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REMARKS ON ISOVARIANT MAPS FOR REPRESENTATIONS

Ikumitsu Nagasaki (大阪大学大学院理学研究科・長崎 生光)
Department of Mathematics, Graduate School of Science
Osaka University

1. INTRODUCTION

In this note we shall discuss an isovariant version of the Borsuk-Ulam theorem, which we call the isovariant Borsuk-Ulam theorem, and give some related results on the isovariant Borsuk-Ulam theorem for $SO(3)$.

We say that a compact Lie group G has the *IB-property* if G has the following property:

- For any (orthogonal) G-representations V, W such that a G-isovariant map $f : V \to W$ exists, the inequality
$$\dim V - \dim V^G \leq \dim W - \dim W^G$$
holds.

An interesting problem is the following.

Problem A. Which compact Lie groups have the IB-property?

By a result of Wasserman [3], any compact solvable Lie group has the IB-property, however this problem is still open for a general compact Lie group. On the other hand, a weaker version of this problem has an affirmative answer for an arbitrary compact Lie group.

Theorem 1.1 (The weak isovariant Borsuk-Ulam theorem). *For an arbitrary compact Lie group, the weak isovariant Borsuk-Ulam theorem holds.*

In section 2 we shall recall this theorem from [2].

In section 3, as an example, we shall discuss further details when $G = SO(3)$, and show the isovariant Borsuk-Ulam theorem holds when the dimension of $SO(3)$-representation is small, that is,

Proposition 1.2. *Let $V = \oplus_{i=0}^{6} a_i U_i \oplus U$ and $W = \oplus_{i=0}^{6} b_i U_i \oplus U$, where a_i, b_i are nonnegative integers, U_i is the $(2i+1)$-dimensional irreducible $SO(3)$-representation and U is any $SO(3)$-representation. If there is an $SO(3)$-isovariant map from V to W, then*
$$\dim V - \dim V^{SO(3)} \leq \dim W - \dim W^{SO(3)}$$

2. A weak version of the isovariant Borsuk-Ulam theorem

We first recall the *prime condition* in order to state Wasserman's result.

Definition 1. We say that a finite group G satisfies the *prime condition* if for every pair of subgroups $H \lhd K$ with K/H simple,

$$\sum_{\substack{p:\text{prime} \\ p \mid |g|}} \frac{1}{p} \leq 1$$

for every $g \in K/H$, where $|g|$ denotes the order of g.

Wasserman's isovariant Borsuk-Ulam theorem is stated as follows.

Theorem 2.1 (The isovariant Borsuk-Ulam theorem). *Every finite group G satisfying the prime condition has the IB-property.*

Remark. All finite groups do not satisfy the prime condition, for example, A_n, $n \leq 11$, satisfies the prime condition, but A_n, $n \geq 12$, does not satisfy the prime condition. The author does not know whether all A_n have the IB-property.

We next consider a weaker version of the isovariant Borsuk-Ulam theorem.

Definition 2. We say that a compact Lie group G has the *WIB-property* if there exists a monotone increasing function $\varphi_G : \mathbb{N}_0 \to \mathbb{N}_0$ ($\mathbb{N}_0 :$ the nonnegative integers) diverging to $+\infty$ with the following property:

- For any (orthogonal) G-representations V, W such that a G-isovariant map $f : V \to W$ exists, the inequality

$$\varphi_G(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

holds.

Remark. In [2] we defined the WIB-property for linear G-spheres, but it is essentially same as above, because one can see that the existence of a G-isovariant map from V to W and the existence of a G-isovariant map from SV to SW are equivalent.

A weak version of Problem A is:

Problem B. Which compact Lie groups have the WIB-property?

The answer is the following:

Theorem 2.2 (The weak isovariant Borsuk-Ulam theorem). *An arbitrary compact Lie group G has the WIB-property.*

The outline of proof is as follows. The full details will appear in [2]. We first note:

Lemma 2.3. *Let*

$$1 \to H \to G \to K \to 1$$

be a short exact sequence of compact Lie groups.

(1) *If H and K have the WIB [IB]-property, then G has the WIB [IB]-property.*
(2) *If G has the WIB [IB]-property, then K has the WIB [IB]-property.*

By this lemma, the problem is reduced to two cases:

(1) G is a finite simple group,
(2) G is a compact, simply-connected, simple Lie group.

Using the (ordinary) Borsuk-Ulam theorem, one can see

Proposition 2.4. *C_p (p : prime) and S^1 have the IB-property.*

Therefore we obtain the following corollary from Lemma 2.3 and Proposition 2.4:

Corollary 2.5. *Any compact solvable Lie group has the IB-property.*

The next result is easy, but plays an important role in the proof of the weak isovariant Borsuk-Ulam theorem.

Lemma 2.6. *Let H be a closed subgroup of G with the IB-property. Assume that there exists a constant $0 < c < 1$ such that $\dim U^H \leq c \dim U$ for all nontrivial irreducible representations U of G. Then G has the WIB-property, and moreover $\varphi_G(n)$ can be taken to be $\langle (1-c)n \rangle$, where $\langle x \rangle = \min\{n \in \mathbb{Z} \mid n \geq x\}$.*

Proof. Let $f : V \to W$ be any G-isovariant map between representations. Let $V = V_G \oplus V^G$ and $W = W_G \oplus W^G$, where V_G [resp. W_G] denotes the orthogonal complement of V^G [resp. W^G]. Since the natural inclusion $i : V_G \to V$ and the projection $p : W \to W_G$ are G-isovariant, we get a G-isovariant map $g := p \circ \tilde{f} \circ i : V_G \to W_G$. Since H has the IB-property, it follows that

$$\dim V_G - \dim V_G^H \leq \dim W_G - \dim W_G^H \leq \dim W_G.$$

By the complete reducibility of G, V_G is isomorphic to a direct sum of nontrivial irreducible representations. Hence by assumption one can see that

$$(1-c) \dim V_G \leq \dim V_G - \dim V_G^H.$$

Setting $\varphi_G(n) = \langle (1-c)n \rangle$, we obtain that $\varphi_G(\dim V_G) \leq \dim W_G$, or equivalently

$$\varphi_G(\dim V - \dim V^G) \leq \dim W - \dim W^G.$$

Clearly φ_G is a monotone increasing function diverging to ∞. This implies that G has the WIB-property.

In the case (1), since there are only finitely many irreducible representations, we have following:

Proposition 2.7. *Let G be a finite simple group. Let H be any nontrivial subgroup of G. Then there exists a constant $0 < c < 1$ such that $\dim U^H \leq c \dim U$ for all nontrivial irreducible representations U.*

In particular, taking H as a cyclic subgroup of prime order, we obtain by Lemma 2.6 that G has the WIB-property.

In the case (2), by representation theory of compact Lie groups, we also see the following:

Proposition 2.8 ([2]). *Let G be a compact, simply-connected, simple Lie group and T a maximal torus. There exists a constant $0 < c < 1$ such that $\dim U^T \leq c \dim U$ for all nontrivial irreducible representations U of G.*

Since T has the IB-property, it follows from Lemma 2.6 that G has the WIB-property. Thus the proof of the weak isovariant Borsuk-Ulam theorem is complete.

Before ending this section, we give a remark on the (weak) isovariant Borsuk-Ulam theorem in semilinear actions.

Definition 3. A closed (smooth) G-manifold M is called a *semilinear G-sphere* if the H-fixed point set M^H is homotopy equivalent to a sphere or empty for every closed subgroup H of G.

We can consider a similar problem in the family of semilinear G-spheres, however the conclusion is different from linear case. For semilinear G-spheres, the (weak) isovariant Borsuk-Ulam theorem does not hold in general. In this case we show in [2] that the (weak) isovariant Borsuk-Ulam theorem holds if and only if G is solvable.

3. Some estimate of φ_G for $G = SO(3)$

In this section we concerned with the function φ_G as in Definition 2.

We set

$$c_G(n) = \max\{\varphi_G(n) \mid \varphi_G \text{ as in Definition 2}\}$$

for $n \geq 1$, and $c_G(0) = 0$ for convenience.

Set $\mathcal{D}_G = \{n \mid n = \dim V - \dim V^G \text{ for some } V\}$. We also define a similar function d_G on $\mathcal{D}_G$, where $d_G(n)$, $n \geq 1$, is defined as the greatest integer with the following property:

- For any representation V with $\dim V - \dim V^G = n$ and for any W, if there is a G-isovariant map $f : V \to W$, then

$$d_G(n) \leq \dim W - \dim W^G$$

holds.

We also define $d_G(0) = 0$. Though the definition of d_G resembles that of c_G, these are different in definition, namely d_G need not be monotonely increasing. (However the author does not have such an example.)

We first note the following.

Lemma 3.1. *The value $c_G(n)$, $n \geq 1$, is equal to the greatest integer with the following property:*

- *For any representation V with $\dim V - \dim V^G \geq n$ and for any W, if there is a G-isovariant map $f : V \to W$, then*

$$c_G(n) \leq \dim W - \dim W^G$$

holds.

Proof. Let $c'_G(n)$ be the greatest integer satisfying the above property. Then c'_G is monotonely increasing and diverging to ∞ by the weak isovariant Borsuk-Ulam theorem. Hence c'_G is one of φ_G and so $c'_G = c_G$.

Remark. From this lemma, c_G is thought of as an isovariant version of the Borsuk-Ulam function b_G defined in [1].

One can easily see the following by definition.

Proposition 3.2. *$\varphi_G(n) \leq c_G(n) \leq d_G(n) \leq n$ for any $n \in \mathcal{D}_G$.*

Proposition 3.3. *The following are equivalent.*

(1) *G has the IB-property.*
(2) *$c_G(n) = n$ for any $n \in \mathcal{D}_G$.*
(3) *$d_G(n) = n$ for any $n \in \mathcal{D}_G$.*

As an example we shall estimate c_G or d_G by finding some function φ_G when $G = SO(3)$. As is well-known, $SO(3)$ has only one (real) $(2k+1)$-dimensional irreducible representation for each $k \geq 0$, which we denote by U_k. Let T ($\cong S^1$) be a maximal torus and N ($\cong O(2)$) the normalizer of T. Each U_k has the weight $1 + t + \cdots + t^k$, where t is the standard irreducible representation of S^1. So we obtain $\dim U_k^T = 1$, moreover we have

$$\dim U_k^N = \begin{cases} 1 & (k : \text{even}) \\ 0 & (k : \text{odd}), \end{cases}$$

and so

$$\frac{\dim U_k^N}{\dim U_k} = \begin{cases} \frac{1}{2k+1} & (k : \text{even}) \\ 0 & (k : \text{odd}). \end{cases}$$

Therefore we obtain

$$\dim V^N \leq \frac{1}{5} \dim V$$

for any representation V with $V^G = 0$. Since N is solvable, by Proposition 2.8 and its proof, we obtain

$$\frac{4}{5}(\dim V - \dim V^G) \leq \dim W - \dim W^G.$$

So φ_G can be taken as

$$\varphi_G(n) = \left\langle \frac{4}{5} n \right\rangle.$$

and hence

$$c_G(n) \geq \left\langle \frac{4}{5} n \right\rangle.$$

For $G = SO(3)$, $\mathcal{D}_G$ consists of the nonnegative integers except $n = 1, 2, 4$. Consequently we have $c_G(3) = 3$, $c_G(5) \geq 4$, $c_G(6) \geq 5$, etc. However this estimate is not very sharp. In fact one can see $c_G(5) = 5$, $c_G(6) = 6$ later.

Remark. The value of φ_G or c_G of $n \notin \mathcal{D}_G$ is not important as well as of $n = 0$ for our purpose.

The following is a partial result on the isovariant Borsuk-Ulam theorem for

Proposition 3.4. *Let $G = SO(3)$. Let $V = \oplus_{i=0}^{6} a_i U_i \oplus U$ and $W = \oplus_{i=0}^{6} b_i U_i \oplus U$, where a_i, b_i are nonnegative integers and U is any representation. If there is a G-isovariant map from V to W, then*

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

We notice some facts for the sake of proof. Firstly it suffices to show the proposition when $a_0 = b_0 = 0$. Secondly, as is well-known, the (closed) proper subgroups of $SO(3)$ are the following: the cyclic group C_n, the dihedral group D_n, the tetrahedral group T, the octahedral group O, the icosahedral group I, $SO(2)$ and $O(2)$. All of these except I are solvable, and I is isomorphic to A_5, whence all proper subgroups of $SO(3)$ have the IB-property. Therefore the isovariant Borsuk-Ulam theorem gives various inequalities between dimensions. We consider them in a general setting. Let $V = \oplus_{i=1}^{n} a_i U_i$ and $W = \oplus_{i=1}^{n} b_i U_i$. Set $\eta = W - V$ and set $\alpha_i = \sum_{k=i}^{n}(b_k - a_k)$, $1 \leq i \leq n$. Then we have

$$\mathrm{Res}_{SO(2)}\eta = \alpha_1 1 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n,$$

and

$$\dim \eta = 3\alpha_1 + 2(\alpha_2 + \cdots + \alpha_n).$$

By the isovariant Borsuk-Ulam theorem, one can easily see the following.

Lemma 3.5. (1) $\dim \eta^{SO(2)} - \dim \eta^{O(2)} = \sum_{k=1}^{n}(-1)^{k-1}\alpha_k \geq 0$.

(2) $\dim \eta - \dim \eta^{C_p} = \sum_{k \not\equiv 0(p)} \alpha_k \geq 0$.

(3) $\dim \eta^{C^2} - \dim \eta^{C^4} = \sum_{\substack{k \equiv 0(2) \\ k \not\equiv 0(4)}} \alpha_k \geq 0$.

(4) *If $i > \frac{n}{3}$, then $\alpha_i \geq 0$.*

Proof. (1)–(3): easy.

(4): By the isovariant Borsuk-Ulam theorem, we have

$$\dim \eta^{C_i} - \dim \eta^{C_{2i}} = 2(\alpha_i + \alpha_{3i} + \alpha_{5i} + \cdots) \geq 0.$$

Since $3i > n$, α_m must be 0 for $m \geq 3i$. Hence $\alpha_i \geq 0$.

Proof of Proposition 3.4. We may suppose that $a_0 = b_0 = 0$. When $n = 6$, by Lemma 3.5, we have inequalities

$$\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 - \alpha_6 \geq 0,$$
$$\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 \geq 0,$$
$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 \geq 0,$$
$$\alpha_2 + \alpha_6 \geq 0.$$

Adding up these inequalities, we have

$$3\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6 \geq 0.$$

Since $\alpha_4 \geq 0$ and $\alpha_6 \geq 0$ by Lemma 3.5 (4), it follows that

$$\dim \eta = 3\alpha_1 + 2(\alpha_2 + \cdots + \alpha_6) \geq 0.$$

Hence $\dim V \leq \dim W$.

Remark. For a general n, it does not seem that the above argument works well though many other inequalities as in Lemma 3.5 exist.

Proposition 3.4 gives some information about $c_{SO(3)}(n)$ or $d_{SO(3)}(n)$ for lower n. For example,

Example 3.6. $d_{SO(3)}(n) = n$ for $n \leq 15$ $(n \in \mathcal{D}_{SO(3)})$.

Proof. When $n \leq 14$, $d_{SO(3)}(n) = n$ follows directly from Proposition 3.4. If $d_{SO(3)}(15) < 15$, there is a G-isovariant G-map $f : S(V) \to S(W)$ for some V, W $(V^G = W^G = 0)$ such that $\dim W < \dim V = 15$, hence W does not include U_k, $k > 6$, by dimensional reason. Since $\alpha_7 = b_7 - a_7 \geq 0$ by Lemma 3.5 (4), V does not also include U_7. Hence $d_{SO(3)}(15) = 15$ by Proposition 3.4.

By a similar argument we also have

Example 3.7. $c_{SO(3)}(n) = n$ for $n \leq 15$ $(n \in \mathcal{D}_{SO(3)})$.

Remark. By a further argument, one can see that the above equality holds for some more large integers. The detail is left to the readers.

Finally we pose

Conjecture. $c_G(n) = d_G(n) = n$ *for each* $n \in \mathcal{D}_G$ *when* $G = SO(3)$.

References

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka 560-0043, Osaka, Japan

E-mail address: nagasaki@math.sci.osaka-u.ac.jp