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Stability of Generic Pseudoplanes

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Problem(Baldwin[B1]) Is there any "generic" structure that is superstable but not ω-stable?

Theorem There is no δ-generic pseudoplane that is superstable but not ω-stable.

1 δ-Generic Pseudoplanes

Let $L=\{R(*,*)\}$ be a language of undirected graphs: It satisfies $\models \forall x(\neg R(x,x))$ and $\models \forall x\forall y(R(x,y)\rightarrow R(y,x))$. Let α be a positive real number. Then

- For a finite graph A, $\delta_{\alpha}(A):=|A|-\alpha|R^{A}|$, where $R^{A}=\{\{a,b\}:A\models R(a,b)\}$.
- $K_{\alpha}:=\{A:A$ is a finite graph, $\forall B\subset A[\delta_{\alpha}(B)\geq 0]\}$.

Definition Let A be a finite subgraph of a graph M
(i) We say A is *closed in* M (in symbol, $A\leq M$), if $\delta_{\alpha}(XA)\geq\delta_{\alpha}(A)$ for any finite $X\subset M-A$.
(ii) The *closure* of A in M, $\mathrm{cl}_{M}(A):=\bigcap\{B:A\subset B\leq M,|B|<\omega\}$.

To simplify our notation, we write $\delta(*)$ in place of $\delta_{\alpha}(*)$. For finite A,B, we write $\delta(A/B)=\delta(AB)-\delta(B)$.

Definition Let $K\subset K_{\alpha}$ be closed under subgraphs. Then a countable graph M is said to be $(K,\leq)$-*generic*, if it satisfies the following:
(i) If A is a finite subset of M, then $A\in K$;

(ii) If $A \leq B \in K$ and $A \leq M$, then there exists $B' \leq M$ such that $B' \cong_A B$.

Definition We say that a graph M is *δ-generic*, if M is $(K, \leq)$-generic for some α and some $K \subset K_\alpha$ such that
(1) M has finite closures (i.e., any finite subset of M has finite closures);
(2) M is saturated.

Definition A pseudoplane P is called *δ-generic*, if there is a δ-generic graph M with $P = (M, M, I)$ where an incidence relation xIy is defined by $R(x, y)$.

Example (i) Hrushovski's pseudoplanes ([H1]) are δ-generic, ω-categorical and strictly stable.
(ii) Baldwin's projective planes ([B2]) are δ-generic and $\aleph_1$-categorical.

Note 1.1 It is an open problem whether there is an ω-categorical projective plane or not (for instance, see [C], [Ho]). In [I], it is proven that there is no δ-generic ω-categorical projective plane.

Definition (i) Given a finite $A \subset M$, define $d_M(A) = \delta(\mathrm{cl}_M(A))$.
(ii) For finite A, B, write $d_M(A/B) = d_M(AB) - d_M(B)$. Define $d_M(A/B)$ for possibly infinite B to be $\inf\{d_M(A/B') : B' \subset B, B' \text{ is finite}\}$.

Fact 1.2 Let $A \leq B \leq M$ and $\bar{a} \in M$. Then $\mathrm{tp}(\bar{a}/B)$ does not fork over A if and only if $d_M(\bar{a}/B) = d_M(\bar{a}/A)$.

Fact 1.3 Let P be a δ-generic pseudoplane.
(i) $\mathrm{Th}(P)$ is stable;
(ii) If α is rational, then $\mathrm{Th}(P)$ is ω-stable.

2 Lemmas

Lemma 2.1 If $\alpha > 0$ is irrational, then $\sup\{d : d = a - b\alpha < 0, a, b \in \mathbf{N}\} = 0$.

Proof Let $X = \{a - b\alpha : a, b \in \mathbf{N}, a - b\alpha < 0\}$ and $Y = \{a - b\alpha : a, b \in \mathbf{Z}, a - b\alpha < 0\}$.
Claim: $\sup Y = 0$.

Proof: For each $k \in \mathbf{Z}$, let $f(k) = k\alpha - \max\{m \in \mathbf{Z} : m \leq k\alpha\}$. Take any $\epsilon > 0$. Since α is irrational, we have $f(k) \neq f(l)$ for any distinct $k, l \in \mathbf{Z}$. So there are distinct $i, j < \omega$ with $0 > f(i) - f(j) > -\epsilon$. Let $d = f(i) - f(j)$. Then we have $d \in Y$. Hence we have $\sup Y = 0$. (End of Proof of Claim)

We assume by way of contradiction that $\sup X = e < 0$. By the claim, there is a strictly monotone increasing sequence $\{d_n\}_{n<\omega}$ of elements of Y such that $\lim_{n\to\infty} d_n = 0$ and $d_n > e$ for each $n < \omega$. Then, for each $n < \omega$, $d_n \notin X$, and therefore we can write $d_n = b_n\alpha - a_n$ where $a_n, b_n \in \mathbf{N}$. Since $\{d_n\}_{n<\omega}$ is strictly monotone increasing, there is $m < \omega$ such that $b_{m+1} > b_m$. Now we have $0 > d_m - d_{m+1} > e$. On the other hand, since $b_{m+1} - b_m \in \mathbf{N}$, we have $d_m - d_{m+1} = (a_{m+1} - a_m) - (b_{m+1} - b_m)\alpha \in X$. This contradicts $\inf X = e$.

Lemma 2.2 If α is irrational with $0 < \alpha < 1$, then for any $\epsilon > 0$ there exists a sequence $\{q_n\}_{1\leq n\leq p}$ of $\mathbf{N}$ such that

(1) $0 > p - q_p\alpha > -\epsilon$;

(2) If $0 < n < p$ then $n - q_n\alpha > 0$;

(3) If $0 < n < m \leq p$ then $(q_m - q_n - 1)\alpha < m - n$.

Proof: By 2.1, for any $\epsilon > o$ there are $p, q < \omega$ with $0 > p - q\alpha > -\epsilon$. Let

$$q_n = \begin{cases} \max\{k \in \mathbf{N} : \alpha \leq \frac{n}{k}\} & \text{if } 0 < n < p \\ q & \text{if } n = p \end{cases}$$

By the definition of q_n, it is clear that (1) and (2) hold. To see (3), we prove two claims.

Claim 1: For any n, m with $0 < n < m \leq p$, $q_m - q_n - 1 \geq 0$.

Proof: By the definition of q_m, we have $\frac{m}{q_m+1} < \alpha$, so $q_m > \frac{n}{\alpha} - 1$. By the definition of q_n, we have $\alpha < \frac{n}{q_n}$, so $q_n < \frac{n}{\alpha}$. By our assumption, we have $0 < \alpha < 1$. It follows that $q_m - q_n - 1 > (\frac{m}{\alpha} - 1) - \frac{n}{\alpha} - 1 = \frac{m-n}{\alpha} - 2 > (m-n) - 2 \geq 1 - 2 = -1$. Hence $q_m - q_n - 1 \geq 0$.

Claim 2: For any n, m with $0 < n < m \leq p$, $(q_m - q_n - 1)\alpha < m - n$.

Proof: If $q_m - q_n - 1 = 0$ then clearly $(q_m - q_n - 1)\alpha < m - n$. So, by claim 1, we can assume that $q_m - q_n - 1 > 0$. By the definition of q_n and q_m, we have $\frac{n}{q_n+1} < \alpha < \frac{m}{q_m}$, so $mq_n - nq_m + m > 0$. Then we have $\frac{m-n}{q_m-q_n-1} - \frac{m}{q_m} = \frac{mq_n-nq_m+m}{(q_m-q_n-1)q_m} > 0$. So $\frac{m-n}{q_m-q_n-1} > \frac{m}{q_m} > \alpha$. Hence $(q_m - q_n - 1)\alpha < m - n$.

Definition Let $AB \in K_\alpha$ with $A \cap B = \emptyset$. Then we say that a pair (B, A) is *biminimal*,if it satisfies the following:

(i) $\delta(B/A) < 0$;
(ii) $\delta(X/A) \geq 0$ for any nonempty proper subset of B;
(iii) $\delta(B/Y) \geq 0$ for any nonempty proper subset of A.

We say that a graph A has *no loops*, if for each $n > 2$ there do not exist distinct $b_1, b_2, ..., b_n \in A$ with $R(b_1, b_2)$, $R(b_2, b_3)$,..., $R(b_{n-1}, b_n)$ and $R(b_n, b_1)$.

Lemma 2.3 If α is irrational with $0 < \alpha < 1$, then for any $\epsilon > 0$ there is a finite graph eBC such that
(1) (C, eB) is biminimal;
(2) $\delta(C/eB) > -\epsilon$;
(3) eBC has no loops;
(4) eB has no relations.

Proof: Take any $\epsilon > 0$. Then there is a sequence $\{q_n\}_{1 \leq n \leq p}$ satisfying (1)–(3) of 2.2. Let $q_0 = -1$. Let $\{c_i : 1 \leq i \leq p\} \cup \{b_i^j : 1 \leq i \leq p, 1 \leq j \leq q_i - q_{i-1} - 1\}$ be a graph with the relations:
(a) $R(c_1, c_2), ..., R(c_{n-1}, c_n)$;
(b) $R(c_i, b_i^j)$ for each i, j with $1 \leq i \leq p$ and $1 \leq j \leq q_i - q_{i-1} - 1$.

Let $e = b_1^1$, $C = \{c_i : 1 \leq i \leq p\}$ and $B = \{b_i^j : 1 \leq i \leq p, 1 \leq j \leq q_i - q_{i-1} - 1\} - \{b_1^1\}$. Clearly eBC satisfies (3) and (4). By the definition of eBC, we have

$$\delta(C/eB) = p - \left\{(p-1) + \sum_{i=1}^{p}(q_i - q_{i-1} - 1)\right\}\alpha = p - q_p\alpha.$$

By (1) of 2.2, we have $0 > \delta(C/eB) > -\epsilon$, so (2) holds.
Claim: If $X(\subset C)$ is connected with $X \neq C$, then $\delta(X/eB) > 0$.
Proof: Let $X = \{c_i\}_{n<i\leq m}$ for some n, m. If $n = 0$, then $\delta(X/eB) = m - q_m\alpha > 0$ by (2) of 2.2. If $n > 0$, then $\delta(X/eB) = (m-n) - (q_m - q_n - 1)\alpha > 0$ by (3) of 2.2. (End of Proof of Claim)

We show (1). Take any $X \subset C$ with $X \neq C$. Let $X = \bigcup X_i$ where each X_i is connected component of X. Then $\delta(X/eB) = \sum \delta(X_i/eB) > 0$ by the claim. Hence (1) holds.

Lemma 2.4 If α is irrational with $0 < \alpha < 1$, then for any $\epsilon > 0$ there is a sequence $\{eB_iC_i\}_{i<\omega}$ of finite graphs such that
(1) D has no loops;
(2) $B_n^* \leq eB_n^*C_n^* \leq D$ for each $n < \omega$;
(3) (C_n, eB_n) is biminimal for each $n < \omega$;

(4) eB^* has no relations;
(5) For each $i, j < \omega$ there is no relation between $B_i\bar{C_i}$ and B_jC_j,
where $B_n^* = \bigcup_{i\leq n} B_i, C_n^* = \bigcup_{i\leq n} C_i, B^* = \bigcup_{i<\omega} B_i, C^* = \bigcup_{i<\omega} C_i$ and $D = eB^*C^*$.

Proof For each $i < \omega$ there is eC_iB_i that satisfies $\delta(C_i/eB_i) > -\frac{1}{2^i}$ and (1)-(4) of 2.3. We can assume that (5) holds. Then (1), (3) and (4) hold. To see (2), we prove two claims. Let X_E denote $X \cap E$ for each X and E.
Claim 1: $eB_n^*C_n^* \leq D$.
Proof: Take any $X \subset D - eB_n^*C_n^*$. Then $\delta(X/eB_n^*C_n^*) = \delta(X/e) = \delta(X_{C^*}/eX_{B^*}) + \delta(X_{B^*}/e) = \delta(X_{C^*}/eX_{B^*}) + |X_{B^*}| \geq \delta(X_{C^*}/eX_{B^*}) + 1 = \sum_i \delta(X_{C_i}/eX_{B_i}) + 1 \geq -\sum_{i=1}^{\omega} \frac{1}{2^i} + 1 \geq 0$.
Claim 2: $B_n^* \leq eB_n^*C_n^*$.
Proof: Take any $X \subset eC_n^*$. To show $\delta(X/B_n^*) \geq 0$ we divide into two cases. Suppose $e \in X$. $\delta(X/B_n^*) = \delta(X/B_n^*e) + \delta(e/B_n^*) = \sum_{i=1}^{n} \delta(X_{C_i}/B_ie) + 1 \geq -\sum_{i=1}^{\omega} \frac{1}{2^i} + 1 \geq 0$.
Suppose $e \notin X$. By biminimality of (C_i, B_ie) it can be seen that $\delta(Y/B_i) > 0$ for any $Y \subset C_i$. So $\delta(X/B_n^*) = \sum_{i=1}^{n} \delta(X_{C_i}/B_i) > 0$.

3 Theorem

Lemma 3.1 Let $P = (M, M, I)$ be a δ-generic pseudoplane. Suppose that a finite graph $A \subset M$ has no loops. Then $A \in K$.

Proof Take any $a_0 \in A$. Let C_0 be a connected component of a_0 in A. As A has no loops, C_0 can be regarded as a tree with $height(a_0) = 0$. Since P is a pseudoplane, M satisfies

- For any $a \in M$ there are infinitely many $b \in M$ with $R(a, b)$;
- For any distinct $a, b \in M$ there are at most finitely many $c \in M$ with $R(a, c) \wedge R(b, c)$.

So, we can inductively construct $C_0^* \subset M$ with $C_0^* \cong C_0$. Take any $a_1 \in A - C_0$. Let C_1 be a connected component of a_1. In the same way, we have $C_1^* \subset M$ with $C_0^*C_1^* \cong C_0C_1$. Iterating this process, we have $A^* \subset M$ with $A^* \cong A$. Hence $A \in K$.

Lemma 3.2 Let $P = (M, M, I)$ be a δ-generic pseudoplane. Then $\alpha < 1$.

Proof Suppose that $\alpha \geq 1$. Take some $a \in M$ with $a \leq M$. Then there is no $b \in M$ with $R(a, b)$. This contradicts axioms of a pseudoplane. Hence $\alpha < 1$.

Theorem There is no δ-generic pseudoplane that is superstable but not ω-stable.

Proof Take any δ-generic pseudoplane $P = (M, M, I)$. Let M be a $(K, \leq)$-generic graph for some $K \subset K_\alpha$. By 1.3, if α is rational, then P is ω-stable. Thus it is enough to show that, if α is irrational then P is not superstable. By 3.2, we have $0 < \alpha < 1$. So we have a sequence $\{eB_iC_i\}_{i<\omega}$ satisfying (1)–(5) of 2.4. Let $D = \bigcup_{i<\omega} eB_iC_i$. Since D has no loops, any finite subset of D belongs to K by 3.1. By genericity of M, we can assume that $D \leq M$.
Claim: $d(e/B_n^*) = \sum_{i\leq n} \delta(C_i/eB_i) + 1$.
Proof: By (2)–(5) of 2.4, we have $d(e/B_n^*) = \mathrm{d}(eB_n^*) - d(B_n^*) = \delta(eC_n^*B_n^*) - \delta(B_n^*) = \delta(eC_n^*/B_n^*) = \delta(C_n^*/eB_n^*) + 1 = \sum_{i\leq n} \delta(C_i/eB_i) + 1$. (End of Proof of Claim)

For each $n < \omega$, $\mathrm{tp}(e/B_{n+1}^*)$ is a forking extension of $\mathrm{tp}(e/B_n^*)$, because $d(e/B_{n+1}^*) = d(e/B_n^*) + \delta(C_{n+1}/eB_{n+1}) < d(e/B_n^*)$ by the claim. Hence $\mathrm{Th}(M)$ is not superstable.

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