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EQUIVARIANT APPROXIMATION TO EQUIVARIANT LOOP SPACES

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Abstract. The "approximation theorem" states that the n-fold loop space $\Omega^n\Sigma^n X$ can be approximated by the configuration space of finite sets in R^n parametrized by X. An equivariant analogue of the approximation theorem holds when X has a finite group action. But this type of approximation theorem no longer holds in the case of positive dimensional Lie transformation groups. In this paper we shall introduce an equivariant configuration space $C(V, X)$ of "smooth submanifolds" (instead of "finite sets") in the orthogonal G-module V parametrized by a countable G-CW complex X and show that there is a weak G-equivalence $C(V, X) \simeq \Omega^V\Sigma^V X$ at least if V contains an infinite-dimensional trivial G-module.

1. Introduction

Let $C(\mathbb{R}^n, X)$ be the configuration space of finite point sets in $\mathbb{R}^n$ $(1 \leq n \leq \infty)$ parametrized by a pointed space X; that is,

$$C(\mathbb{R}^n, X) = \{(c, x)\},$$

where c is a finite subset of $\mathbb{R}^n$ and $x\colon c \to X$ is a map. But (c, x) is identified with (c', x') if $c \subset c'$, $x'|c = x$, and $x'(p) = *$ when $p \notin c$. Then the classical "approximation theorem" states that

Theorem 1 (May, Segal). *There exists an approximation map*

$$C(\mathbb{R}^n, X) \to \Omega^n\Sigma^n X,$$

which is an equivalence if X is path-connected and in general is a group-completion. (When $n = \infty$ this yields a form of the Barratt-Priddy-Quillen theorem.)

The aim of this work is to establish an equivariant generalization of the theorem above in the compact Lie case. More precisely, we shall construct a sort of "equivariant configuration space" $C(V, X)$ and a weak G-equivalence

$$C(V, X) \simeq_G \Omega^V\Sigma^V X,$$

where G is a compact Lie group, V is an orthogonal G-module containing the trivial G-module $\mathbb{R}^\infty$, and X is a pointed G-space.

Remark 2. (a) When G is finite, this can be achieved by taking $C(V,X)$ to be the usual configuration space of finite point sets in V parametrized by X, and letting G act on $C(V,X)$ in the obvious manner.

(b) J. Caruso and S. Waner [1] gives a group-completion

$$C_G(V,X) \to (\Omega^V\Sigma^V X)^G,$$

where V is an orthogonal G-module such that $V \geq \mathbb{R}^\infty$ and $C_G(V,X)$ is the configuration space of finite G-orbits in V parametrized by a pointed G-space X.

But this is definitely a non-equivariant result and never imply "equivariant approximation," for there exists no reasonable G-equivariant model $C(V,X)$ satisfying

$$C(V,X)^H \simeq C_H(V,X), \quad H \leq G.$$

(c) Our approximation theorem implies that

$$C(V,X)^H \simeq (\Omega^V\Sigma^V X)^H, \quad H \leq G$$

holds for any (not necessarily G-connected) X, and is related to Caruso-Waner's result via group-completion maps

$$C_H(V,X) \to C(V,X)^H, \quad H \leq G.$$

(d) Caruso and Waner [loc. cit.] asked:

Can we construct a manageable global model $C(W,X)$ so that

$$(C(W,X))^H = C_H(W,X) \textit{ for all } H \leq G$$

(as for the case where G is finite)?

The previous remark says that the answer is YES if we replace $C_H(W,X)$ by its (naturally constructed) group-completion. But the answer will be NO if we stick to Caruso-Waner's $C_H(W,X)$.

2. The space $C(V,X)$

Definition 3. Given an orthogonal G-module V and a pointed G-space X let $C(V,X)$ denote the set of pairs (P,f), where P is a *smooth submanifold* of V and f is a map $P \to X$; but (P_0,f_0) is identified with (P_1,f_1) if there exists a submanifold $P \subset P_0 \cap P_1$ such that

$$P_i - f_i^{-1}(*) \subset P \ (i=0,\,1), \quad f_0|P \equiv f_1|P.$$

Here the closure $\overline{P}$ of P should be a compact smooth submanifold, with possible corners, such that $\overline{P} - P$ is a closed submanifold of $\partial\overline{P}$. Furthermore every component of $\overline{P}$ should be of finite-dimensional, although different components may have different dimensions.

To define a topology on $C(V,X)$ let $\mathcal{P}$ be the set of pairs (K,L) consisting of a finite polyhedron $K \subset \mathbb{R}^\infty$ and its subpolyhedron L, and consider the space

$$B(V,X) = \coprod_{(K,L)\in\mathcal{P}} \{(K,L)\} \times C^\infty(K,V) \times \mathrm{Map}(K,X)/\sim .$$

Here $C^\infty(K,V)$ is the space of piecewise differentiable maps from K to V, and $\sim$ is the least equivalence relation such that

$$((K_0,L_0),i_0,f_0) \sim ((K_1,L_1),i_1,f_1)$$

if there exists a simplicial map $\varphi\colon K_0 \to K_1$ satisfying the following conditions:

(C1) For $\epsilon = 0$, 1 let $i_\epsilon \ltimes f_\epsilon$ denote the composite

$$K_\epsilon \xrightarrow{(i_\epsilon,f_\epsilon)} V \times X \to V \times X/V \times * = V \ltimes X.$$

Then we have

$$\begin{aligned} i_0 \ltimes f_0(K_0) - i_0 \ltimes f_0(L_0) \subset i_1 \ltimes f_1(K_1) - i_1 \ltimes f_1(L_1) \\ \subset i_1 \ltimes f_1(K_1 - L_1) \subset i_0 \ltimes f_0(K_0 - L_0) \end{aligned}$$

(C2) The maps

$$\begin{aligned} K_0 - L_0 - \varphi^{-1}(L_1) &\to K_1 - L_1, \\ \varphi^{-1}(L_1) &\to L_1 \cap \varphi(K_0), \\ \varphi^{-1}(L_1) \cap L_0 &\to L_1 \cap \varphi(K_0) \end{aligned}$$

induced by φ are "contractible," in the sense that the inverse image of a point in the target space is always a compact contractible set.

Definition 4. We denote by $C(V,X)'$ the subspace of $B(V,X)$ consisting of those classes $[(K,L),i,f]$ where $i\colon K \to V$ is an embedding such that $i(K)$ is a smooth manifold and $i(L)$ is a closed submanifold of $\partial i(K)$.

With respect to the action

$$g[(K,L),i,f]) = [(K,L),gi,gf], \quad g \in G$$

$C(V,X)'$ is a pointed G-space with basepoint $\emptyset$.

By (C1), (C2) and the "Hauptvermutung" for smooth manifolds we see that the correspondence

$$((K,L),i,f) \mapsto (i(K) - i(L), fi^{-1})$$

induces a well-defined bijection

$$C(V,X)' \cong C(V,X),$$

hence $C(V,X)$ can be regarded as a pointed G-space.

Clearly, the correspondence $X \mapsto C(V,X)$ defines a G-equivariant continuous functor of the category of pointed G-spaces and pointed maps, with G acting by conjugation, to itself.

Proposition 5. (a) $C(V,-)$ *preserves G-homotopy.*
(b) *If A is a pointed G-NDR of X then*

$$C(V,A) \to C(V,X) \to C(V, X \cup CA)$$

is a G-homotopy fibration sequence.

Observe that if X has a disjoint basepoint then for any subgroup H the fixpoint set $C(V,X)^H$ can be identified with the set of pairs (P,f), where P is an H-invariant submanifold of V and $f\colon P \to X$ is an H-equivariant map such that $* \notin f(P)$.

For general X, we can study $C(V,X)$ by using the G-homotopy fibration sequence

$$C(V,S^0) \to C(V,X_+) \to C(V, X \cup CS^0) \simeq_G C(V,X).$$

Here $S^0 \to X_+ = X \cup S^0$ is the pointed map which takes the non-basepoint of S^0 to the original basepoint of X (which is assumed to be non-degenerate).

3. The group $\pi_0 C(V,S^0)^G$

Let us write $C(V) = C(V,S^0)$. Then each element of $C(V)^G$ can be identified with a G-invariant smooth submanifold of V. For given P, $Q \in C(V)^G$ we write $P \sim Q$ if they belong to the same path-component of $C(V)^G$, i.e. $[P]=[Q]$ in $\pi_0 C(V)^G$.

Example 6. Show that the following holds in $C(\mathbb{R}^n)$.

(a) $[0,1) \sim \emptyset$. In fact $\lim_{t\to 1}[t,1) = \emptyset$ in $C(\mathbb{R}^\infty)$. Here

$$(K_0,L_0) = ([0,1],\{1\}), \quad (K_1,L_1) = (\{1\},\{1\}),$$

and $\varphi\colon [0,1] \to \{1\}$ is the evident map.

(b) $[0,1) \sim S^1$. Here

$$(K_0,L_0) = ([0,1],\{1\}), \quad (K_1,L_1) = (\partial\Delta^2,\emptyset),$$

and $\varphi\colon [0,1] \to \partial\Delta^2 \cong S^1$ is the exponential map.

(c) Let us write $P = K - L$ where K is a smooth triangulation of $\overline{P}$ and L is a subcomplex of K. Let Λ be the set of open cells contained in $K - L$. Then P is equivalent to the disjoint union $\coprod_{\sigma\in\Lambda}\sigma$.

Let B^n denote the n-dimensional open ball. Then $B^2 \sim B^2 \cup S^1 = \bar{B}^2 \sim$ point, and hence

$$B^{2n} \sim \text{point}, \quad B^{2n+1} \sim B^1.$$

Thus even-dimensional open balls are equivalent with each other, and similarly for odd-dimensional open balls.

On the other hand, even-dimensional open balls are never equivalent to odd-dimensional open balls. For we have

$$\chi^c(B^{2n}) = 1, \quad \chi^c(B^{2n+1}) = -1,$$

where $\chi^c(-)$ denotes the Euler characteristic computed with Alexander-Spanier cohomology with compact support. Note that if P has a cellular decomposition then its Euler characteristic is given by the formula

$$\chi^c(P) = \sum\nolimits_{n\geq 0}(-1)^n b_n$$

where b_n is the number of open n-cells in the decomposition of P. It follows by (C2) that $\sim$ preserves Euler characteristics, hence

$$\chi^c(P) \neq \chi^c(Q) \quad \text{implies} \quad P \not\sim Q.$$

Now we have a well-defined homomorphism

$$\pi_0 C(\mathbb{R}^n) \to \mathbb{Z}, \quad [P] \mapsto \chi^c(P)$$

As every P (or more precisely, its closure $\overline{P}$) admits a smooth triangulation, P is equivalent in $C(\mathbb{R}^n)$ to the union of m distinct points and n distinct open intervals, where $m - n = \chi^c(P)$. Hence we can show

Proposition 7. *The correspondence $P \mapsto \chi^c(P)$ induces an isomorphism*

$$\pi_0 C(\mathbb{R}^n) \cong \mathbb{Z}, \quad 1 \leq n \leq \infty.$$

For general G we can use the G-CW decomposition of smooth G-manifold to show that there is a well-defined monomorphism:

$$\Phi\colon \pi_0 C(V)^G \to \bigoplus\nolimits_{(H)} \mathbb{Z}, \quad \Phi([P]) = (\chi^c(P^H))$$

Here (H) ranges over conjugacy classes of closed subgroups of G such that $|NH : H| < \infty$.

Proposition 8. *Let G be a compact Lie group and V an orthogonal G-module. If V is sufficiently large then Φ induces an isomorphism of $\pi_0 C(V)^G$ to the Burnside ring $A(G)$.*

Proof. It suffices to show that the image of Φ coincides with the image of the inclusion $A(G) \subset \bigoplus_{(H)}\mathbb{Z}$. By definition, elements of $A(G)$ are the equivalence classes of closed G-manifolds. Hence $A(G) \subset \mathrm{Im}\Phi$. Conversely, if $P \in C(V)^G$ then by attaching $(\overline{P} - P) \times S^1$ to P along $\overline{P} - P$ we obtain a compact G-ENR whose H-fixpoint set has the same Euler characteristic as P^H. By the alternative description of $A(G)$ as the set of equivalence classes of compact G-ENR's not just closed manifolds, we see that P represents an element of $A(G)$. □

4. Statement of the main result

If W is a finite-dimensional G-module let $\overline{C}(W,X)$ denote the space of "thick submanifolds" in W parametrized by X, i.e. $\overline{C}(W,X)$ consists of pairs (ν, f) where ν is an ε-neighborhood of some $P \subset W$ and $(P,f) \in C(W,X)$. Then there is a diagram of pointed G-spaces

$$C(W,X) \xleftarrow{\gamma_W} \overline{C}(W,X) \xrightarrow{\alpha_W} \Omega^W\Sigma^W X,$$

where $\gamma_W(\nu,f) = (P,f)$ and $\alpha_W(\nu,f)$ is the composite

$$S^V \to T\nu \to T(\nu \oplus \tau) \cong S^V P_+ \to S^V X.$$

If V is the direct limit of its finite-dimensional subspaces then we define $\overline{C}(V,X) = \lim \overline{C}(W,X)$, and

$$\gamma_V = \lim \gamma_W \colon \overline{C}(V,X) \to C(V,X) = \lim C(W,X),$$
$$\alpha_V = \lim \alpha_W \colon \overline{C}(V,X) \to \Omega^V\Sigma^V X = \lim \Omega^W\Sigma^W X,$$

where W ranges over finite-dimensional G-subspaces of V.

Now we have a diagram of pointed G-spaces

$$C(V,X) \xleftarrow{\gamma_V} \overline{C}(V,X) \xrightarrow{\alpha_V} \Omega^V\Sigma^V X,$$

and the main result can be stated as follows:

Theorem 9. *Let X be a countable G-CW complex. If V contains an infinite-dimensional trivial G-module $\mathbb{R}^\infty$ then both γ_V and α_V are weak G-equivalences, hence*

$$C(V,X) \simeq_G \Omega^V\Sigma^V X.$$

5. Outline of the proof

We need to show that for any closed subgroup $H \leq G$ the arrows γ^H and α^H in the diagram below are weak equivalences.

$$C(V,X)^H \xleftarrow{\gamma^H} \overline{C}(V,X)^H \xrightarrow{\alpha^H} (\Omega^V\Sigma^V X)^H$$

But the argument for the case $H = G$ automatically applies to general H. Hence we need only treat the case $H = G$. Also, as γ^G is clearly an equivalence, we shall concentrate on α^G.

Now the proof consists of two parts:

(1) Apply the standard argument using orbit-type families to reduce the problem to the non-equivariant case, that is, the case $G = e$.
(2) Validate the non-equivariant case.

Part 1. If $\mathcal{F}$ is an orbit-type family, let $C(V,X)^G_{\mathcal{F}}$ denote the subspace of $C(V,X)^G$ consisting of those elements (P,f) such that all the conjugacy classes of isotropy subgroups of the points of P belong to $\mathcal{F}$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be orbit-type families such that $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_2 - \mathcal{F}_1$ consists of just one conjugacy class (H). Let NH be the normalizer of H in G. Then both V^H and X^H are NH-spaces, and there is a diagram

$$\begin{array}{ccccc} \overline{C}(V,X)^G_{\mathcal{F}_1} & \xrightarrow{i} & \overline{C}(V,X)^G_{\mathcal{F}_2} & \xrightarrow{p} & \overline{C}(V^H,X^H)^{NH}_{\mathcal{F}_2} \\ \downarrow{\scriptstyle \alpha^G} & & \downarrow{\scriptstyle \alpha^G} & & \downarrow{\scriptstyle \alpha^{NH}} \\ (\Omega^V\Sigma^V X)^G_{\mathcal{F}_1} & \xrightarrow{i'} & (\Omega^V\Sigma^V X)^G_{\mathcal{F}_2} & \xrightarrow{p'} & (\Omega^{V^H}\Sigma^{V^H} X^H)^{NH}_{\mathcal{F}_2} \end{array}$$

in which both rows are homotopy fibration sequences.

Therefore, if we can show that α^{NH} is an equivalence then we can proceed by induction with respect to some cofinal sequence of adjacent families

$$\{1\} \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \cdots$$

But if (H) is maximal in $\mathcal{F}$ we can construct a commutative diagram

$$\begin{array}{ccc} \overline{C}(V^H,X^H)^{NH}_{\mathcal{F}} & \xrightarrow{\simeq} & \overline{C}(\mathbb{R}^\infty, EJ_+ \wedge_J S^L X^H) \\ \downarrow{\scriptstyle \alpha^{NH}} & & \downarrow{\scriptstyle \alpha} \\ (\Omega^{V^H}\Sigma^{V^H} X^H)^{NH}_{\mathcal{F}} & \xrightarrow{\simeq} & \Omega^\infty\Sigma^\infty(EJ_+ \wedge_J S^L X^H) \end{array}$$

where $J = NH/H$ and L is the Lie algebra of J. *Thus everything can be reduced to the non-equivariant case.*

Part 2. We need to show that

$$\alpha\colon \overline{C}(\mathbb{R}^\infty, X) \to \Omega^\infty\Sigma^\infty X$$

is a weak equivalence for any X.

A submanifold of $\mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R}$ is called a "vertical interval" if it is of the form $\{v\}\times J$, where $v \in \mathbb{R}^{n-1}$ and $J \subset \mathbb{R}$ is a bounded interval. Let $I(\mathbb{R}^n, X)$ be the subset of $C(\mathbb{R}^n, X)$ consisting of those elements (P,f) such that P is the disjoint union of finite vertical intervals in $\mathbb{R}^n$. Similarly, let $\overline{I}(\mathbb{R}^n, X)$ be its thickened version. Then there is a natural equivalence $\overline{I}(\mathbb{R}^n,X) \to I(\mathbb{R}^n,X)$.

Lemma 10. *The inclusion* $I(\mathbb{R}^\infty,X) \to C(\mathbb{R}^\infty,X)$ *is a weak equivalence, hence so is* $\overline{I}(\mathbb{R}^\infty,X) \to \overline{C}(\mathbb{R}^\infty,X)$.

Consequently, the main theorem is a consequence of the following result due to S. Okuyama [2].

Theorem 11 (S. Okuyama). *Let* $1 \le n \le \infty$. *For any pointed space* X,

$$\alpha\colon \overline{I}(\mathbb{R}^n,X) \to \Omega^n\Sigma^n X$$

is a weak equivalence.

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