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Group configurations in simple theories (Part.1)

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1. Introduction

Group configuration theorem is one of the most important theorems of geometric stability theory. This theorem is proved in full generality by Hrushovski [1] following ideas of Zilber. It states that if some dependence / independence situation exists, there is a non-trivial group behind it. Recently, this theorem is generalized to the simple theory context by Wagner et al [2] [3]. In this note, we introduce the results.

In stable case, the proof can be decomposed into two main steps ;

1. Obtain a generic group chunk whose elements are germs of generic functions, and whose product is the composition.
2. Apply the Weil-Hrushovski generic group chunk theorem.

In simple theories, the theorem is also proved through the two steps. There are two papers accordingly. In this note, we give a summary of the next paper at first [2] ;

"Group configurations and germs in simple theories." by Itay Ben-Yaacov. This paper is concerned with the generalization of the first step in the stable case. But they could not obtain a generic group chunk this time. They construct a generic polygroup chunk, that is a generic group chunk whose product is defined up to a bounded number of possible values.

2. Germs of generic actions

Alike in the stable case, generic functions and their compositions are treated from now on. But in the simple context, compositions of complete types (as generic functions) are not necessarily complete. It is due to the lack of stationarity of types. Therefore generic functions (actions) are defined as partial types. And their compositions are also partial types.

Definition 1 A partial type $\pi(x)$ over A has *definable independence* if for any partial type $\pi'(y)$ over A, "$\pi(x) \wedge \pi'(y) \wedge x \mathop{\smile\!\!\!\!\!\!\:|}_A y$" is type-definable.

Remark 2 Every complete type has definable independence. If $\pi(x)$ has definable independence and E is a hyperdefinable equivalence relation, then $\pi(x)/E$ has definable independence.

Definition 3 Let $\pi(x, y, z)$ be a partial type in three hyperimaginary variables over a hyperimaginary parameter. We say that π is a *generic action*, if :
1. $\pi\lceil_x$, $\pi\lceil_y$ and $\pi\lceil_z$ have definable independence.
2. $\pi(x, y, z)$ implies that x, y, z are pairwise independent.
3. For any f, a there is at most boundedly many b such that $\pi(f, a, b)$, that is $\pi(f, a, z)$ is a bounded (possibly inconsistent) type. We note $f(a)$ the set of all such b.

We note $\mathrm{Func}(\pi) = \pi\lceil_x$, $\mathrm{Arg}(\pi) = \pi\lceil_y$, $\mathrm{Val}(\pi) = \pi\lceil_z$, namely the functions, arguments and values of π. If f is a function, we note $\mathrm{Gr}(f)(y, z) = \pi(f, y, z)$. Note that $f \in \mathrm{Func}(\pi) \iff \mathrm{Gr}(f) \neq \emptyset$.

Definition 4
1. We say that π is *trivial* if $\pi(x, y, z)$ implies that x, y, z are an independent triplet.
2. We say that π is *invertible* if every function sends at most boundedly many arguments to any given value.
3. We say that π is *complete* if for any $f \in \mathrm{Func}(\pi)$, $\mathrm{Gr}(f)$ is a Lascar strong type. (i.e. an amalgamation base) over f.
4. We say that π is *reduced* if it is complete, and whenever $\mathrm{Gr}(f)$ and $\mathrm{Gr}(g)$ have a common non-forking extension, then $f = g$.

Definition 5 Suppose that $\pi(x, y, z), \pi'(t, z, w)$ are generic actions. Then we define $\pi' \circ \pi(xt, y, w)$ to be the partial type such that $\pi' \circ \pi(fg, a, c)$ if and only if:
1. f, g, a are independent.
2. $c \in g \circ f(a)$, that is, there is b such that $b \in f(a)$ and $c \in g(b)$.

Proposition 6 *$\pi' \circ \pi$ always exists (provided that the sorts match) and it is a generic action.*

Proof. $\pi' \circ \pi$ is the partial type "$x \mathop{\smile\!\!\!\!\!\!|}\, t \wedge \exists z\,[y \mathop{\smile\!\!\!\!\!\!|}\, xt \wedge \pi(x, y, z) \wedge \pi(t, z, w)\,]$"∎

Proposition 7
1. *Suppose π is a generic action, and note $\pi^{-1}(x, y, z) = \pi(x, z, y)$. Then π is invertible if and only if π^{-1} is a generic action.*
2. *Any composition of two invertible functions is invertible, and $(\pi' \circ \pi)^{-1} = \pi^{-1} \circ \pi'^{-1}$.*

Proposition 8 *Let π, π' be generic actions on sorts such that $\pi' \circ \pi$ is defined. Suppose furthermore that Arg(π') =Val(π) and these are Lascar strong types. Then for any independent f $\in$Func(π) and g $\in$Func(π'), we have $g \circ f$ $\in$Func($\pi' \circ \pi$). If furthermore π' is non-trivial, the so is the composition.*

Proof. Now $f \perp g$ and Arg(π') =Val(π) is a Lascar strong type. By the independence theorem, there is b such that $b \perp fg$ and $b \models$Arg(π') =Val(π). Thus there are a and c such that $b \in f(a)$ and $c \in g(b)$. We may assume that $a \perp_{bf} g$. By $b \perp fg$, $a \perp fg$. Then $g \circ f$ is defined on a. ∎

As we saw above, the composition of two complete actions need not be complete. Thus graphs need not have the same type. However the passage to germs requires the action to be complete. In case it is not, we construct its "completion". Every graph of a function is replaced by Lascar strong types over the same parameter.

Fact 9 *For any two variables x and y, let $LS(x,y,x',y')$ be the partial type saying that $y = y'$ and there are y-indiscernible sequence $(x_i : i < \omega)$ and $(x'_i : i < \omega)$ such that $x = x_0, x' = x'_0$ and $x_1 = x'_1$.*
Then we have (taking LS on the right sorts) :
1. LS is a (type-definable) equivalence relation on the sort of x, y.
2. $\mathrm{lstp}(a/b) = \mathrm{lstp}(a'/b)$ *if and only if* $LS(a,b,a',b)$.

Definition 10 Let $\pi(x,y,z)$ be a generic action. Consider the hyperdefinable equivalence relation $LS(yz,x,y'z',x')$ from Fact 9 on the sort of yz, x. Note the quotient sort by $\underline{x} = (yz,x)/LS$. An element of this sort can be viewed as a pair that we shall note f_p, where f is an element in the sort of x, and p is a Lascar strong type over f in the variables yz. Now let $\underline{\pi}(\underline{x},y,z)$ be defined as :

$$\underline{\pi}((y'z',x)/LS,y,z) = \pi(x,y,z) \wedge LS(yz,x,y'z',x)$$

So $\models \underline{\pi}(f_p,a,b)$ if and only if $b \in f(a)$ and $\mathrm{lstp}(ab/f) = p$.
We call $\underline{\pi}$ the *completion* of π. For a function f, we write $\underline{f} = \{(ab,f)_{LS} : b \in f(a)\} = \{f_{lstp(ab/f)} : b \in f(a)\}$, that is the set of consistent completions of f, or the set of extensions of Gr(f) to a complete Lascar strong type over f. Note that this is a bounded hyperdefinable set.

Remark 11 As $\underline{x}$ is a bounded extension of x, all the properties of independence and definable independence are preserved.

After the "*completion*", we can pass to germs. This procedure, called "*reduction*", is essentially the same as in the stable case. We replace each function with the canonical base for its graph.

Definition 12 Let π be a complete generic action. We can construct its "*reduction*" as follows : If $f, g \models$Func(π), define $f \sim g$ if $\pi(f,x,y)$ and $\pi(g,x,y)$ have a common non-forking extension. By the usual argument of simple theory, the transitive closure of $\sim$ is a type-definable equivalence relation, noted $E(x,y)$. Putting $\bar{f} = f_E$ and $\bar{\pi}(\bar{x},y,z) = \exists x'[\pi(x',y,z) \wedge E(x,x')]$, we obtain $\models \bar{\pi}(\bar{f},a,b)$ if and only if $\models \pi(f,a,b)$ and $\bar{f}$ =Cb(ab/f). We call $\bar{f}$ the *germ* of a function f. If π is not complete, $\hat{f}$ is the set of germs of all the completions of f. That is $\hat{f} = \{\bar{f'} : f' \in \underline{f}\}$. The set of germs of π is Germ(π) =Func($\bar{\pi}$).

Definition 13 1. We say that $\pi(x,y,z)$ and $\pi'(x',y,z)$ are *isomorphic* if there is a (hyper)definable bijection $\tilde{\phi}$: Func(π) $\to$ Func(π') such that Gr(f) =Gr($\phi(f)$) for every $f \in$Func(π).
2. We say that two generic actions are *equivalent* if their reductions are isomorphic. We note it $\pi \approx \pi'$.

3. Elimination in compositions

We pass to considering compositions. In order to get a suitable set of germs, we need a generic action $\hat{\pi}$ such that a germ of the composition $\hat{\pi} \circ \hat{\pi}$ is also a germ of $\hat{\pi}$. In stable case this is done by defining $\hat{\pi} = \pi^{-1} \circ \pi$. For this $\hat{\pi}$, it is proved that the two middle terms can be eliminated from the composition $(\pi^{-1} \circ \pi) \circ (\pi^{-1} \circ \pi)$. The stable proof fails in the simple case since a non-forking extension of a Lascar strong type is not necessarily Lascar strong. Therefore they defined the technical notion of a generic action being strong on the left or on the right. And they realized the required "elimination" under some strong assumption.

Lemma 14 *Let $f \in$Func(π), $g \in$Func(π'), $f \perp g$, and $h \in$Germ($\pi' \circ \pi$). Suppose furthermore that $abc \models$Gr(f)$(x,y)\cup$Gr(g)$(y,z)\cup$Gr(h)(x,z). Then the following are equivalent :*
1. $h \in \mathrm{bdd}(fg)$ and $af \perp_b cg$.
2. $h \in \mathrm{bdd}(fg)$ and $ac \perp_h fg$.
3. $a \perp fgh$.

Proof. 1 $\Longrightarrow$ 3, 3 $\Longrightarrow$ 2, and 2 $\Longrightarrow$ 1 are proved in turn by forking calculation and property of canonical base. ∎

Definition 15 When *abc* satisfy the conditions of Lemma 14 (that is, all of the initial assumptions as well as any of the equivalent conditions 1 $\sim$ 3), we say that they *witness* that $h \in \widehat{g \circ f}$.

Lemma 16 *Under the hypothesis of Lemma14, $h \in \widehat{g \circ f}$ if and only if there are witnesses to it.*

Definition 17 1. We say that a generic action π is *connected* if $\pi(x, y, z)$ is a complete type.

2. We say that a generic action π is *strong on the left(resp. on the right)* if $\pi(x, y, z)$ implies that $\mathrm{tp}(xz/y)$(*resp.* $\mathrm{tp}(xy/z)$) is a Lascar strong type.

3. We say that a composition $\pi' \circ \pi$ is *generic* if for every independent $f \in$Func(π), $g \in$Func(π'), and for every $h \in \widehat{g \circ f}$, h is independent both of f and g.

Lemma 18 *Consider a composition $\pi' \circ \pi$, and $f \in$Func(π), $g \in$Func(π'), $h \in$Germ$(\pi' \circ \pi)$.*
1. If π is invertible, and $f \perp g$, $f \perp h$, then : $h \in \widehat{g \circ f} \iff g \in \widehat{h \circ f^{-1}}$, and abc witness the first if and only if bac witness the second.
2. If π' is invertible, and $f \perp g$, $h \perp g$, then : $h \in \widehat{g \circ f} \iff f \in \widehat{g^{-1} \circ h}$, and abc witness the first if and only if acb witness the second.

Proof. 1. The first statement is equivalent to $a \perp fgh$, while the second to $b \perp fgh$. We can prove their equivalence easily.
2. Both statements are equivalent to $a \perp fgh$. ∎

We now see when a generic action and its inverse can be eliminated from a composition.

Definition 19 1. Let π, π', π'' be generic actions, π' invertible, on sorts such that the compositions $\pi'^{-1} \circ \pi$ and $\pi'' \circ \pi'$ exist and generic. Then we say that they form an *elimination context*.

2. Let π, π', π'' form an elimination context. Suppose $f \in$Germ(π'), $g_0 \in$ Germ(π), $g_1 \in$Germ(π''), $h_0 \in$Germ$(\pi'^{-1} \circ \pi)$, $h_1 \in$Germ$(\pi'' \circ \pi')$. Suppose that adb witness $h_0 \in \widehat{f^{-1} \circ g_0}$, and bdc witness $h_1 \in \widehat{g_1 \circ f}$. Suppose furthermore that $ag_0h_0 \perp_{bdf} cg_1h_1$. Then $abcdfg_0g_1h_0h_1$ form an *elimination diagram*.

3. Let π, π', π'' form an elimination context. For independent $h_0 \in$Germ $(\pi'^{-1} \circ \pi)$, $h_1 \in$Germ$(\pi'' \circ \pi')$, Elim(h_0, h_1) is the set of all the germs of $h_1 \circ h_0$ obtained by elimination diagrams, i.e. germs of lstp(ac/h_0h_1) taken from an elimination diagram $abcdfg_0g_1h_0h_1$.

Lemma 20 *Let π, π', π'' form an elimination context, and $abcdfg_0g_1h_0h_1$ be an elimination diagram for it. Then :*
1. abd witness $g_0 \in \widehat{f \circ h_0}$, and dbc witness $g_1 \in \widehat{h_1 \circ f^{-1}}$.
2. $g_0 \perp g_1$, $h_0 \perp h_1$, and $a \perp fg_0g_1h_0h_1$.
3. The germs of $g_1 \circ g_0$ and of $h_1 \circ h_0$ given by ac are the same.
4. Note this common germ h. Then abc witness $h \in \widehat{h_1 \circ h_0}$, and adc witness $h \in \widehat{g_1 \circ g_0}$.
5. h is independent of each of g_0, g_1, h_0, h_1.
6. $Elim(h_0, h_1) \subseteq \widehat{h_1 \circ h_0} \cap Germ(\pi'' \circ \pi)$.

Sketch of the proof.
From Lemma 18 and the genericity of $\pi'^{-1} \circ \pi$ and $\pi'' \circ \pi'$. These are proved by the argument about witness and forking calculation. ∎

Theorem 21 *Let π, π', π'' form an elimination context and suppose furthermore that π' is connected and strong on the left.*

1. *For independent $h_0 \in Germ(\pi'^{-1} \circ \pi)$, $h_1 \in Germ(\pi'' \circ \pi')$:*
   $Elim(h_0, h_1) = \widehat{h_1 \circ h_0}$
2. *The composition $(\pi'' \circ \pi') \circ (\pi'^{-1} \circ \pi)$ is generic, and :*
   $Germ((\pi'' \circ \pi') \circ (\pi'^{-1} \circ \pi)) \subseteq Germ(\pi'' \circ \pi)$
3. *If we further suppose that $= Val(\pi) \wedge Arg(\pi'') \vdash Val(\pi')$, we have equality, that is :*
   $(\pi'' \circ \pi') \circ (\pi'^{-1} \circ \pi) \ \approx \ \pi'' \circ \pi$

Sketch of the proof. 1. We show $\widehat{h_1 \circ h_0} \subseteq$Elim$(h_0, h_1)$. Let $h \in \widehat{h_1 \circ h_0}$, $h_0 \in \widehat{f^{-1} \circ g_0}$, $h_1 \in \widehat{g_1 \circ f'}$. We re-choose their witness by Lemma 18 and the independence theorem. And we can get an elimination diagram for $h \in$Elim(h_0, h_1).
2. By Lemma 20. 5, 6 and 1. above.
3. By 2. above, we show Germ$((\pi'' \circ \pi') \circ (\pi'^{-1} \circ \pi)) \supseteq$Germ$(\pi'' \circ \pi)$. Let $h \in \widehat{g_1 \circ g_0}$ where $g_0 \in$Func(π), $g_1 \in$Func(π'') and $g_0 \perp g_1$. After some argument, we can get an elimination diagram containing h_0 and h_1 where $h \in h_1 \circ h_0$, $h_0 \in$Germ$(\pi'^{-1} \circ \pi)$ and $h_1 \in$Germ$(\pi'' \circ = \pi')$. ∎

4. Generic multi-chunk

Definition 22 We say that a generic action π is a *generic multi − chunk* if π is reduced, Arg(π) is Lascar strong, π is invertible satisfying $\pi = \pi^{-1}$, and the composition $\pi \circ \pi$ is generic satisfying $\pi^2 \approx \pi$.

So Theorem 21 gives :

Corollary 23 *Let π be an invertible generic action, and let π' be possibly another invertible generic action which is connected and strong on the left, such that Arg(π) and Val(π) are Lascar strong, and $\pi^{-1} \circ \pi' \approx \pi'^{-1} \circ \pi \approx \pi^{-1} \circ \pi$ are all generic compositions (so Arg(π') =Arg(π)). Note $\hat{\pi} = \overline{\pi^{-1} \circ \pi}$. Then π, π', π^{-1} form an elimination context, and $\hat{\pi}$ is a generic multi-chunk. If π is non-trivial, so is $\hat{\pi}$.*

Proof. By Theorem 21 and Proposition 7, we can see by letting $\{\pi,\ \pi',\ \pi^{-1}\}$ be $\{\pi,\ \pi',\ \pi''\}$ of Theorem 21. ∎

Usually we would have $\pi' = \pi$.

We will show that a generic multi-chunk almost satisfies the hypothesis of the Weil-Hrushovski group chunk theorem, except that multiplication is many-valued. They call such a structure $\langle P,\ *\rangle$ a *generic polygroup chunk.*

Theorem 24 *Let π be a generic multi-chunk. Let P =Germ(π). Then the composition $\pi^2 \approx \pi$ induces a hyperdefinable function $* \ :\ P \times P \longrightarrow P$, which is defined up to a bounded non-zero number of possible values. This function satisfies the hypothesis of the generalized Hrushovski-Weil theorem [4], in the following sense :*

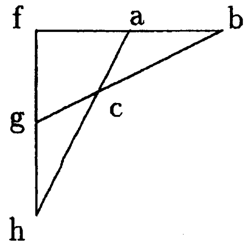
1. *Generic independence : If $f \mathop{\smile\!\!\!\!\!\mid} g$ and $h \in f * g$, then f, g, h are pairwise independent.*
2. *Generic associativity : Suppose f, g, h are independent. Then $f * (g * h) = (f\ *\ g) * h$ (as sets).*
3. *Generic surjectivity : For any independent f, g, there is h such that $g \in f\ *\ h$. Moreover, for any f, g, $h\ :\ g \in f\ *\ h \iff h \in f^{-1}\ *\ g$.*

Sketch of the proof. Let $f\ *\ g = \widehat{g \circ f}$.
1. Clear by definition.
2. Both sides of inclusion are proved by Lemma 14 and the independence theorem. It is too long to contain here.
3. Easily checked by Proposition 8 and Lemma 18. ∎

5. Quadrangle

Up to this time, we defined a generic function generally, and deduced a generic polygroup chunk. In stable case, a generic group chunk is obtained from group configuration, i.e. from some quadrangle structure. In simple theory, we can also start the argument from a quadrangle structure.

Definition 25 Let e be some hyperimaginary parameter, and (f, g, h, a, b, c) a tuple whose elements we put on a diagram as follows :



Then (f, g, h, a, b, c) is a *algebraic quadrangle* over e if it satisfies the following conditions :
1. Every non-collinear triplet is e-independent.
2. $\mathrm{bdd}(fge) = \mathrm{bdd}(fhe) = \mathrm{bdd}(ghe)$ (i.e. any two of f, g, h are e-interbounded over the third).
3. a, b are fe-interbounded, b, c are ge-interbounded, a, c are he-interbounded.
4. f is e-interbounded with $\mathrm{Cb}(ab/fe)(=\mathrm{Cb}(\mathrm{lstp}(ab/fe))$, g is e-interbounded with $\mathrm{Cb}(bc/ge)$, h is e-interbounded with $\mathrm{Cb}(ac/he)$.

Fact 26 *If (f, g, h, a, b, c) is an algebraic quadrangle over e as above, and (f', g', h', a', b', c') is such that each primed element is interbounded over e with the corresponding unprimed element, then (f', g', h', a', b', c') is also an algebraic quadrangle over e. In such a case we say that these quadrangles are algebraically equivalent over e.*

We obtain more or less immediately :

Theorem 27 *Let (f, g, h, a, b, c) be an algebraic quadrangle over e. Let $a' = \mathrm{dcl}(fabe) \cap \mathrm{bdd}(ae)$ and $b' = \mathrm{dcl}(fabe) \cap \mathrm{bdd}(be)$. Then (f, g, h, a', b', c) is algebraically equivalent over e to the original quadrangle. Take $\pi = \mathrm{lstp}(fa'b'/e)$. Then, π is strong on both sides, and it satisfies the assumptions of Corollary23, with $\pi = \pi'$, yielding a generic multi-chunk $\hat{\pi} = \overline{\pi^{-1} \circ \pi}$ (over $\mathrm{bdd}(e)$).*

References

[1] Ehud Hrushovski, *Contributions to stable model theory*, Ph.D.thesis, University of California at Berkeley, 1986.

[2] Itay Ben-Yaacov,
Group configurations and germs in simple theories, preprint, 2000.

[3] Itay Ben-Yaacov, Ivan Tomasic, and Frank O.Wagner, *Constructing an almost hyperdefinable group*, preprint, 2001.

[4] Frank O.Wagner, *Simple theories*, Kluwer Academic Publishers, 2000.