation to $SO(7)$. Then there are two singular orbits, $G_2/SU(3) \simeq S^6, G_2/U(2)$. The principal orbit type is $G_2/SU(2)$.

4.4 $G = Sp(2)$

$M = S^7 \times_{Sp(1)} P_2(\mathbf{C})$. $H^*(M;\mathbf{Q}) \simeq H^*(Q_4;\mathbf{Q})$. $Sp(2)$ acts canonically on $S^7 \simeq Sp(2)/Sp(1)$. $Sp(1)$ acts right side product on $Sp(2)/Sp(1)$. $Sp(1)$ acts on $P_2(\mathbf{C}) = P(\mathbf{R}^3 \otimes_{\mathbf{R}} \mathbf{C})$ through double covering $\pi : Sp(1) \to SO(3)$. Then there are two singular isotropy groups, $Sp(1) \times U(1)$, $Sp(1) \times \pi^{-1}(S(O(2) \times O(1)))$. The principal isotropy group is $Sp(1) \times \{1, -1, \mathbf{i}, -\mathbf{i}\}$.

5 Preliminary of classification

In this section we put $H = \cap_{x \in M} G_x$.

Definition (essentially isomorphic) *If the induced effective actions* $(G/H, M)$ *and* $(G'/H', M')$ *are eqivalent diffeomorphic, then we call* (G, M) *and* (G', M') *are essentially isormorphic.*

Because we classify up to essentilly isomorpic, we can assume that

$$G = G_1 \times \cdots \times G_k \times T$$

for some simply connected simple Lie groups G_i and some toral group T.

Lemma 5.1 (**[5]**) *If $G = G_1 \times \cdots \times G_k \times T$ then the maximal rank subgroup of G is $G' = G'_1 \times \cdots \times G'_k \times T$. Here G'_i is G_i or the maximal rank subgroup of G_i.*

To classify such a pairs (G, M) up to essentially isomorphic, we can assume that G acts almost effectively on M. Here we say that G acts almost effectively on M, if $H = \cap_{x \in M} G_x$ is a finite group. In this case G acts almost effectively on the principal orbit G/K, and hence

(*) K dose not contain any positive dimensional closed normal subgroup of G.

Lemma 5.2 (**[6]**) *Let $f, f' : \partial X_1 \to \partial X_2$ be G-equivariant diffeomorphisms. Then $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G-manifolds, if one of the following conditions is satisfied:*

1. *f is G-diffeotopic to f'*
2. *$f^{-1}f'$ is extendable to a G-equivariant diffeomorphism on X_1*
3. *$f'f^{-1}$ is extendable to a G-equivariant diffeomorphism on X_2*

Lemma 5.3 (**[6]**) *If $k_1 = 2$, then*

$$H^*(G/K_s^o; \boldsymbol{Q}) = q_s^* H^*(G/K_s; \boldsymbol{Q}) + Ker(p_s^{o*})$$

Here $p_s^o : G/K^o \to G/K_s^o, q_s : G/K_s^o \to G/K_s$.

Lemma 5.4 (**[6]**) *Write $J = \oplus_k J_k = \oplus_k q_2^k H^k(G/K_2; \boldsymbol{Q})$, and denote by $e(p_2^o)$ the rational Euler class of the orientable (k_2-1)-sphere bundle $K_2^o/K^o \to G/K^o \to G/K_2^o$. Then*

$$Ker(p_2^{o*}) = J \cdot e(p_2^o) + J \cdot e(p_2^o)^2.$$

Next we compute the Poincaré polynomial $P(G/U; t)$. Here G is compact connected simple Lie group and U is its closed connected subgroup, with $rankG = rankU$. All pairs (G, U) are known if U is maximal([5]) or if G is classical([7]). So we can compute $P(G/U; t)$ by making use of [5] Section 7,Theorem 3.21. We have the following propositions

Proposition 5.1 ([6]) *If* $P(G/U;t) = 1 + t^{2a}$, *then the pair* (G,U) *is pairwise locally isomorphic to*

$$(SO(2a+1), SO(2a)) \quad or \quad (G_2, SU(3)), a = 3.$$

Proposition 5.2 ([6]) *If* $P(G/U;t) = 1+t^2+\cdots+t^{2b}$, *then the pair* (G,U) *is pairwise locally isomorphic to*

$$\begin{aligned}
&(SU(b+1), S(U(b) \times U(1))),\\
&(SO(b+2), SO(b) \times SO(2)), b = 2m+1,\\
&(Sp(\frac{b+1}{2}), Sp(\frac{b-1}{2}) \times U(1)), b = 2m+1,\\
&(G_2, U(2)), b = 5.
\end{aligned}$$

Proposition 5.3 ([6]) *If* $P(G/U;t) = (1+t^{2a})(1+t^2+\cdots+t^{2b})$, *then the pair* (G,U) *is pairwise locally isomorphic to*

$$\begin{aligned}
&(SO(2t+2), SO(2t) \times SO(2)), a = b = t,\\
&(SO(2t+3), SO(2t) \times SO(2)), a = t, b = 2t+1,\\
&(SO(7), U(3)), a = b = 3,\\
&(SO(9), U(4)), a = 3, b = 7,\\
&(SU(3), T^2), a = 1, b = 2,\\
&(SO(10), U(5)), a = 3, b = 7,\\
&(SU(5), S(U(2) \times U(3))), a = 2, b = 4,\\
&(Sp(3), Sp(1) \times Sp(1) \times U(1)), a = 2, b = 5,\\
&(Sp(3), U(3)), a = b = 3,\\
&(Sp(4), U(4)), a = 3, b = 7,\\
&(G_2, T^2), a = 1, b = 5,\\
&(F_4, Spin(7) \cdot T^1), a = 4, b = 11,\\
&(F_4, Sp(3) \cdot T^1), a = 4, b = 11.
\end{aligned}$$

Proposition 5.4 *If* $P(G/U;t) = 1 + t^4 + t^8 + t^{12}$, *then the pair* (G,U) *is pairwise locally isomorphic to*

$$(Sp(4), Sp(1) \times Sp(3)).$$

By Theorem 3.1, only these four Poincaré polynominals are possible.

6 The two singular orbits are non-orientable

In this section we shall prove that this case is not occur. By Theorem 3.1

$$P(G/K_s;t) = 1 + t^2 + t^4, \quad P(G/K_s^o;t) = (1+t^2)(1+t^2+t^4).$$

So $rankG = rankK_s^o$.

6.1 G/K_s^o is indecomposable

A manifold is called *decomposable* if it is a product of positive dimensional manifolds. By Proposition 5.3, this case is

$$\begin{aligned} G &= SU(3) \times G' \times T^h, \\ K_s^o &= T_s^2 \times G' \times T^h. \end{aligned}$$

Here T_s^2 is a maximal torus of $SU(3)$ and G' is a product of compact simply connected simple Lie groups.

Now $k_s = 2$, hence $K_s^o/K^o \simeq S^1$. Therefore K_s^o acts on S^1 through the representation $\rho : K_s^o \to SO(2)$. So $Ker(\rho) = K^o \triangleleft K_s^o$. Consequently $G' = \{e\}, h = 0$ or 1 by (*).

We consider the slice representation $\sigma_s : K_s \to O(2)$. Since G/K_s is non-orientable, there is the element $g_s \in K_s - K_s^o$ with

$$\sigma_s(g_s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The centralizer of $\sigma_s(g_s)$ in $O(2)$ is a finite group, hence $h = 0$. Then we know $N(K_s^o;G)/K_s^0 \simeq S_3$, where S_3 is the symmetric group of degree 3. Because G/K_s is non-orientable, $K_s/K_s^o \simeq \mathbf{Z}_2$, so we can put

$$g_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in K_1 - K_1^o \subset SO(3).$$

We can assume that

$$K_1^o = \{ \begin{pmatrix} \bar{u}\bar{v} & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix} \in SU(3) | u, v \in U(1)\} \ni (u,v).$$

The centralizer of g_1 in K_1 is

$$\{ \begin{pmatrix} \bar{u}^2 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix}, \begin{pmatrix} -\bar{u}^2 & 0 & 0 \\ 0 & 0 & u \\ 0 & u & 0 \end{pmatrix} | v \in U(1)\}.$$

However by the slice representation

$$\sigma_1 : (u,v) \mapsto \begin{pmatrix} cos(a\theta) & sin(a\theta) \\ -sin(a\theta) & cos(a\theta) \end{pmatrix},$$

we see that

$$\sigma_1(g_1(u,v)g_1^{-1}) = \begin{pmatrix} cos(a\theta) & -sin(a\theta) \\ sin(a\theta) & cos(a\theta) \end{pmatrix}.$$

This gives $a = 0$. This contradicts of $a \neq 0$.

6.2 G/K_1^o is decomposable

By Theorem 5.1(a=1), 5.2(b=2), we know that

$$G = SU(2) \times SU(3) \times G' \times T^h,$$
$$K_1^o = T^1 \times S(U(2) \times U(1)) \times G' \times T^h.$$

Now we can prove easily G/K_2 is decomposable. Hence $K_1^o \simeq K_2^o$.

Now $k_s = 2$, hence $G' = \{e\}, h = 0$ by a proof similar that when G/K_s^o is indecomposable. Since G/K_s is non-orientable, $K_s \simeq N(T^1; SU(2)) \times S(U(2) \times U(1))$. For the slice representation $\sigma_1 : K_1 \to O(2)$, there exists $g_1 \in K_1 - K_1^o$ such that

$$\sigma_1(g_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here the centralizer of $\sigma_1(g_1)$ in $O(2)$ is finite group. So the slice representation $\sigma_s : K_s \to O(2)$ can be composable

$$\sigma_s : K_s \to N(SO(2); SO(3)) \to O(2).$$

Therefore there is an equivariant decomposition

$$M \simeq P_2(\mathbf{C}) \times (SU(2) \times_{N(T^1)} S^2).$$

Here $N(T^1) = N(T^1; SU(2))$. This contradicts the assumption that M is indecomposable.

7 One singular orbit is orientable, the other is non-orientable

We can assume G/K_1 is orientable, G/K_2 is non-orientable.
By Theorem 3.1

$$G/K_1 \sim P_{2n-1}(\mathbf{C}), \quad P(G/K_2^o;t) = (1+t^n)(1+t^{2n}).$$

In this case G/K_1 is indecomposable. We see that $K_1^o = K_1$. Since $k_1 = 2$, we can assume that $G = H \times T^h, K_1 = H_{(s)} \times T^h$ ($h = 0$ or 1). By Proposition 5.2, 5.3, 5.4, we know that $n = 2$ or 4 and

$$\begin{aligned}(G, K_s^o) &\sim (SU(4), S(U(3)\times U(1))\ (n=2)\ or \\ &\quad (Sp(2), Sp(1)\times U(1))\ (n=2)\ or \\ &\quad (SO(5), SO(3)\times SO(2)) \sim (Sp(2), U(2))\ (n=2), \\ (G, K_1, K_2^o) &\sim (Sp(4), Sp(3)\times U(1), Sp(1)\times Sp(3))\ (n=4).\end{aligned}$$

Since G/K_2 is non-orientable, $G = SU(4), Sp(4)$ is not occur (so $h = 0$). Consequently G/K_2^o is indecomposable.

7.1 $G = Sp(2), K_s^o \simeq Sp(1)\times U(1)$

Since G/K_1 is orientable and G/K_2 is non-orientable, $K_1 = Sp(1)\times U(1) = K_1^o$ and $K_2 = N(K_2^o;G)$. Since $K_s/K \simeq S^1$, we have $K = Sp(1)\times F$ (where F is a finite subgroup of U(1)). If $K_2^o = K_1 = Sp(1)\times U(1)$, then $K_2/K \simeq N(U(1);Sp(1))/F \simeq S^1 \oplus S^1$. This contradicts of $K_2/K \simeq S^1$. So (in particular) we can put $K_2^o = Sp(1)\times U(1)_j$, where $U(1)_j = \{a + b\mathbf{j} | a^2+b^2 = 1\}$. If $K_2^o = Sp(1)\times U(1)_j$, then $K_2 = Sp(1)\times (U(1)_j \cup U(1)_j\mathbf{i})$. $K_1 \cap K_2 = Sp(1)\times\{1,-1,\mathbf{i},-\mathbf{i}\}$. Since $K_2/K \simeq K_1/K \simeq S^1$, we have $F = \{1,-1,\mathbf{i},-\mathbf{i},\}$.

The slice representation σ_1 has a following decomposition

$$\sigma_1 : K_1 \to U(1) \overset{\rho_1}{\to} SO(2).$$

Here we can put

$$\rho_1(exp(i\theta)) = \begin{pmatrix} cos(4\theta) & -sin(4\theta) \\ sin(4\theta) & cos(4\theta)\end{pmatrix},$$

since $Ker(\rho_1) = F$. So the slice representation ρ_1 is uniquely up to equiva-

The slice representation σ_2 has a following decomposition

$$\sigma_2 : K_2 \to N(U(1)_j; Sp(1)) = U(1)_j \cup U(1)_j\mathbf{i} \overset{\rho_2}{\to} O(2).$$

Since $K_2/K \simeq S^1$ and $Ker(\rho_2|_{U(1)_j}) = \mathbf{Z}_2$,

$$\rho_2(i) = \rho_2(-i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So the slice representation ρ_2 is uniquely up to equivalence.

Now $N(K;G)/K \simeq Sp(1) \times Sp(1)$ is connected. So this case is satisfied the assumption of Lemma 5.2 1. Hence (G, M) is unique up to essentially isomorphic. Such an example of (G, M) was constructed by in Section 4.4.

7.2 $G = Sp(2), K_s^o = U(2)$

Since G/K_1 is orientable, $K_1 = U(2)$. So $K^o = SU(2)$ because $K_1/K \simeq S^1$. Since G/K_2 is non-orientable, $K_2 \simeq N(U(2); Sp(2))$ (K_2 has two components). If $K_1 = K_2^o$, then $K_2/K \simeq S^1 \oplus S^1$. This contradicts of $K_2/K \simeq S^1$. However $K \subset K_1 \cap K_2$, so $K_1 = K_2^o$. Hence this case does not occur.

8 The two singular orbits are orientable

8.1 $G/K_1 \sim P_{2n-1}(\mathbf{C}), G/K_2 \sim S^{2n}$

In this case $G/K_1, G/K_2$ are indecomposable. Since $k_1 = 2$ and $k_2 = 2n$ ($n \geq 2$), $G = H \times T^h$ and $K_1^o = K_1 = H_1 \times T^h$ ($h = 0$ or 1). By Proposition 5.2,

$$\begin{aligned}(H, H_1) \quad \sim \quad & (SU(2n), S(U(2n-1) \times U(1))) \textit{ or} \\ & (SO(2n+1), SO(2n-1) \times SO(2)) \textit{ or} \\ & (Sp(n), Sp(n-1) \times U(1)) \textit{ or} \\ & (G_2, U(2)) : n = 3.\end{aligned}$$

By Lemma 5.3 and Lemma 5.4, we can easily show that

$$P(G/K_2^o; t) = P(G/K_2; t).$$

We can put $K_2^o = H_2 \times T^h$. By Proposition 5.1,

$$\begin{aligned}(H, H_2) \quad \sim \quad & (SO(2n+1), SO(2n)) \textit{ or} \\ & (G_2, SU(3)) : n = 3.\end{aligned}$$

Since $K_2^o/K^o \simeq S^{2n-1}$, we have $h = 0$. Hence

$$G = Spin(2n+1) \quad \textit{or} \quad G_2 : n = 3.$$

8.1.1 $G = Spin(2n+1)$

In this case $K_1 = Spin(2n-1)\cdot T^1, K_2^o = Spin(2n), K^o = Spin(2n-1)$.

Since G/K_2 is orientable, $K_2 = K_2^o$. So $K = K^o$. Hence the slice representation $\sigma_1 : K_1 \to SO(2)$ is decomposed

$$\sigma_1 : K_1 = Spin(2n-1)\cdot T^1 \stackrel{proj}{\to} T^1 \stackrel{\rho}{\to} SO(2).$$

Since $Ker(\sigma_1) = K$, ρ is an isomorphism. So the slice representation σ_1 is uniquely up to equivalence.

Next we consider the slice representation $\sigma_2 : K_2 \to SO(2n)$.

Since $\mathbf{Z}_2 \subset Ker(\sigma_2) \subset \sigma_2^{-1}(SO(2n-1)) = K$, σ_2 is decomposed

$$\sigma_2 : K_2 = Spin(2n) \stackrel{proj}{\to} SO(2n) \stackrel{\rho}{\to} SO(2n).$$

Since $SO(2n)$ acts transitively on S^{2n-1}, ρ is an isomorphism by making use of [3]. Hence the slice representation σ_2 is uniquely up to equivalence.

Now we show that

> any equivariant diffeomorphism of $G/K = \partial(G \times_{K_2} D^{2n})$ is extendable to an equivariant diffeomorphism of $G \times_{K_2} D^{2n}$.

proof *In this case $N(K,G)$ has two components. So we can assume $N(K,G)/N(K,G)^o \simeq$ $\mathbf{Z}_2 =< y >$ $(y \in Spin(2n+1))$ such that*

$$p(y) = \begin{pmatrix} -I_{2n} & 0 \\ 0 & 1 \end{pmatrix}.$$

Here $p : Spin(2n+1) \to SO(2n+1)$ is the natural projection. It suffices to prove that the right translation R_y on G/K is extendable. Because y is in the center of K_2, we have the following commutative diagram:

$$\begin{array}{ccc} G \times_{K_2} K_2/K & \to & G/K \\ \downarrow R_y \times 1 & & \downarrow R_y \\ G \times_{K_2} K_2/K & \to & G/K \end{array}$$

Here $G \times_{K_2} K_2/K = \partial(G \times_{K_2} D^{2m})$. It is clear that $R_y \times 1$ is extendable. ■

Consequently (G, M) is unique up to essentially isomorphic. Such an example of (G, M) was constructed in Section 4.1.

8.1.2 $G = G_2$

In this case $K_1 \simeq U(2), K_2^o \simeq SU(3), K^o \simeq SU(2), n = 3$.

The exceptional Lie group $G_2 = Aut(\mathbf{Cay})$. Here **Cay** is a Cayley number generated by **R**-basis $\{1, e_1, \cdots, e_7\}$. It is well known that $G_2 \subset SO(7)$, G_2 acts on **Cay** which fix the **R**-basis 1.

Now we can consider that $K_2^o = \{A \in G_2 | A(e_1) = e_1\} \simeq SU(3)$. Then $N(K_2^o, G)$ has two components. Since G/K_2 is orientable, $K_2 = K_2^o$. So $K = K^o$.

We denote the slice representation $\sigma_2 : K_2 \to SO(6)$. Because K_2 acts transitively on $K_2/K \simeq S^5$ via σ_2, so the slice representation σ_2 is uniquely determined up to equivalence. Then we see that $\sigma_2^{-1}(SO(5)) = \{B \in K_2 | B(e_2) = e_2\} = K \simeq SU(2)$.

Next we denote the slice representation $\sigma_1 : K_1 \to SO(2)$. Since $Ker(\sigma_1) = K \simeq SU(2)$, σ_1 is decomposed that

$$\sigma_1 : K_1 \to U(1) \overset{\rho}{\to} SO(2).$$

Here ρ is an isomorphism. So the slice representation σ_1 is uniquely determined up to equivalence.

This implies $N(K, G)/K \simeq SO(3)$. Consequently (G, M) is unique up to essentially isomorphism by Lemma 5.2. Such an example of (G, M) was constructed in Section 4.3.

8.2 $G/K_s \sim P_n(\mathbf{C})$

In this case we can compute similary. We see this case is Section 4.2.

8.3 $P(G/K_1; t) = a(2n-1) + t^{n-1} + t^{3n-1}$

This case is Theorem 3.1 (5),(6). We can easily see that this case does not occur.

8.4 $P(G/K_1; t) = (1 + t^{k_2 - 1})a(n)$:$k_2$ is odd.

In this case we see $K_1 = K_1^o$ by $k_2 > 2$. We can assume that $G = G' \times G"$, $K_1 = K_1' \times G"$.

8.4.1 G/K_1 is decomposable

In this case we can assume that

$$G = H_1 \times H_2 \times G", K_1 = H_{(1)} \times H_{(2)} \times G".$$

Here $H_1/H_{(1)} \sim S^{k_2-1}, H_2/H_{(2)} \sim P_n(\mathbf{C})$. By Proposition 5.2,5.3.

$$\begin{aligned}(H_1, H_{(1)}) &= (Spin(k_2), Spin(k_2-1)) \ or \\ &= (G_2, SU(3)) \ (k_2 = 7), \\ (H_2, H_{(2)}) &= (SU(n+1), S(U(n) \times U(1))) \ or \\ &= (Spin(n+2), Spin(n) \cdot T^1) \ (n: odd) \ or \\ &= (Sp(\frac{n+1}{2}), Sp(\frac{n-1}{2}) \times U(1)) \ (n: odd) \ or \\ &= (G_2, U(2)) \ (n = 5).\end{aligned}$$

By lemma 8.1, $H_{(1)} \times H_{(2)}$ acts transitively on $K_1/K \simeq S^{k_1-1}$.

Lemma 8.1 *$H_1 = SU(2), H_2 = SU(3)$, or $H_{(2)}$ acts transitively on K_1/K.*

If $H_{(2)}$ does not act transitively on K_2/K. Then $k_1 = 2, k_2 = 3, n = 2$

$$\begin{aligned}G &= SU(2) \times SU(3) \times G", \\ K_1 &= T^1 \times S(U(2) \times U(1)) \times G".\end{aligned}$$

Then we see $G" = \{e\}$ by $G"$ acting non-transitively on $K_1/K \simeq S^1$. Since $K_2/K \simeq S^2$, $K_2^o = A \cdot N, K^o = A' \cdot N$. Here $(A, A') \sim (SU(2), T^1)$. Consider the slice representation $\sigma_1 : K_1 = T^1 \times S(U(2) \times U(1)) \to SO(2)$.

By $Ker(\sigma_1) = K$, $K^o \supset 1 \times SU(2) \times 1$. So $K^o = (1 \times SU(2) \times 1) \cdot T^1$. Hence $K_2^o = (1 \times SU(2) \times 1) \cdot SU(2), K^o = T^1 \times SU(2)$. But this is a contradiction. So we see $H_{(2)}$ acts transitively on K_2/K.

Let $p_t : G \to H_t, p'_t : G \to H_t \times G"$ be the natural projection, and let $h_t : H_t \to G, h'_t : H_t \times G" \to G$ be the natural inclusion. Put

$$L_{st} = p_t(K_s),\ L_t = p_t(K),\ L'_{st} = p'_t(K_s),\ L'_t = p'_t(K),$$
$$N_{st} = h_t^{-1}(K_s),\ N_t = h_t^{-1}(K),\ N'_{st} = h'^{-1}_t(K_s),\ N'_t = h'^{-1}_t(K).$$

Since $H_{(1)} \times G" \subset K$, we have $L'_1 = L'_{11} = H_{(1)} \times G"$ and $H_{(2)}/N_2 \simeq K_1/K \simeq S^{k_1-1}$. We see easily that L_2/N_2 acts freely on $H_{(2)}/N_2 \simeq S^{k_1-1}$ by right translation, and $L_2/N_2 \simeq L'_1/N'_1$. Here we have from [2]

$$dim(L'_1/N'_1) \leq 3. \qquad (1)$$

We can prove

$$\begin{aligned}L_{21} &= H_1, &(2)\\ N_1 &\neq H_{(1)}. &(3)\end{aligned}$$

By Proposition 5.1

$$\begin{aligned}(H_1, H_{(1)}) &= (Spin(k_2), Spin(k_2-1))) \; or \\ & \quad (G_2, SU(3)) : k_2 = 7.\end{aligned}$$

If $k_2 \geq 5$, then $H_{(1)}$ is simple Lie group. Since $N_1' \lhd L_1' = H_{(1)} \times N"$ and the equation (1), So $N_1 = Sp(1)$ and $dim(N_1) > 0$. Hence we get $N_{21} = L_{21} = H_1$ and $K_2 = H_1 \times N_{22}'$.

Therefore $N_1 = L_1 = H_{(1)}$. This contradicts of (3). Consequantly $k_2 = 3$. Hence $(H_1, H_{(1)}) = (SU(2), T^1)$.

This gives $k_1 = 2n-2$. So $H_{(2)}$ acts transitively S^{2n-3}.

By Proposition 5.2 and making use of [3], we have $k_1 = 2n-2, k_2 = 3$,

$$\begin{aligned} G &= SU(2) \times Sp(\frac{n+1}{2}) \times G", \\ K_1 &= T^1 \times Sp(\frac{n-1}{2}) \times U(1) \times G", \end{aligned}$$

and $n = 9$, $G = SU(2) \times Spin(11) \times G"$.

These cases we can easily see that $G" = \{e\}$. and $K_2 = K_2^o$.

If $G = SU(2) \times Sp(\frac{n+1}{2})$, the slice representation

$$\sigma_1 : K_1 \to SO(2n-2)$$

is unique up to equivalence and $Ker(\sigma_1) \supset T^1 \times \{e\} \times U(1)$. So $K = T^1 \times Sp(\frac{n-3}{2}) \times U(1)$. Since $K_2/K \simeq S^2$ and $P(G/K_2; t)$, we get

$$K_2 = SU(2) \times Sp(\frac{n-3}{2}) \times U(1).$$

Hence the slice representation $\sigma_2 : K_2 \to SO(3)$ is unique up to equivalence.

$N(K;G)/K = N(T^1; SU(2))/T^1 \times Sp(1) \times N(U(1); Sp(1))/U(1)$. If $N(U(1); Sp(1))/U(1) \simeq Z_2 =< a >$, then $xa = a\bar{x}$ for all $x \in U(1)$. We consider the next diagram

$$\begin{array}{ccc} G \times_{K_2} K_2/K & \xrightarrow{f} & G/K \\ \downarrow R_a \times 1 & & \downarrow R_a \\ G \times_{K_2} K_2/K & \xrightarrow{f} & G/K \end{array}$$

Here $f([g, kK]) = gkK$. We have $tK = \bar{t}K$ for all $t \in \{e\} \times \{e\} \times U(1) \subset K$. So this diagram is commutative. Hence any eqivalent diffeomorphism on G/K is extendable to an equivalent diffeomorphism on $X_2 = G \times_{K_2} D^{k_2}$. In this case we can put $M = Sp(k+1)/U(1) \times_{Sp(k)} S^{4k+2}$, with $k = \frac{n-1}{2}$. However we can prove $H^*(M) \neq H^*(Q_{4k+2})$. This is a contradiction.

If $G = SU(2) \times Spin(11)$, then we see similary this case does not occur.

8.4.2 G/K_1 is indecomposable

Also we can prove this case is not occur.

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