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# Regular surfaces with genus two fibrations after Horikawa
## — Obstructed surfaces with ample canonical bundle —

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## 0 Introduction

The purpose of the present note is to show that some surfaces of general type with a genus two fibration are obstructed, that is, the Kuranishi space of deformations is singular. More precisely, it has two irreducible components meeting normally one of which parametrizes surfaces with genus two fibrations while the other does not. Surfaces with such a property already appeared in his famous series of papers "Small $c_1^2$" by Horikawa and, indeed one of the highlights was to see that they form a bridge connecting realms of surfaces with weak canonical map and those with birational canonical map, that is, the other component corresponds to canonical surfaces (see [1], [3] and [4]). The calculations presented here are only a mimic of his, [1] among others, done as an exercise when I learned Horikawa's works at the begining of my research on surfaces of general type. This explains a reason why it has not been submitted to a journal for a long time, though I already completed around 1989. As time goes by, I become to think that it may have a certain meaning to gather my sporadicting notes and put them in the preprint format.

I want to emphasize here again the importance of the still misterious line $K^2 = 4p_g - 12$ appearing in Miles Reid's Quadric Hull Conjecture [6] to which I refereed several times in my papers, because our obstructed surfaces live in the region bounded from below by Reid's line and the "unknown" component of the Kuranishi space seems a new world of canonical surfaces.

**Main Theorem.** *Let $S$ be a minimal regular surface of general type whose numerical characters satisfy $K_S^2 < \min\{4p_g + 10, 5p_g + 2\}$ and $K_S$ is ample. Assume that $S$ has a genus two fibration with generic branch locus and that the canonical image of $S$ is a cone over a rational curve. Let $p : \mathscr{S} \to M$ be the Kuranishi family of deformations of $S$. Then*

(1) *$M = M_1 \cup M_2$, where the $M_i$'s are complex manifolds with $\dim M_1 = 11p_g + 6 - 2K_S^2$, $\dim M_2 = 10p_g + 10 - 2K_S^2$.*

(2) *$N = M_1 \cap M_2$ is a complex manifold of dimension $10p_g + 9 - 2K_S^2$.*

(3) *For $t \in M_1$, $S_t$ has a genus two fibration. For $t \in M_2 \setminus N$, $S_t$ does not have a genus two fibration.*

I do not know who live in $M_2 \setminus N$. Please let me know if you get acquainted with them !

# 1 Assumptions and the fundamental calculations

We refer the readers to [2] and [7] for the general theory of surfaces with a genus two fibration. In [2], singular fibres are classified into six types (0), (I) through (V).

Let $S$ be a minimal regular surface of general type over $\mathbb{C}$ with a genus two fibration $f : S \to \mathbb{P}^1$. Let $g : S \to W := \mathbb{P}(f_*\omega_{S/\mathbb{P}^1})$ be the relative canonical map which is a dominant rational map of degree two. The target surface $W$ is isomorphic to the Hirzebruch surface $\Sigma_n$ for some $n$. We denote by $\Delta_0$ and $\Gamma$ a minimal section and a fibre on $\Sigma_n$, respectively. Let $B$ denote the branch locus of $g$. We begin with the following lemma in which Reid's line appears naturally.

**Lemma 1.1.** *Let $S$ be a minimal regular surface of general type with a genus two fibration. Assume that $p_g \geq 4$ and that the canonical image of $S$ is a cone over a rational curve. Then $n = p_g - 2$. Furthermore, either $K_S^2 \geq 4p_g(S) - 12$ or $(p_g, K_S^2) = (5,6)$, $(5,7)$, $(6,8)$.*

*Proof.* See §1 of [3]. It follows from [2] that

$$K_S^2 - (2p_g - 4) = \sum_k (2k-1)\{\nu(I_k) + \nu(III_k)\} + \sum_k 2k\{\nu(II_k) + \nu(IV_k)\} + \nu(V)$$

where $\nu(A)$ stands for the number of singular fibres of type $A$. Put $B \sim 6\Delta_0 + 2(m+d+2)\Gamma$ on $\Sigma_n$. Then there is a divisor $\mathfrak{c}$ such that

$$p_g(S) = h^0(\Sigma_n, [\Delta_0 + (m - \deg\mathfrak{c})\Gamma]), \quad q(S) = h^1(\Sigma_n, [\Delta_0 + (m - \deg\mathfrak{c})\Gamma]).$$

The condition that the canonical image is a cone is equivalent to $m - \deg\mathfrak{c} = n$. Hence $p_g = n + 2$.

If we put

$$r = 2\deg\mathfrak{c} - (K_S^2 - 2p_g + 4) = \sum_k \{\nu(I_k) + \nu(III_k)\} + \nu(V),$$

then $B$ contains exactly $r$ fibres $\Gamma_1, \ldots, \Gamma_r$. Let $B_h = B - \Gamma_1 - \cdots - \Gamma_r$ be the horizontal part. Then we have $\Delta_0 B_h = -6n + 2(m+n+2) - r = K_S^2 - 4p_g + 12$. Assume that $K_S^2 < 4p_g - 12$. Then $\Delta_0 B_h < 0$ and it follows that $\Delta_0$ is a component of $B_h$. We set $C = B_h - \Delta_0$. Since $B_h$ has no multiple component, we must have $0 \leq \Delta_0 C = K_S^2 - 3p_g + 10$. Furthermore, since $\Delta_0$ is a component of $B_h$, we have $\nu(III_k) = \nu(IV_k) = 0$ and $\Delta_0 C \geq \sum_k (4k-2)\nu(I_k) + \sum_k 4k\nu(II_k) + 3\nu(V) = 2K_S^2 - 4p_g + 8 + \nu(V) \geq 2K_S^2 - 4p_g + 8$. It follows that $K_S^2 \leq p_g + 2$. Since we have Noether's inequality, $K_S^2 \geq 2p_g - 4$, we get $(p_g, K_S^2) = (5,6)$, $(5,7)$, $(6,8)$. □

Throughout the paper, we assume that the following two conditions are satisfied:

(A1) Any fibre of $f$ do not contain $(-2)$-curves, that is, the relative canonical bundle $K_{S/\mathbb{P}^1}$ is ample.

(A2) The canonical map of $S$ is a rational map of degree two onto its image which is a cone over a rational normal curve. In particular, $p_g(S) \geq 4$.

It follows from (A1) that any singular fibres of $f$ are of type (0) or $(I_1)$. We have $n = p_g(S) - 2$ by (A2). Write $K_S^2 = 2p_g - 4 + r$ with a non-negative integer $r$. Then $B$ is linearly equivalent to $6\Delta_0 + (4n + 4 + 2r)\Gamma$ on $\Sigma_n$. We can find $r$ distinct fibres $\Gamma_1, \ldots, \Gamma_r$ of $\Sigma_n$ such that they are contained in $B$ and, on each $\Gamma_i$, there exist two triple points $p_i^+$, $p_i^-$ of $B_h = B - \sum \Gamma_i$. We denote by $q : \tilde{W} \to \Sigma_n$ the blowing up with center $\{p_i^\pm\}_{i=1}^r$ and put $E_i^\pm = q^{-1}(p_i^\pm)$. Then the proper transform $\tilde{B}_h$ of $B_h$ is linearly equivalent to $q^*(6\Delta_0 + (4n + 4 + r)\Gamma) - 3E$ on $\tilde{W}$, where $E = \sum_i (E_i^+ + E_i^-)$. Let $\tilde{\Gamma}_i$ be the proper transform of $\Gamma_i$, and put $\tilde{B} = \tilde{B}_h + \sum_{i=1}^r \tilde{\Gamma}_i$. By (A1), $\tilde{B}$ is non-singular. We denote by $\tilde{g} : \tilde{S} \to \tilde{W}$ the finite double covering with branch locus $\tilde{B}$ constructed in a natural way in the total space of the bundle $[q^*(3\Delta_0 + (2n + 2 + r)\Gamma) - 2E]$. Since $\tilde{\Gamma}_i$ is a part of the branch locus, we can write $\tilde{g}^*\tilde{\Gamma}_i = 2E_i$. Then the $E_i$'s are $(-1)$-curves. We remark that the canonical bundle of $\tilde{S}$ is given by $[h^*(\Delta_0 + n\Gamma) + 2\sum_i E_i]$, where $h = \tilde{g} \circ q : \tilde{S} \to W$. If we blow down all such $(-1)$-curves $E_i$, then we obtain the original $S$ by the uniqueness of the minimal model. We denote the blow down map by $\pi : \tilde{S} \to S$. We put $D = \pi_*(h^*\Gamma)$ and $G = \pi_*(h^*\Delta_0)$. Then the canonical bundle of $S$ is given by $K_S = [G + nD]$. Needless to say $D$ stands for a fibre of $f : S \to \mathbb{P}^1$. We remark that $|K_S|$ has exactly $r$ base points $e_i = \pi(E_i)$ all of which are on $G$. Note that $S$ has a canonical involution which induces the hyperelliptic involution on a general fibre. The $\mathbb{Z}_2$-action has $\{e_1, \ldots, e_r\}$ as the set of isolated fixed points, and the quotient space $W^\flat = S/\mathbb{Z}_2$ can be obtained from $\tilde{W}$ by contracting all the $(-2)$ curves $\tilde{\Gamma}_i$, $1 \leq i \leq r$.

$$\begin{tikzcd}
 & \tilde{S} \arrow[dl, "\pi"'] \arrow[d, "\tilde{g}"] \arrow[dr, "h"] & \\
S \arrow[d, "f"] \arrow[dr] & \tilde{W} \arrow[r, "q"] \arrow[d] & W \arrow[dll] \\
\mathbb{P}^1 & W^\flat \arrow[l] &
\end{tikzcd}$$

Before going further, we give a few comments on the branch locus. Let $C$ be a subdivisor of $B_h \in |6\Delta_0 + (4n + 4 + r)\Gamma|$ consisting of horizontal components. We denote by $\tilde{C}$ the proper transform of $C$ by $q : \tilde{W} \to W$. Put $C \sim a\Delta_0 + b\Gamma$ and let $m_i$ denote the multiplicity of $C$ at $p_i^+$. Then $a - m_i$ is the multiplicity of $C$ at $p_i^-$. We can assume that $0 \leq m_i \leq a - m_i \leq 3$. We put $r_j = r_j(C) = \mathrm{card}\{i | m_i = j,\ 1 \leq i \leq r\}$ for $0 \leq j \leq [a/2]$. We have

$$\tilde{C} \sim q^*(a\Delta_0 + b\Gamma) - \sum_{i=1}^{r} (m_i E_i^+ + (a - m_i)E_i^-).$$

Since $\tilde{B}_h$ is non-singular, we have $\tilde{C}(\tilde{B}_h - \tilde{C}) = 0$, that is,

$$(n + 4)a + (2b - na)(3 - a) + \sum_{i=1}^{r} \{(a - m_i)^2 + m_i^2\} = 2ar.$$

When $a = 1$, we have $m_i = 0$ and $r = 3n + 4 + 4(b - n)$. Hence $K_S^2 = 5p_g - 6 + 4(b - n)$. It follows that $K_S^2 \geq 5p_g - 6$ if $C \neq \Delta_0$. If $C = \Delta_0$, we have $n + r = 4$, that is, $(n, r) = (4, 0)$, $(3, 1)$, $(2, 2)$. In all such cases, $\tilde{\Delta}_0$ is a $(-4)$-curve which gives us a $(-2)$-curve on $S$ and, hence, $K_S$ is not ample. When $a = 2$, we get $r_1 = 2n + 4 + (b - 2n)$, $r_0 + r_1 = r$. It follows that $K_S^2 = 4p_g - 4 + (b - 2n) + r_0$. When $a = 3$, we get $3p_g + 6 + 4r_0 = r$ and $K_S^2 = 5p_g + 2 + 4r_0$.

Therefore, when $K_S^2 < 5p_g + 2$, there are no subdivisor with $a = 3$ in $B_h$ and we have the following types of branch loci.

(B1) $B_h$ is irreducible

(B2) $B_h$ consists of two irreducible components $C_1$ and $C_2$ with $C_j\Gamma = 2j$ $(j = 1, 2)$, $K_S^2 \geq 4p_g - 4$.

(B3) $B_h$ consists of three irreducible components $C_1$, $C_2$ and $C_3$ with $C_j\Gamma = 2$ $(j = 1, 2, 3)$, $K_S^2 \geq 4p_g$.

(B4) $B_h$ has $C_1 \sim \Delta_0 + (n + \epsilon)\Gamma$ as an irreducible component, $K_S^2 = 5p_g + 4\epsilon - 6$ $(r = 3n + 4\epsilon + 4, \epsilon = 0, 1)$.

  (i) $B_h$ consists of two irreducible components $C_1$, $C_2$ with $C_2 \sim 5\Delta_0 + (6n + 3\epsilon + 8)\Gamma$.

  (ii) $B_h$ consists of three irreducible components $C_1, C_2, C_3$ with $C_2 \sim \Delta_0 + (n + \epsilon)\Gamma$ and $C_3 \sim 4\Delta_0 + (5n + 2\epsilon + 8)\Gamma$.

(B5) $B_h$ consists of two irreducible components $C_1 = \Delta_0$ and $C_2 \sim 5\Delta_0 + (3n + 8)\Gamma$, $n \leq 4$. In this case, $K_S$ is not ample and $(p_g, K_S^2) = (4, 6)$, $(5, 7)$ or $(6, 8)$.

We say that the branch locus $B$ is *generic* if the following conditions are satisfied.

(A3) $\Delta_0$ is not a component of $B$.

(A4) Every triple point of $B_h$ is ordinary.

(A5) $B$ and $\Delta_0$ meet normally. In particular, no singular points of $B_h$ are on $\Delta_0$.

As we have already seen, both (A1) and (A3) are satisfied automatically when $K_S$ is ample. The following examples show that (A4) and (A5) are also harmless at least when $K_S^2 \leq 4p_g$.

**Example.** Though we already have in literatures a lot of examples of surfaces with a genus two fibration, we give here ones for the convenience of readers.

(1) We take an irreducible non-singular member $C \in |2\Delta_0 + 2n\Gamma|$ on $\Sigma_n$ and choose $r$ distinct fibres $\Gamma_i$, $1 \leq i \leq r$, so that the intersection $C \cap \Gamma_i$ consists of two distinct points $p_i^+, p_i^-$. Let $\tilde{C}$ be the proper transform of $C$ by $q : \tilde{W} \to W$. Then

$$\begin{aligned} & q^*(6\Delta_0 + (4n + 4 + r)\Gamma) - 3E \\ \sim\ & q^*(6\Delta_0 + (4n + 4 - 2r)\Gamma) + 3\textstyle\sum_{i=1}^r \tilde{\Gamma}_i \\ \sim\ & 3\tilde{C} + (r + 4 - 2n)q^*\Gamma \end{aligned}$$

Note that we have $C \cap \Delta_0 = \emptyset$ and $\tilde{C} \cap \tilde{\Gamma}_i = \emptyset$ for any $i$. Hence, if $2n - 4 \le r \le 2n + 2$, then $|q^*(6\Delta_0 + (4n + 4 + r)\Gamma) - 3E|$ is free from base points and we can take its non-singular member $\tilde{B}_h$. In this way, we can construct desired surfaces in the range $4p_g - 12 \le K^2 \le 4p_g - 6$. Since a general member of the linear subsystem $3\tilde{C} + |(r + 4 - 2n)q^*\Gamma|$ meets $q^*\Delta_0$ transversally, we can assume that so does $\tilde{B}_h$. Furthermore, since the restriction of $|\tilde{B}_h|$ to $E_i^{\pm}$ contains special members of the forms $(3 - k)(\tilde{C} \cap E_i^{\pm}) + k(\tilde{\Gamma}_i \cap E_i^{\pm})$ for $0 \le k \le 3$, we see that $\tilde{B}_h$ can be assumed to meet each $E_i^{\pm}$ at three distinct points.

(2) We assume that $r \ge 2n$ and write $r = 2n + c_1 + c_2$ with two non-negative integers $c_1, c_2$ such that $0 \le c_1 \le c_2$. We take irreducible members $C_i \in |2\Delta_0 + (r - c_i)\Gamma|$ so that $C_1 \cap C_2$ consists of $2r$ distinct points and there exist $r$ fibres $\Gamma_i$ each of which contains just two points $p_i^+, p_i^-$ in $C_1 \cap C_2$. To see that this is possible, we fix a section $\Delta \in |\Delta_0 + n\Gamma|$ and let $\zeta_0 \in H^0([\Delta_0])$ and $\zeta_1 \in H^0([\Delta_0 + \Gamma])$ define $\Delta_0$ and $\Delta$, respectively. Then any element in $H^0([2\Delta_0 + m\Gamma])$, $m \ge 2n$, can be written as $a_m\zeta_0^2 + a_{m-n}\zeta_0\zeta + a_{m-2n}\zeta^2$, where the $a_i$'s are homogeneous forms of degree $i$ on $\mathbb{P}^1$. We choose $C_i$ whose equation is of the form $a_{r-c_i}\zeta_0^2 + a_{r-c_i-2n}\zeta^2$. Then, if we choose the coefficients generic, the desired properties are satisfied. Furthermore, we can assume that $C_1 \cap C_2 \cap \Delta_0 = \emptyset$.

Let $\tilde{C}_i$ be the proper transform of $C_i$ by $q$. Then $\tilde{C}_1 \cap \tilde{C}_2 = \emptyset$ and

$$\begin{aligned} & q^*(6\Delta_0 + (4n + 4 + r)\Gamma) - 3\textstyle\sum_i(E_i^+ + E_i^-) \\ \sim\ & 3\tilde{C}_1 + (4 + c_1 - 2c_2)q^*\Gamma \\ \sim\ & 3\tilde{C}_2 + (4 - 2c_1 + c_2)q^*\Gamma \end{aligned}$$

Hence, if $2c_2 \le c_1 + 4$ and $2c_1 \le c_2 + 4$, then we can find a non-singular member $\tilde{B}_h$. In this way, we can fill the range $4p_g - 8 \le K^2 \le 4p_g$, $K^2 \ne 4p_g - 1$. In order to fill the gap and to give examples of types (B2) and (B3), we put $c_2 = 4$. Then $\tilde{C}_2(q^*(4\Delta_0 + (4n + 8)\Gamma) - 2E) = 0$. We can take a non-singular member $\tilde{B}_1 \in |q^*(4\Delta_0 + (4n + 8)\Gamma) - 2E|$, because

$$\begin{aligned} & q^*(4\Delta_0 + (4n + 8)\Gamma) - 2E \\ \sim\ & 2\tilde{C}_1 \\ \sim\ & 2\tilde{C}_2 + (8 - 2c_1)q^*\Gamma \end{aligned}$$

Then $\tilde{B}_h = \tilde{B}_1 + \tilde{C}_2$ is a non-singular curve, giving us surfaces with $K_S^2 = 4p_g - (4 - c_1)$, $0 \le c_1 \le 4$. Note that, when $c_1 = 4$, $B_h$ consists of three members in the pencil spanned by $C_1$ and $C_2$.

**Lemma 1.2.** *For $k \in \mathbb{Z}$,*

$$h^p(S, [kD]) = \begin{cases} (k+1)_0 & \text{if } p = 0, \\ (-k-1)_0 + (k-1)_0 + (k-1-n)_0 & \text{if } p = 1, \\ (-k+1)_0 + (n-k+1)_0 & \text{if } p = 2 \end{cases}$$

*Here $(m)_0 = m$ if $m \ge 0$ and $(m)_0 = 0$ if $m < 0$.*

*Proof.* We have $H^p(S, [kD]) \simeq H^p(\tilde{S}, k[h^*\Gamma]) \simeq H^p(\tilde{W}, k[q^*\Gamma]) \oplus H^p(\tilde{W}, [kq^*\Gamma - \tilde{B}/2])$ and $H^p(\tilde{W}, k[q^*\Gamma]) \simeq H^p(W, [k\Gamma]) \simeq H^q(\mathbb{P}^1, \mathcal{O}(k))$. Since $kq^*\Gamma - \tilde{B}/2 \sim K_{\tilde{W}} - q^*(\Delta_0 + (n + r - k)\Gamma) +$

$\sum_{i=1}^r(E_i^+ + E_i^-)$, we have $H^p(\tilde{W}, [kq^*\Gamma - \tilde{B}/2]) \simeq H^{2-p}(\tilde{W}, [q^*(\Delta_0 + (n-k)\Gamma) + \sum_{i=1}^r \tilde{\Gamma}_i])^\vee$ by duality theorem. It follows from the cohomology long exact sequence for

$$0 \to \mathcal{O}_{\tilde{W}}(q^*[\Delta_0 + (n-k)\Gamma]) \to \mathcal{O}_{\tilde{W}}([q^*(\Delta_0 + (n-k)\Gamma) + \sum_{i=1}^r \tilde{\Gamma}_i]) \to \bigoplus_{i=1}^r \mathcal{O}_{\tilde{\Gamma}_i}(-1) \to 0$$

that

$$H^{2-p}(\tilde{W}, [q^*(\Delta_0 + (n-k)\Gamma) + \sum_{i=1}^r \tilde{\Gamma}_i]) \simeq H^{2-p}(\tilde{W}, q^*[\Delta_0 + (n-k)\Gamma])$$

Since the last group is isomorphic to $H^{2-p}(\Sigma_n, [\Delta_0 + (n-k)\Gamma])$, we get the assertion. □

For a coherent seaf $\mathcal{F}$ on $S$, we sometimes denote by $\vec{h}(\mathcal{F})$ the triple $(h^0(S,\mathcal{F}), h^1(S,\mathcal{F}), h^2(S,\mathcal{F}))$. Since $f^*\Theta_{\mathbb{P}^1} = \mathcal{O}_S([2D])$, we get the following from Lemma 1.2:

**Corollary 1.3.** $\vec{h}(f^*\Theta_{\mathbb{P}^1}) = (3, 1, n-1)$.

**Lemma 1.4.** $h^0(G, [G]|_G) = 0$ *and* $h^1(G, [G]|_G) = n$.

*Proof.* It follows from Lemma 1.2 that $\vec{h}(G) = (1, n-1, n+1)$, since $h^p(S, [G]) = h^{2-p}(S, [nD])$ by the duality theorem. We consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_S \to \mathcal{O}_S([G]) \to \mathcal{O}_G([G]|_G) \to 0.$$

Since $S$ is a regular surface, we have $\vec{h}(\mathcal{O}_S) = (1, 0, n+2)$. Hence we get the assertion. □

**Lemma 1.5.** $h^0(G, [D+G]|_G) = 0$ *and the restriction map* $H^1(S, [D+G]) \to H^1(G, [D+G]|_G)$ *is an isomorphism between vector spaces of dimension* $n-2$.

*Proof.* We consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_S([D]) \to \mathcal{O}_S([D+G]) \to \mathcal{O}_G([D+G]) \to 0.$$

From Lemma 1.2, we have $\vec{h}(D) = (2, 0, n)$ and $\vec{h}(D+G) = (2, n-2, n)$, since $h^p(S, [D+G]) = h^{2-p}(S, [(n-1)D])$. □

**Lemma 1.6.** $h^0(G, [2D+G]|_G) = 1$ *and* $h^1(G, [2D+G]|_G) = (n-3)_0$. *In particular, the restriction map* $H^1(S, [2D+G]) \to H^1(G, [2D+G]|_G)$ *is an isomorphism between vector spaces of dimension* $(n-3)_0$.

*Proof.* Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_S([2D]) \to \mathcal{O}_S([2D+G]) \to \mathcal{O}_G([2D+G]) \to 0.$$

We have $\vec{h}(2D) = (3, 1, n-1)$ and $\vec{h}(2D+G) = ((3-n)_0 + 3, (n-3)_0, n-1)$ by Lemma 1.2 and the duality theorem. The assertion for $n \leq 3$ follow. Assume that $n \geq 4$. Then $(h^0(G, [2D +$

$G]|_G), h^1(G, [2D+G]_G)) = (0, n-4)$ or $(1, n-3)$. It suffices to show that $H^1(S, [2D]) \to H^1(S, [2D+G])$ is the zero map.

On $\tilde{S}$, we have $\pi^*(2D+G) = h^*(\Delta_0 + 2\Gamma) + \sum_i E_i \sim \tilde{g}^*L - \sum_i E_i$, where $L = q^*(\Delta_0 + (r+2)\Gamma) - \sum_i (E_i^+ + E_i^-)$. From the cohomology long exact sequence for

$$0 \to \mathscr{O}_{\tilde{S}}(\pi^*[G+2D]) \to \mathscr{O}_{\tilde{S}}(\tilde{g}^*L) \to \bigoplus_{i=1}^{r} \mathscr{O}_{E_i}(-1) \to 0,$$

we see that $H^p(\tilde{S}, \pi^*[G+2D]) \to H^p(\tilde{S}, \tilde{g}^*L)$ is an isomorphism for any $p$, which is equivariant with respect to the action of $\mathrm{Gal}(\tilde{S}/\tilde{W})$ because the $E_i$'s are invariant divisors. We have the eigen space decomposition

$$H^p(\tilde{S}, \tilde{g}^*L) \simeq H^p(\tilde{W}, L) \bigoplus H^p(\tilde{W}, L - \frac{1}{2}\tilde{B})$$

We have $L - (1/2)\tilde{B} \sim K_{\tilde{W}} + (2-n)q^*\Gamma$. So the duality theorem gives us $H^p(\tilde{W}, L - (1/2)\tilde{B})^\vee \simeq H^{2-p}(\tilde{W}, (n-2)[q^*\Gamma]) \simeq H^{2-p}(\mathbb{P}^1, \mathscr{O}(n-2))$. It follows that the $(-1)$-eigen space $H^1(\tilde{W}, \mathscr{L} - (1/2)\tilde{B})$ vanishes. In sum, we see that $H^1(\tilde{S}, \pi^*[G+2D])$ is $\mathrm{Gal}(\tilde{S}/\tilde{W})$-invariant. On the other hand, $H^1(\tilde{S}, [2\pi^*D])$ is clearly anti-invariant. Since $H^1(\tilde{S}, [2\pi^*D]) \to H^1(\tilde{S}, \pi^*[G+2D])$ is an equivariant homomorphism, we conclude that it must be the zero map. □

**Lemma 1.7.** *The linear map*

$$H^1(S, [G]) \to \mathrm{Hom}_{\mathbb{C}}(H^0(S, [(n-2)D]), H^1(S, [G+(n-2)D]))$$

*induced from the multiplication map* $\mathscr{O}_S([G]) \otimes \mathscr{O}_S([(n-2)D]) \to \mathscr{O}_S([G+(n-2)D])$ *is injective.*

*Proof.* Since the assertion is trivial for $n = 2$, we assume that $n \geq 3$. We shall show that $H^1(S, [G+iD]) \to \mathrm{Hom}_{\mathbb{C}}(H^0(S, [D]), H^1(S, [G+(i+1)D]))$ is injective for any $i$ with $0 \leq i \leq n-3$. Since $|D|$ is a base-point-free pencil, it suffices to show that the Koszul map

$$H^0(S, [G+(i+1)D]) \otimes H^0(S, [D])^\vee \to H^0(S, [G+(i+2)D]) \otimes \bigwedge^2 H^0(S, [D])^\vee$$

is surjective for any $i$ with $0 \leq i \leq n-3$ (see [5]). This is equivalent to showing that the multiplication map $H^0(S, [G+(i+1)D]) \otimes H^0(S, [D]) \to H^0(S, [G+(i+2)D])$ is surjective. By the free-pencil-trick, its kernel is isomorphic to $H^0(S, [G+iD])$. We have

$$2h^0(S, [G+(i+1)D]) = 2(i+2) = (i+1) + (i+3) = h^0(S, [G+iD]) + h^0(S, [G+(i+2)D])$$

by Lemma 1.2, which completes the proof. □

## 2 Deformations of genus two fibrations

We denote by $\mathscr{T}_{\tilde{S}/W}$ the cokernel of the natural map $\Theta_{\tilde{S}} \to h^*\Theta_W$, where $\Theta_X$ denotes the tangent sheaf of a complex manifold $X$.

We first study when $H^1(\tilde{S}, \mathscr{T}_{\tilde{S}/W})$ vanishes. Recall that we have an exact sequence

$$0 \to \mathscr{T}_{\tilde{S}/\tilde{W}} \to \mathscr{T}_{\tilde{S}/W} \to \tilde{g}^* \mathscr{T}_{\tilde{W}/W} \to 0,$$

and isomorphisms (cf. [1]):

$$\tilde{g}_* \mathscr{T}_{\tilde{S}/\tilde{W}} \simeq \mathscr{N}_{\tilde{B}}, \quad \mathscr{T}_{\tilde{W}/W} \simeq \bigoplus_{i=1}^{r} (\mathscr{O}_{E_i^+}(1) \oplus \mathscr{O}_{E_i^-}(1)),$$

where $\mathscr{N}_{\tilde{B}}$ denotes the normal sheaf of $\tilde{B}$ in $\tilde{W}$. Hence taking direct images, we have

$$0 \to \mathscr{N}_{\tilde{B}} \to \tilde{g}_* \mathscr{T}_{\tilde{S}/W} \to \mathscr{T}_{\tilde{W}/W} \oplus \mathscr{T}_{\tilde{W}/W}(-\frac{1}{2}\tilde{B}) \to 0.$$

Since $\tilde{B}$ is of degree 4 on $E_i^{\pm}$, we have $H^p(\mathscr{T}_{\tilde{W}/W}(-\frac{1}{2}\tilde{B})) = 0$ for any $p$, and $H^1(\mathscr{T}_{\tilde{W}/W}) = 0$. Hence $H^1(\tilde{S}, \mathscr{T}_{\tilde{S}/W})$ vanishes if and only if the coboundary map $\delta : H^0(\tilde{W}, \mathscr{T}_{\tilde{W}/W}) \to H^1(\tilde{B}, \mathscr{N}_{\tilde{B}})$ is surjective. Apparently, $\delta$ factors as

$$H^0(\tilde{W}, \mathscr{T}_{\tilde{W}/W}) \to H^0(\tilde{B}, \mathscr{T}_{\tilde{W}/W}|_{\tilde{B}}) \to H^1(\tilde{B}, \mathscr{N}_{\tilde{B}})$$

where the first map is the restriction and the last map is the coboundary map coming from

$$0 \to \mathscr{N}_{\tilde{B}} \to \mathscr{T}_{\tilde{B}/W} \to \mathscr{T}_{\tilde{W}/W}|_{\tilde{B}} \to 0.$$

which consists of the following two parts

$$0 \to \bigoplus_{i=1}^{r} \mathscr{N}_{\tilde{\Gamma}_i} \to \bigoplus_{i=1}^{r} \mathscr{T}_{\tilde{\Gamma}_i/W} \to \bigoplus_{i=1}^{r} \mathscr{T}_{\tilde{W}/W}|_{\tilde{\Gamma}_i} \to 0$$

and

$$0 \to \mathscr{N}_{\tilde{B}_h} \to \mathscr{T}_{\tilde{B}_h/W} \to \mathscr{T}_{\tilde{W}/W}|_{\tilde{B}_h} \to 0.$$

We have $\mathscr{N}_{\tilde{\Gamma}_i} \simeq \mathscr{O}_{\mathbb{P}^1}(-2)$ and $\mathscr{T}_{\tilde{\Gamma}_i/W} \simeq \mathscr{O}_{\mathbb{P}^1}$, since $\tilde{\Gamma}_i \to W$ is an embedding. It follows that $H^1(\mathscr{T}_{\tilde{\Gamma}_i/W}) = 0$ for any $i$, $1 \le i \le r$.

We assume that any singular point of $B_h$ is an ordinary triple point. Then $\tilde{B}_h$ meets $E_i^{\pm}$ normally at distinct points and $\mathscr{T}_{\tilde{B}_h/W}$ is an invertible sheaf, because $\tilde{B}_h \to W$ is a local embedding. For each $i$, $1 \le i \le r$, we put $\tilde{B}_h \cap E_i^{\pm} = \{q_{i1}^{\pm}, q_{i2}^{\pm}, q_{i3}^{\pm}\}$ and $\tilde{\Gamma}_i \cap E_i^{\pm} = \{q_{i4}^{\pm}\}$.

**Lemma 2.1.** *Assume that $B$ satisfies* (A3) *and* (A4). *If $r < \min\{2n + 18, 3n + 12\}$, then $H^1(\tilde{B}_h, \mathscr{T}_{\tilde{B}_h/W}) = 0$ and one can assume that the $3r$-dimensional subspace*

$$\bigoplus_{i=1}^{r} (\mathbb{C}_{q_{i1}^+} \oplus \mathbb{C}_{q_{i1}^-} \oplus \mathbb{C}_{q_{i2}^+}) \subset H^0(\mathscr{T}_{\tilde{W}/W}|_{\tilde{B}})$$

*is in the image of $H^0(\mathscr{T}_{\tilde{B}_h/W}) \to H^0(\mathscr{T}_{\tilde{W}/W}|_{\tilde{B}_h})$.*

*Proof.* Note that the map $H^0(\mathscr{T}_{\tilde{B}_h/W}) \to H^0(\mathscr{T}_{\tilde{W}/W}|_{\tilde{B}_h})$ can be identified with the restriction map. We consider case by case according to the types (B1) through (B4) of branch loci given in the previous section. We denote by $\eta_j$ a divisor on an irreducible component $\tilde{C}_j$ of $\tilde{B}_h$.

We consider the case (B1) that $B_h$ is irreducible. Then we have $(\tilde{B}_h)^2 = 6(2n+8-r)$, $g(\tilde{B}_h) = 5n+15-r$. Since $\deg \mathscr{T}_{\tilde{B}_h/W} - 2g(\tilde{B}_h)+2 = 12n+48-(10n+28-2r) = 2n+20+2r$, we have $H^1(\tilde{B}_h, \mathscr{T}_{\tilde{B}_h/W}(-\eta_1)) = 0$ for any divisor $\eta_1$ with $\deg \eta_1 \leq 2n+19+2r$. We have $2n+19+2r \geq 3r$ when $r \leq 2n+19$.

Assume that $B_h = C_1 + C_2$ as in (B2). Put $C_1 \sim 2\Delta_0 + (2n+\epsilon)\Gamma$ with a non-negative integer $\epsilon$ and $r_j = r_j(C_1)$. Then $r_1 = 2n+4+\epsilon$. Since $q$ blows up $2r_1$ simple points and $r_0$ double points of $C_1$, we have $(\tilde{C}_1)^2 = C_1^2 - 2r_1 - 4r_0 = 8n+8+6\epsilon-4r$ and $g(\tilde{C}_1) = p_a(C_1) - r_0 = n+\epsilon-1-r_0$. We have $\deg \mathscr{T}_{\tilde{C}_1/W} - 2g(\tilde{C}_1)+2 = 8n+8+6\epsilon-2r-(2n+2\epsilon-2r_0-4) = r_1+\epsilon$. It follows that $H^1(\tilde{C}_1, \mathscr{T}_{\tilde{C}_1/W}(-\eta_1)) = 0$ when $\deg \eta_1 < r_1+\epsilon$. Similarly, since $q$ blows up $C_2 \sim 4\Delta_0+(4n+8+r_0)\Gamma$ at $2r_1$ double points, $r_0$ triple points and simple points, we have $(\tilde{C}_2)^2 = C_2^2 - 8r_1 - 10r_0 = -8\epsilon+32-2r_0$ and $g(\tilde{C}_2) = n+13-2\epsilon$. We have $\deg \mathscr{T}_{\tilde{C}_2/W} - 2g(\tilde{C}_2)+2 = 8n+48-4\epsilon+2r_0-(4n+24-4\epsilon) = 4n+24+2r_0$. It follows that we get $H^1(\tilde{C}_2, \mathscr{T}_{\tilde{C}_2/W}(-\eta_2)) = 0$ when $\deg \eta_2 < 4n+24+2r_0$. We see $(r_1+\epsilon-1)+(4n+23+2r_0) = 2n+18+2r$ is not less than $3r$ when $r \leq 2n+18$.

Assume that $B_h = C_1+C_2+C_3$ as in (B3). We put $C_j \sim 2\Delta_0 + (2n+\epsilon_j)\Gamma$ with non-negative integers $\epsilon_j$, $1 \leq j \leq 3$. Then $\epsilon_1+\epsilon_2+\epsilon_3 = r-2n+4$ and $r_1(C_j) = 2n+4+\epsilon_j$. As in the previous case, we have $H^1(\tilde{C}_j, \mathscr{T}_{\tilde{C}_j/W}(-\eta_j)) = 0$ when $\deg \eta_j \leq r_1(C_j)+\epsilon_j-1$. Since $\sum_{j=1}^{3}(r_1(C_j)+\epsilon_j-1) = 2n+17+2r$, we see that it is not less than $3r$ when $r \leq 2n+17$.

Assume that $B_h = C_1+C_2$ as in (B4), (i). Put $C_1 \sim \Delta_o + (n+\epsilon)$, where $\epsilon = 0,1$. Then $r = 3n+4\epsilon+4$. We have $(\tilde{C}_1)^2 = n+2\epsilon-r$, $g(\tilde{C}_1) = 0$ and $\deg \mathscr{T}_{\tilde{C}_1/W} = n+2\epsilon$. Hence $H^1(\tilde{C}_1, \mathscr{T}_{\tilde{C}_1/W}(-\eta_1)) = 0$ if $\deg \eta_1 \leq n+2\epsilon+1$. Since $q$ blows up $r$ double points and $r$ triple points of $C_2$, we have $(\tilde{C}_2)^2 = 35n+30\epsilon+80-13r$, $g(\tilde{C}_2) = 14n+12\epsilon+28-4r$ and $\deg \mathscr{T}_{\tilde{C}_2/W} - 2g(\tilde{C}_2)+2 = 7n+6\epsilon+26$. It follows that $H^1(\tilde{C}_2, \mathscr{T}_{\tilde{C}_2/W}(-\eta_2)) = 0$ if $\deg \eta_2 \leq 7n+6\epsilon+25$. Now $(n+2\epsilon+1)+7n+6\epsilon+25 \geq 3r$ holds when $n \leq 14-4\epsilon$, that is, when $r \leq 2n+18$.

Assume that $B_h = C_1+C_2+C_3$ as in (B4), (ii). For $j = 1,2$, we put $C_j \sim \Delta_0+(n+\epsilon)\Gamma$, where $\epsilon = 0,1$. Then $(\tilde{C}_j)^2 = n+2\epsilon-r$, $g(\tilde{C}_j) = 0$ and $H^1(\tilde{C}_j, \mathscr{T}_{\tilde{C}_j/W}(-\eta_j)) = 0$ if $\deg \eta_j \leq n+2\epsilon+1$. Since $q$ blows up $2(r-n)$ double points, $n$ triple points and $n$ simple points of $C_3 \sim 4\Delta_0+(5n+2\epsilon+8)\Gamma$, we have $(\tilde{C}_3)^2 = 22n+16\epsilon+64-8r$, $g(\tilde{C}_3) = 8n+6\epsilon+21-2r$ and $\deg \mathscr{T}_{\tilde{C}_3/W} - 2g(\tilde{C}_3)+2 = 6n+4\epsilon+24$. It follows that $H^1(\tilde{C}_3, \mathscr{T}_{\tilde{C}_3/W}(-\eta_3)) = 0$ if $\deg \eta_3 \leq 6n+4\epsilon+23$. Now $(n+2\epsilon+1)+(n+2\epsilon+1)+(6n+4\epsilon+8) \geq 3r$ holds when $n \leq 13-4\epsilon$, that is, when $r \leq 2n+17$

In any cases, we can put $\eta_j = 0$ to see $H^1(\mathscr{T}_{\tilde{B}_h/W}) = 0$. Furthermore, if $r \leq 2n+17$, then we can choose the $\eta_j$'s so that $\sum_j \eta_j = \sum_{i=1}^{r}(q_{1i}^+ + q_{2i}^+ + q_{1i}^-)$ and $H^1(\mathscr{T}_{\tilde{B}_h/W}(-\sum_j \eta_j)) = 0$ hold. Hence, we can assume that the $3r$-dimensional subspace $\oplus_{i=1}^{r}(\mathbb{C}_{q_{i1}^+} \oplus \mathbb{C}_{q_{i1}^-} \oplus \mathbb{C}_{q_{i2}^+})$ is in the image of $H^0(\mathscr{T}_{\tilde{B}_h/W})$. □

**Lemma 2.2.** *Assume that $B$ satisfies* (A3) *and* (A4). *If $r < \min\{2n+18, 3n+12\}$, then $H^1(\mathscr{T}_{\tilde{S}/W}) = 0$.*

*Proof.* By Lemma 2.1, we have $H^1(\mathscr{T}_{\tilde{B}_h/W}) = 0$ and get the commutative diagram

$$\begin{array}{ccccccccc} & & & & H^0(\tilde{W}, \mathscr{T}_{\tilde{W}/W}) & & & & \\ & & & & \downarrow{\scriptstyle rest.} & \searrow^{\delta} & & & \\ 0 \to H^0(\mathscr{N}_{\tilde{B}}) & \to & H^0(\mathscr{T}_{\tilde{B}/W}) & \longrightarrow & H^0(\mathscr{T}_{\tilde{W}/W}|_{\tilde{B}}) & \longrightarrow & H^1(\mathscr{N}_{\tilde{B}}) & \to & 0 \end{array}$$

where the bottom sequence is exact. Therefore, if $H^0(\mathscr{T}_{\tilde{W}/W}|_{\tilde{B}})$ is generated by the images of $H^0(\mathscr{T}_{\tilde{W}/W})$ and $H^0(\mathscr{T}_{\tilde{B}/W})$, then $\delta : H^0(\mathscr{T}_{\tilde{W}/W}) \to H^1(\mathscr{N}_{\tilde{B}})$ is surjective and it follows $H^1(\mathscr{T}_{\tilde{S}/W}) = 0$.

In the $8r$-dimensional vector space $H^0(\mathscr{T}_{\tilde{W}/W}|_{\tilde{B}})$, we can assume that the $3r$-dimensional subspace $\oplus_i(\mathbb{C}_{q_{i1}^+} \oplus \mathbb{C}_{q_{i1}^-} \oplus \mathbb{C}_{q_{i2}^+})$ is in the image of $H^0(\mathscr{T}_{\tilde{B}_h/W})$ by Lemma 2.1. Similarly, the $2r$-dimensional subspaces $\oplus_i(\mathbb{C}_{q_{i3}^+} \oplus \mathbb{C}_{q_{i4}^+})$ and $\oplus_i(\mathbb{C}_{q_{i2}^-} \oplus \mathbb{C}_{q_{i3}^-})$ are in the image of $\oplus_i H^0(E_i^+, \mathscr{O}(1))$ and $\oplus_i H^0(E_i^-, \mathscr{O}(1))$, respectively, and the $r$-dimensional subspace $\oplus_i \mathbb{C}_{q_{i4}^-}$ is in the image of $\oplus_i H^0(\mathscr{T}_{\tilde{\Gamma}_i/W})$. $\square$

In the rest of the section, we assume that $H^1(\mathscr{T}_{\tilde{S}/W}) = 0$ and follow the considerations in [1]. It follows from the cohomology long exact sequence for

$$0 \to \Theta_{\tilde{S}} \to \pi^*\Theta_S \to \bigoplus_{i=1}^{r} \mathscr{O}_{E_i}(1) \to 0,$$

that $H^2(\tilde{S}, \Theta_{\tilde{S}}) \simeq H^2(\tilde{S}, \pi^*\Theta_S) \simeq H^2(S, \Theta_S)$ and $h^1(\tilde{S}, \Theta_{\tilde{S}}) = h^1(S, \Theta_S) + 2r$. We consider the cohomology long exact sequence for

$$0 \to \Theta_{\tilde{S}} \to h^*\Theta_W \to \mathscr{T}_{\tilde{S}/W} \to 0.$$

Since the support of $\mathscr{T}_{\tilde{S}/W}$ is (at most) one-dimensional, we have $H^2(\tilde{S}, \mathscr{T}_{\tilde{S}/W}) = 0$. Hence assuming that $H^1(\tilde{S}, \mathscr{T}_{\tilde{S}/W}) = 0$, we get $H^2(\tilde{S}, \Theta_{\tilde{S}}) \simeq H^2(\tilde{S}, h^*\Theta_W)$ and the exact sequence

$$0 \to H^0(\tilde{S}, h^*\Theta_W) \to H^0(\tilde{S}, \mathscr{T}_{\tilde{S}/W}) \to H^1(\tilde{S}, \Theta_{\tilde{S}}) \to H^1(\tilde{S}, h^*\Theta_W) \to 0.$$

We study $H^p(\tilde{S}, h^*\Theta_W)$ with the cohomology long exact sequence for

$$0 \to h^*[2\Delta_0 + n\Gamma] \to h^*\Theta_W \to h^*[2\Gamma] \to 0.$$

As we have already seen in Corollary 1.3, we have $\vec{h}(\tilde{S}, 2h^*\Gamma) = (3, 1, n-1)$. We have

$$H^p(\tilde{S}, h^*[2\Delta_0 + n\Gamma]) \simeq H^p(\tilde{W}, q^*[2\Delta_0 + n\Gamma]) \bigoplus H^p(\tilde{W}, q^*[2\Delta_0 + n\Gamma] - \frac{1}{2}\tilde{B}).$$

Since $H^p(\tilde{W}, q^*[2\Delta_0 + n\Gamma]) \simeq H^p(W, [2\Delta_0 + n\Gamma])$, we get $\vec{h}(\tilde{W}, q^*[2\Delta_0 + n\Gamma]) = (n+2, n-1, 0)$. We have $q^*(2\Delta_0 + n\Gamma) - \frac{1}{2}\tilde{B} \sim -q^*(\Delta_0 + (n+2+r)\Gamma) + 2E$. We put $L_i = -q^*(\Delta_0 + (n+2+r)\Gamma) + iE$ for $i = 0, 1, 2$. Then, since $E_i^{\pm} L_i = -i$, we have the exact sequence

$$0 \to L_{i-1} \to L_i \to \bigoplus_{i=1}^{r} (\mathscr{O}_{E_i^+}(-i) \oplus \mathscr{O}_{E_i^-}(-i)) \to 0.$$

It follows that $H^p(\tilde{W}, L_1) \simeq H^p(\tilde{W}, L_0) = 0$ for any $p$, and we get

$$H^p(\tilde{W}, q^*(2\Delta_0 + n\Gamma) - \frac{1}{2}\tilde{B}) = H^p(\tilde{W}, L_2) \simeq \oplus_{i=1}^r (H^p(E_i^+, \mathscr{O}(-2)) \oplus H^p(E_i^-, \mathscr{O}(-2))).$$

Hence $\vec{h}(\tilde{W}, q^*(2\Delta_0 + n\Gamma) - \frac{1}{2}\tilde{B}) = (0, 2r, 0)$. We have shown that $\vec{h}(\tilde{S}, q^*(2\Delta_0 + n\Gamma)) = (n+2, n-1+2r, 0)$. Since $H^0(W, \Theta_W) \to H^0(W, 2\Gamma)$ is surjective, we get $\vec{h}(\tilde{S}, q^*\Theta_W) = (n+5, n+2r, n-1)$.

We have $h^0(\tilde{S}, \Theta_{\tilde{S}}) = 0$ and $h^2(\tilde{S}, \Theta_{\tilde{S}}) = n-1$. Since it follows from the Riemann-Roch theorem that we have $\chi(\Theta_X) = 2K_X^2 - 10\chi(\mathscr{O}_X)$ for any compact complex surface $X$, we get $h^1(\tilde{S}, \Theta_{\tilde{S}}) = 7n + 29$. We have shown the following:

**Lemma 2.3.** *Assume that $h^1(\tilde{S}, \mathscr{T}_{\tilde{S}/W}) = 0$. Then $h^0(\tilde{S}, \mathscr{T}_{\tilde{S}/W}) = 7n + 34 - 2r$, $\vec{h}(\tilde{S}, \Theta_{\tilde{S}}) = (0, 7n + 29, n - 1)$ and $\vec{h}(S, \Theta_S) = (0, 7n + 29 - 2r, n - 1)$.*

**Lemma 2.4.** *Assume that $h^1(\tilde{S}, \mathscr{T}_{\tilde{S}/W}) = 0$. Then the natural map $F^p : H^p(S, \Theta_S) \to H^p(S, f^*\Theta_{\mathbb{P}^1})$ is surjective for $p = 1$ and it is an isomorphism for $p = 2$.*

*Proof.* We have the commutative diagram

$$\begin{array}{ccc} H^p(\tilde{S}, \Theta_{\tilde{S}}) & \longrightarrow & H^p(\tilde{S}, h^*\Theta_W) \\ \downarrow & & \downarrow \\ H^p(\tilde{S}, \pi^*\Theta_S) & \longrightarrow & H^p(\tilde{S}, \pi^* f^*\Theta_{\mathbb{P}^1}) \end{array}$$

Note that we have already shown that the composite $H^1(\tilde{S}, \Theta_{\tilde{S}}) \to H^1(\tilde{S}, h^*\Theta_W) \to H^1(\tilde{S}, h^*[2\Gamma]) = H^1(\tilde{S}, \pi^* f^*\Theta_{\mathbb{P}^1})$ is surjective. Hence $F^1$ is surjective. Since $H^2(\tilde{S}, \Theta_{\tilde{S}}) \simeq H^2(\tilde{S}, \pi^*\Theta_S)$ and $H^2(\tilde{S}, \Theta_{\tilde{S}}) \simeq H^2(\tilde{S}, h^*\Theta_W) \simeq H^2(\tilde{S}, \pi^* f^*\Theta_{\mathbb{P}^1})$, we see that $F^2$ is an isomorphism. $\square$

Then the following hold by the deformation theory of holomorphic maps as in [1].

**Lemma 2.5.** *Assume that $B$ satisfies* (A3) *and* (A4). *If $r < \min\{2n + 18, 3n + 12\}$, then there exists a family $p_1 : \mathscr{S}_1 \to M_1$ of deformations of $f : S \to \mathbb{P}^1$ such that the Kodaira-Spencer map gives an isomorphism onto* $\mathrm{Ker}(F^1)$, *where $M_1$ is a complex manifold of dimension $7n + 28 - 2r$. Furthermore,*

(1) *Any fibre of $p_1$ is a surface with a genus two fibration.*

(2) *Any family of deformations of $S$ all members of which have a genus two fibration is induced from $p_1$.*

## 3 A filtration on the tangent space of the Kuranishi space

We assume that (A1) through (A5) are satisfied. In particular, $B$ is generic. We further assume $r < \min\{2n + 18, 3n + 12\}$ so that all the results in the previous section are valid.

**Lemma 3.1.** *$G$ is an irreducible non-singular curve of genus $r-n+1$, $G^2 = r-2n$ and $DG = 2$.*

*Proof.* We have $(h^*\Delta_0)^2 = 2\Delta_0^2 = -2n$. Hence $G^2 = r-2n$. We have $DG = 2$ by $(h^*\Delta_0)(h^*\Gamma) = 2\Delta_0\Gamma = 2$. By (A3) and (A5), $h^*\Delta_0$ is irreducible and non-singular. □

**Lemma 3.2.** *Let $\mathscr{N}_G$ denote the normal sheaf of $G$ in $S$. Then the map $\zeta_* : H^1(S,\Theta_S) \to H^1(G,\mathscr{N}_G)$ induced from the composite $\Theta_S \to \Theta_S|_G \to \mathscr{N}_G$ is surjective. Furthermore,* $\mathrm{Ker}(\zeta_*)$ *is contained in* $\mathrm{Ker}(F^1)$.

*Proof.* By Lemma 5.5 of [4], we have $\mathrm{Ker}(\zeta_*) \subset \mathrm{Ker}(F^1)$ and

$$\dim \mathrm{Ker}(\zeta_*) \leq \dim \mathrm{Ker}\{H^1(\Theta_{\tilde{S}}) \to H^1(h^*\Theta_W)\}.$$

By a direct calculation, we see that the right hand side equals $h^1(\Theta_S) - n$. Then $n = h^1(G,\mathscr{N}_G) \geq \mathrm{rank}(\zeta_*) = h^1(S,\Theta_S) - \dim\mathrm{Ker}(\zeta_*) \geq n$. Hence $\zeta_*$ is surjective. □

**Lemma 3.3.** *Let $s \in H^0(G,[2G+(n+2)D]|_G)$ be an element vanishing at all the base points $e_1,\ldots,e_r$ of $|K_S|$. If $s$ is the restriction of an element of $H^0(S,[2G+(n+2)D])$, then either $s=0$, or $n=2$ and $(s) = 2e_1+\cdots+2e_r$.*

*Proof.* We have

$$\begin{aligned}
&H^0(S,[2G+(n+2)D]) \simeq H^0(\tilde{S},\pi^*[2G+(n+2)D])\\
\simeq\;& H^0(\tilde{S},h^*[2\Delta_0+(n+2)\Gamma]+\tilde{g}^*\textstyle\sum_{i=1}^r\tilde{\Gamma}_i)\\
\simeq\;& H^0(\tilde{W},[q^*(2\Delta_0+(n+2)\Gamma)+\textstyle\sum_{i=1}^r\tilde{\Gamma}_i]) \oplus H^0(\tilde{W},[q^*(2\Delta_0+(n+2)\Gamma)+\sum_{i=1}^r\tilde{\Gamma}_i-\frac{1}{2}\tilde{B}])
\end{aligned}$$

Since $q^*(2\Delta_0+(n+2)\Gamma)+\sum_{i=1}^r\tilde{\Gamma}_i-\frac{1}{2}\tilde{B} \sim -q^*[\Delta_0+n\Gamma]+E$, it follows from the cohomology long exact sequence for

$$0 \to \mathscr{O}_{\tilde{W}}(-q^*[\Delta_0+n\Gamma]) \to \mathscr{O}_{\tilde{W}}([-q^*(\Delta_0+n\Gamma)+E]) \to \bigoplus_{i=1}^r(\mathscr{O}_{E_i^+}(-1)\oplus\mathscr{O}_{E_i^-}(-1)) \to 0$$

that $H^0(\tilde{W},[q^*(2\Delta_0+(n+2)\Gamma)+\sum_{i=1}^r\tilde{\Gamma}_i-\frac{1}{2}\tilde{B}]) = 0$. It follows that $H^0(S,[2G+(n+2)D])$ is the $(+1)$-eigen space with respect to the action of the hyperelliptic involution. So if $s$ were the restriction of an element of $H^0(S,[2G+(n+2)D])$, then it would vanish twice on the base points $e_1,\ldots,e_r$ of $|K_S|$ because they are also ramification points of $f|_G : G \to \mathbb{P}^1$. On the other hand, we have $G(2G+(n+2)D) = 2(r-n+2) < 2r$ when $n \geq 3$. Hence we conclude that $s=0$ when $n \geq 3$. If $n=2$ and $s \neq 0$, then $(s) = 2(e_1+\cdots+e_r)$. □

We let $\{U_i\}$ be a sufficiently fine open covering of $S$. Let $\{d_{ij}\}$, $\{\zeta_{ij}\}$ and $\{\kappa_{ij}\}$ be systems of transition functions of $[D]$, $[G]$ and $K_S$, respectively. We may assume that $\kappa_{ij} = d_{ij}^n\zeta_{ij}$ holds for any $i,j$. Let $x=\{x_i\}$ and $y=\{y_i\}$ be the basis for $H^0(S,[D])$, and $\zeta_i$ the local equation of $G$ in $U_i$; $\zeta_i = \zeta_{ij}\zeta_j$.

Let $\rho \in H^1(S, \Theta_S)$ and let $\{\rho_{ij}\}$ be the corresponding 1-cocycle with coefficients in $\Theta_S$. Then the 1-cocycle $\{y_i\rho_{ij} \cdot x_i - x_i\rho_{ij} \cdot y_i\}$ is nothing but the representative of the image of $\rho$ under the natural homomorphism $F^1 : H^1(S, \Theta_S) \to H^1(S, f^*\Theta_{\mathbb{P}^1})$. Recall that $\nu_{ij} = -\mathrm{div}\rho_{ij}$ satisfies

$$\nu_{ij} + \nu_{jk} + \nu_{ki} = -\rho_{jk} \cdot \log \kappa_{ij}$$

Furthermore, for $\phi = \{\phi_i\} \in H^0(S, K_S)$, $\{\rho_{ij} \cdot \phi_i - \nu_{ij}\phi_i\}$ is a 1-cocycle with coefficients in $\mathscr{O}_S(K_S)$. We consider two elements $\{x_i^n\zeta_i\}$ and $\{y_i^n\zeta_i\}$ in $H^0(S, K_S)$. We have

$$\begin{aligned} & \rho_{ij} \cdot (x_i^n\zeta_i) - \nu_{ij}(x_i^n\zeta_i) \\ = \; & n x_i^{n-1}\zeta_i\rho_{ij} \cdot x_i + x_i^n\rho_{ij} \cdot \zeta_i - \nu_{ij}x_i^n\zeta_i \\ = \; & x_i^{n-1}(n\zeta_i\rho_{ij} \cdot x_i + x_i\rho_{ij} \cdot \zeta_i - \nu_{ij}x_i\zeta_i) \end{aligned}$$

and similarly for $y_i^n\zeta_i$. Hence, if we put $X_{ij} = n\zeta_i\rho_{ij} \cdot x_i + x_i\rho_{ij} \cdot \zeta_i - \nu_{ij}x_i\zeta_i$ and $Y_{ij} = n\zeta_i\rho_{ij} \cdot y_i + y_i\rho_{ij} \cdot \zeta_i - \nu_{ij}y_i\zeta_i$, then the collections $\{X_{ij}\}$ and $\{Y_{ij}\}$ are 1-cocycles with coefficients in $\mathscr{O}_S([G+D])$. We denote by $X$ and $Y$ their cohomology classes, respectively. We have

$$y_iX_{ij} - x_iY_{ij} = n\zeta_i(y_i\rho_{ij} \cdot x_i - x_i\rho_{ij} \cdot y_i).$$

It follows that $(y_iX_{ij} - x_iY_{ij})|_G = 0$.

**(a) The case $n = 2$.**
We assume that $n = 2$. Then $H^1(S, [G + D]) = 0$. Therefore, there exist 0-cochains $\{\alpha_i\}$, $\{\beta_i\}$ satisfying $X_{ij} = d_{ij}\zeta_{ij}\alpha_j - \alpha_i$ and $Y_{ij} = d_{ij}\zeta_{ij}\beta_j - \beta_i$. Since $(y_iX_{ij} - x_iY_{ij})|_G = 0$, we see that $\{(y_i\alpha_i - x_i\beta_i)|_G\}$ is a 0-cocycle with coefficients in $\mathscr{O}_G([G + 2D]|_G)$. We denote its cohomology class by $\gamma(\rho) \in H^0(G, [G + 2D]|_G) = h^0(G, K_S|_G)$.

**Lemma 3.4.** *Assume that $n = 2$ and let $s \in H^0(G, K_S|_G)$ be a section vanishing at all the base points of $|K_S|$. Then there is a section $\tilde{s} \in H^0(S, K_S)$ such that $\tilde{s}|_G = s$.*

*Proof.* We have the exact sequence

$$0 \to H^0(S, K_S - G) \to H^0(S, K_S) \to H^0(G, K_S|_G)$$

We infer that the restriction map $H^0(S, K_S) \to H^0(G, K_S|_G)$ is of rank one. Recall that $K_S = [G + 2D]$, $G^2 = r - 4$ and $GD = 2$. Hence $K_S|_G$ is of degree $r$ which coincides with the number of base points of $|K_S|$. Therefore, for a section $\tilde{s}$ of $K_S$ which does not vanish identically on $G$, $\tilde{s}|_G$ vanishes exactly on the base points. If $s \in H^0(G, K_S|_G)$ vanishes on the base points, then it is a constant multiple of $\tilde{s}|_G$ □

**Lemma 3.5.** *Let $F^1 : H^1(S, \Theta_S) \to H^1(S, f^*\Theta_{\mathbb{P}^1})$ and $\zeta_* : H^1(S, \Theta_S) \to H^1(G, \mathscr{N}_G)$ be the natural maps. Let $\rho \in H^1(S, \Theta_S)$.*

(1) *$F^1(\rho) = 0$ if and only if $\gamma(\rho)$ vanishes at all the base points of $|K_S|$.*

(2) *$\zeta_*(\rho) = 0$ if and only if $\gamma(\rho)$ vanishes identically on $G$.*

*Proof.* (1) Assume that $F^1(\rho) = 0$. Then there exists a 0-cochain $\{a_i\}$ such that $y_i\rho_{ij} \cdot x_i - x_i\rho_{ij} \cdot y_i = d_{ij}^2 a_j - a_i$. Since $K_S = [G + 2D]$ when $n = 2$, we have

$$y_i\alpha_i - x_i\beta_i - 2\zeta_i a_i = \kappa_{ij}(y_j\alpha_j - x_j\beta_j - 2\zeta_j a_j)$$

Hence $\{y_i\alpha_i - x_i\beta_i - 2\zeta_i a_i\} \in H^0(S, K_S)$ and we see that $\gamma(\rho)$ must vanish at $\mathrm{Bs}|K_S|$. Conversely, assume that $\gamma(\rho)$ vanishes at $\mathrm{Bs}|K_S|$. It follows from the previous lemma that there exists an element $\tilde{\gamma} \in H^0(S, K_S)$ such that $\tilde{\gamma}|_G = \gamma(\rho)$. Putting $\tilde{\gamma} = \{\tilde{\gamma}_i\}$, $\tilde{\gamma}_i = \kappa_{ij}\tilde{\gamma}_j$, we can write $y_i\alpha_i - x_i\beta_i - \tilde{\gamma}_i = \zeta_i u_i$ with a holomorphic function $u_i$ on $U_i$. Hence

$$2\zeta_i(y_i\rho_{ij} \cdot x_i - x_i\rho_{ij} \cdot y_i) = \kappa_{ij}(y_j\alpha_j - x_j\beta_j - \tilde{\gamma}_j) - (y_i\alpha_i - x_i\beta_i - \tilde{\gamma}_i) = \zeta_i(d_{ij}^2 u_j - u_i)$$

which shows that $F^1(\rho) = 0$.

(2) If $\zeta_*(\rho) = 0$, then we can find holomorphic functions $v_i$ on $U_i \cap G$ such that $(\rho_{ij} \cdot \zeta_i)|_G = \zeta_{ij}v_j - v_i$ on $U_i \cap U_j \cap G$. On the other hand, we have $x_i\rho_{ij} \cdot \zeta_{ij}|_G = d_{ij}\zeta_{ij}\alpha_j - \alpha_i$ and $y_i\rho_{ij} \cdot \zeta_{ij}|_G = d_{ij}\zeta_{ij}\beta_j - \beta_i$ on $U_i \cap U_j \cap G$. It follows that $\{\alpha_i|_G - x_i v_i\}$ and $\{\beta_i|_G - y_i v_i\}$ represent elements of $H^0(G, [G + D]|_G)$ which is zero by Lemma 1.5. Hence $\alpha_i|_G = x_i v_i$ and $\beta_i|_G = y_i v_i$ on $U_i \cap G$. Therefore, $\gamma(\rho)$ vanishes identically on $G$. Conversely, if $\gamma(\rho)$ vanishes identically on $G$, then we have $y_i\alpha_i = x_i\beta_i$ on $U_i \cap G$. Since $x$ and $y$ have no common zero, $v_i = \alpha_i/x_i = \beta_i/y_i$ gives us a holomorphic function on $U_i \cap G$. Then $(\rho_{ij} \cdot \zeta_i)|_G = \zeta_{ij}v_j - v_i$ implying that $\zeta_*(\rho) = 0$. □

**(b) The case $n \geq 3$.**

We assume that $n \geq 3$. We have the Koszul exact sequence

$$0 \to \mathcal{O}_S([G]) \otimes \bigwedge^2 H^0(S, [D]) \to \mathcal{O}_S([G + D]) \otimes H^0(S, [D]) \to \mathcal{O}_S([G + 2D]) \to 0.$$

By Lemma 1.2 and the duality theorem, we have $2h^0(S, [G + D]) = 4 = h^0(S, [G + 2D]) + h^0(S, [G])$ since $n \geq 3$. It follows that the multiplication map $H^0(S, [G + D]) \otimes H^0(S, [D]) \to H^0(S, [G + 2D])$ is surjective. Hence we get the exact sequence

$$0 \to H^1(S, [G]) \otimes \bigwedge^2 H^0(S, [D]) \to H^1(S, [G + D]) \otimes H^0(S, [D]) \to H^1(S, [G + 2D])$$

Via the last map, $X \otimes y + Y \otimes x$ goes to $yX - xY$. Recall that we have $(yX - xY)|_G = 0$. Since $H^1(S, [G + 2D]) \to H^1(G, [G + 2D]|_G)$ is an isomorphism by Lemma 1.6, we see that $yX - xY = 0$ in $H^1(S, [G + 2D])$. Hence the above exact sequence gives us an element $Z = Z(\rho) \in H^1(S, [G])$ such that $X = xZ$ and $Y = yZ$. We let $\{Z_{ij}\}$ be the 1-cocycle with coefficients in $\mathcal{O}_S([G])$ whose class is $Z$. Then there exist 0-cochains $\{\alpha_i\}$, $\{\beta_i\}$ satisfying

$$X_{ij} = n\zeta_i\rho_{ij} \cdot x_i + x_i\rho_{ij} \cdot \zeta_i - \nu_{ij}x_i\zeta_i = x_i Z_{ij} + d_{ij}\zeta_{ij}\alpha_j - \alpha_i \tag{3.1}$$

$$Y_{ij} = n\zeta_i\rho_{ij} \cdot y_i + y_i\rho_{ij} \cdot \zeta_i - \nu_{ij}y_i\zeta_i = y_i Z_{ij} + d_{ij}\zeta_{ij}\beta_j - \beta_i \tag{3.2}$$

Then substituting them to $y_iX_{ij} - x_iY_{ij}$, we get

$$n\zeta_i(y_i\rho_{ij}\cdot x_i - x_i\rho_{ij}\cdot y_i) = d_{ij}^2\zeta_{ij}(y_j\alpha_j - x_j\beta_j) - (y_i\alpha_i - x_i\beta_i) \tag{3.3}$$

By restricting (3.3) to $G$, we see that the collection $\{(y_i\alpha_i - x_i\beta_i)|_G\}$ is a 0-cocycle with coefficients in $\mathcal{O}_G([G+2D]|_G)$. We denote its cohomology class by $\gamma = \gamma(\rho)$. We remark that it depends only on the cohomology class $\rho$ of $\{\rho_{ij}\}$.

**Lemma 3.6.** *Assume that $n \geq 3$ and let $F^1 : H^1(S, \Theta_S) \to H^1(S, f^*\Theta_{\mathbb{P}^1})$ be the natural map. Then $F^1(\rho) = 0$ if and only if $\gamma(\rho) = 0$.*

*Proof.* Assume that $F^1(\rho) = 0$. Then there exists a 0-cochain $\{a_i\}$ such that $y_i\rho_{ij}\cdot x_i - x_i\rho_{ij}\cdot y_i = d_{ij}^2a_j - a_i$. It follows form (3.3) that

$$y_i\alpha_i - x_i\beta_i - n\zeta_ia_i = d_{ij}^2\zeta_{ij}(y_j\alpha_j - x_j\beta_j - n\zeta_ja_j)$$

so that $\{y_i\alpha_i - x_i\beta_i - n\zeta_ia_i\} \in H^0(S, [G+2D])$. Since $n \geq 3$, the restriction map $H^0(S, [G+2D]) \to H^0(G, [G+2D]|_G)$ is the zero map. Hence $\gamma(\rho) = 0$.

Conversely, assume that $\gamma(\rho) = 0$. Then we have a collection $\{a_i\}$ such that $y_i\alpha_i - x_i\beta_i = \zeta_ia_i$. It follows from (3.3) that $n\zeta_i(y_i\rho_{ij}\cdot x_i - x_i\rho_{ij}\cdot y_i) = \zeta_i(d_{ij}^2a_j - a_i)$. Hence $y_i\rho_{ij}\cdot x_i - x_i\rho_{ij}\cdot y_i = d_{ij}^2a_j/n - a_i/n$ implying that $F^1(\rho) = 0$. □

We define a map $\zeta_* : H^1(S, \Theta_S) \to H^1(G, [G]|_G)$ by $\zeta_*(\rho) = \{(\rho_{ij}\cdot\zeta_i)|_G\}$. By restricting (3.1) and (3.2) to $G$, we get the following equalities on $G \cap U_i \cap U_j$:

$$x_i(\rho_{ij}\cdot\zeta_i - Z_{ij})|_G = d_{ij}\zeta_{ij}\alpha_j - \alpha_i, \quad y_i(\rho_{ij}\cdot\zeta_i - Z_{ij})|_G = d_{ij}\zeta_{ij}\beta_j - \beta_i.$$

Hence we have $x\zeta_*(\rho) = xZ(\rho)|_G$ and $y\zeta_*(\rho) = yZ(\rho)|_G$ in $H^1(G, [G+D]|_G)$.

**Lemma 3.7.** *Assume that $n \geq 3$. Then $\zeta_*(\rho) = 0$ if and only if $Z(\rho) = 0$ and $\gamma(\rho) = 0$.*

*Proof.* Assume that $\zeta_*(\rho) = 0$. Since $H^1(S, [G+D]) \to H^1(G, [G+D]|_G)$ is an isomorphism, we see that $xZ = yZ = 0$ in $H^1(S, [G+D])$. This implies that $Z = 0$ in $H^1(S, [G])$ as we saw in the proof of Lemma 1.7. Since $\zeta_*(\rho) = 0$, we automatically have $F^1(\rho) = 0$ and, hence, $\gamma(\rho) = 0$ by the previous lemma.

Conversely, assume that $Z(\rho) = 0$ and $\gamma(\rho) = 0$. We put $Z_{ij} = \zeta_{ij}z_j - z_i$. It follows from $\gamma(\rho) = 0$ that $y_i\alpha_i = x_i\beta_i$ on $G \cap U_i$. Hence $a_i = \alpha_i/x_i = \beta_i/y_i$ is a holomorphic function on $G \cap U_i$. Substituting $\alpha_i = x_ia_i$ and $\beta_i = y_ia_i$, we get

$$x_i\rho_{ij}\cdot\zeta_i|_G = x_i(\zeta_{ij}(z_j + a_j) - (z_i + a_i)), \quad y_i\rho_{ij}\cdot\zeta_i|_G = y_i(\zeta_{ij}(z_j + a_j) - (z_i + a_i)).$$

Since $x$ and $y$ have no common zeros, we conclude that $\rho_{ij}\cdot\zeta_i|_G = d_{ij}\zeta_{ij}(a_j + z_j) - (z_i + a_i)$ on $G \cap U_i \cap U_j$. Hence $\zeta_*(\rho) = 0$.

We remark that $h^1(S,[G+(n-2)D]) = 1$. We take a non-zero element $\xi \in H^1(S,[G+(n-2)D])$ which is represented by a 1-cocycle $\{\xi_{ij}\}$. Since $H^1(S,[G+(n-1)D]) = 0$, we can write

$$x_i\xi_{ij} = d_{ij}^{n-1}\zeta_{ij}u_j - u_i, \quad y_i\xi_{ij} = d_{ij}^{n-1}\zeta_{ij}v_j - v_i. \tag{3.4}$$

We put $\delta_i = y_iu_i - x_iv_i$. Then $\{\delta_i\}$ determines an element $\delta \in H^0(S, K_S)$.

**Lemma 3.8.** *$\delta|_G$ is not identically zero.*

*Proof.* Assume that we have $y_iu_i = x_iv_i$ on $G \cap U_i$. Then $w_i = u_i/x_i = v_i/y_i$ is a well-defined holomorphic function on $G \cap U_i$. By substituting $u_i = x_iw_i$ and $v_i = y_iw_i$, we get $x_i\xi_{ij}|_G = x_i(d_{ij}^{n-2}\zeta_{ij}w_j - w_i)$ and $y_i\xi_{ij}|_G = y_i(d_{ij}^{n-2}\zeta_{ij}w_j - w_i)$ on $G \cap U_i \cap U_j$. It follows that $\xi_{ij}|_G = d_{ij}^{n-2}\zeta_{ij}w_j - w_i$, implying that $\xi|_G = 0$ in $H^1(G,[G+(n-2)D]|_G)$.

We shall show that the restriction map $H^1(S,[G+(n-2)D]) \to H^1(G,[G+(n-2)D]|_G)$ is an isomorphism to see that $\xi = 0$ contradicting to the choice of $\xi$. Note that we have $h^1(G,[G+(n-2)D]|_G) = h^0(G,[G+2D]|_G) = 1$ by the duality theorem and Lemma 1.6. We consider the cohomology long exact sequence for

$$0 \to \mathscr{O}_S([(n-2)D]) \to \mathscr{O}_S([G+(n-2)D]) \to \mathscr{O}_G([G+(n-2)D]|_G) \to 0.$$

Since we have $\vec{h}((n-2)D) = (n-1, n-3, 3)$ and $\vec{h}(G+(n-2)D) = (n-1, 1, 3)$, we see that $H^1(S,[G+(n-2)D]) \to H^1(G,[G+(n-2)D]|_G)$ is an isomorphism. □

**Lemma 3.9.** *Assume that $n \geq 3$. If $F^1(\rho) \neq 0$ for $\rho \in H^1(S,\Theta_S)$, then $\gamma(\rho)\delta|_G$ is not the restriction of an element in $H^0(S,[2G+(n+2)D])$.*

*Proof.* By the assumption, $\gamma(\rho) \neq 0$ and $\delta|_G \neq 0$. Furthermore, $\delta$ vanishes at all the base points $e_1, \ldots, e_r$ of $|K_S|$. By Lemma 3.3, if $\gamma(\rho)\delta|_G$ is the restriction of an element of $H^0(S,[2G+(n+2)D])$, then it must be zero, a contradiction. □

We put $\phi^m = x^{n-m}y^m\zeta$ for $0 \leq m \leq n$ and $\phi^{n+1} = \delta$. Then these form a basis for $H^0(S, K_S)$. We have for $m$ with $0 \leq m \leq n$,

$$\begin{aligned}
&\rho_{ij}\cdot\phi_i^m - \nu_{ij}\phi_i^m \\
&= (n-m)x_i^{n-1-m}y_i^m\zeta_i\rho_{ij}\cdot x_i + mx_i^{n-m}y_i^{m-1}\zeta_i\rho_{ij}\cdot y_i + x_i^{n-m}y_i^m\rho_{ij}\cdot\zeta_i - \nu_{ij}x_i^{n-m}y_i^m\zeta_i \\
&= \tfrac{n-m}{n}x_i^{n-m-1}y_i^m(n\zeta_i\rho_{ij}\cdot x_i + x_i\rho_{ij}\cdot\zeta_i - \nu_{ij}x_i\zeta_i) + \tfrac{m}{n}x_i^{n-m}y_i^{m-1}(n\zeta_i\rho_{ij}\cdot y_i + y_i\rho_{ij}\cdot\zeta_i - \nu_{ij}y_i\zeta_i) \\
&= \tfrac{n-m}{n}x_i^{n-m-1}y_i^m(x_iZ_{ij} + d_{ij}\zeta_{ij}\alpha_j - \alpha_i) + \tfrac{m}{n}x_i^{n-m}y_i^{m-1}(y_iZ_{ij} + d_{ij}\zeta_{ij}\beta_j - \beta_i) \\
&= x_i^{n-m}y_i^mZ_{ij} + \tfrac{n-m}{n}(\kappa_{ij}x_j^{n-m-1}y_j^m\alpha_j - x_i^{n-m-1}y_i^m\alpha_i) + \tfrac{m}{n}(\kappa_{ij}x_j^{n-m}y_j^{m-1}\beta_j - x_i^{n-m}y_i^{m-1}\beta_i)
\end{aligned}$$

where we understand that, in the last expression, the second (resp. third) term is zero when $m = n$ (resp. $m = 0$). Hence

$$\begin{aligned}
&\rho_{ij}\cdot\phi_i^m - \nu_{ij}\phi_i^m \\
= \;& x_i^{n-m}y_i^mZ_{ij} + \kappa_{ij}(\tfrac{n-m}{n}x_j^{n-m-1}y_j^m\alpha_j + \tfrac{m}{n}x_j^{n-m}y_j^{m-1}\beta_j) - (\tfrac{n-m}{n}x_i^{n-m-1}y_i^m\alpha_i + \tfrac{m}{n}x_i^{n-m}y_i^{m-1}\beta_i)
\end{aligned}$$

Since $Z = Z(\rho) \in H^1(S,[G])$, we have $x^{n-2-k}y^kZ \in H^1(S,[G+(n-2)D])$ for any $k \in \mathbb{Z}$, $0 \le k \le n-2$. Hence there is a 0-cohain $\{w_i^{(k)}\}$ satisfying

$$x_i^{n-2-k}y_i^kZ_{ij} = A^{(k)}\xi_{ij} + d_{ij}^{n-2}\zeta_{ij}w_j^{(k)} - w_i^{(k)}$$

By Lemma 1.7, we see that $(A^{(0)},\dots,A^{(n-2)}) \in \mathbb{C}^{n-1}$ is not zero provided that $Z$ is not zero.

**Lemma 3.10.** *For $1 \le k \le n-2$, the collection $\{A^{(k)}u_i - A^{(k-1)}v_i + x_iw_i^{(k)} - y_iw_i^{(k-1)}\}$ determines an element in $H^0(S,[G+(n-1)D])$. In particular, it vanishes identically on $G$.*

*Proof.* We have $x_i^{n-1-k}y_i^kZ_{ij} = x_i^{n-2-k}y_i^k(x_iZ_i) = x_i^{n-2-(k-1)}y_i^{k-1}(y_iZ_i)$. Since $x_i\xi_{ij} = d_{ij}^{n-1}\zeta_{ij}u_j - u_i$ and $y_i\xi_{ij} = d_{ij}^{n-1}\zeta_{ij}v_j - v_i$, it follows that

$$d_{ij}^{n-1}\zeta_{ij}(A^{(k)}u_j + x_jw_j^{(k)}) - (A^{(k)}u_i + x_iw_i^{(k)}) = d_{ij}^{n-1}\zeta_{ij}(A^{(k-1)}v_j + y_jw_j^{(k-1)}) - (A^{(k-1)}v_i + y_iw_i^{(k-1)})$$

which shows the first assertion. Since $H^0(S,[G+(n-1)D])$ is spanned by $x^{n-1-m}y^m\zeta$, $0 \le m \le n-1$, any element vanishes identically on $G$. □

Note that $x_i^{n-m}y_i^mZ_{ij}$ can be expressed in four forms in general:

$$x_i^{n-m}y_i^mZ_{ij} = \begin{cases} \kappa_{ij}(A^{(m)}x_ju_j + x_j^2w_j^{(m)}) - (A^{(m)}x_iu_i + x_i^2w_i^{(m)}) \\ \kappa_{ij}(A^{(m-1)}y_ju_j + x_jy_jw_j^{(m-1)}) - (A^{(m-1)}y_iu_i + x_iy_iw_i^{(m-1)}) \\ \kappa_{ij}(A^{(m-1)}x_jv_j + x_jy_jw_j^{(m-1)}) - (A^{(m-1)}x_iv_i + x_iy_iw_i^{(m-1)}) \\ \kappa_{ij}(A^{(m-2)}y_jv_j + y_j^2w_j^{(m-2)}) - (A^{(m-2)}y_iv_i + y_i^2w_i^{(m-2)}) \end{cases} \tag{3.5}$$

but Lemma 3.10 shows the differences are not essential.

## 4 Kuranishi space.

Let $p : \mathscr{S} \to M$ be the Kuranishi family of deformations of $S = p^{-1}(0)$, $0 \in M$. Then $M$ is an analytic subset of a small open disc $\mathbf{D}$ around the origin of $\mathbb{C}^\mu$, where $\mu = \mu(S) = h^1(S,\Theta_S)$. We take a Hermitian metric on $S$, and let $\vartheta$ and $\mathbf{G}$ denote respectively the adjoint operator and Green's operator, with respect to this metric, on the space of $(0,q)$-forms with coefficients in $\Theta_S$. We define the Poisson bracket $[\cdot,\cdot]$ of two $(0,1)$-forms $\varphi$ and $\psi$ by the formula

$$[\varphi,\psi] = \sum_{\alpha,\beta}\left(\varphi^\alpha \wedge \frac{\partial\psi^\beta}{\partial z^\alpha} + \psi^\alpha \wedge \frac{\partial\varphi^\beta}{\partial z^\alpha}\right)\frac{\partial}{\partial z^\beta}$$

if

$$\varphi = \sum_\alpha \varphi^\alpha\frac{\partial}{\partial z^\alpha}, \quad \psi = \sum_\alpha \psi^\alpha\frac{\partial}{\partial z^\alpha}$$

are local expessions of $\varphi$ and $\psi$. With the notation, there exists a $(0,1)$-form $\varphi(t)$ with coefficients in $\Theta_S$ which depends holomorphically on $t \in \mathbf{D}$ and satisfies

$$\varphi(0) = 0, \quad \varphi(t) - \frac{1}{2}\vartheta\mathbf{G}[\varphi(t),\varphi(t)] = \varphi_1(t)$$

where $\varphi_1(t)$ denotes the linear term of $\varphi(t)$. Let $\mathbf{H}$ denote the projection onto the space of harmonic forms. Then $M$ is defined by the equation

$$\mathbf{H}[\varphi(t), \varphi(t)] = 0.$$

We let $\varphi_1(t) = \sum_\lambda \rho_\lambda t_\lambda$. We may assume that $\{\rho_\lambda\}$ forms a basis of the space of harmonic $(0,1)$-forms. We denote the cohomology class in $H^1(S, \Theta_S)$ of $\rho_\lambda$ by the same symbol. Note that we have

$$\mathbf{H}[\varphi(t), \varphi(t)] \equiv \sum_{\lambda,\sigma} \mathbf{H}[\rho_\lambda, \rho_\sigma] t_\lambda t_\sigma \mod (t^3)$$

We have the following lemma as in [1], Lemma 30.

**Lemma 4.1.** *If $\rho_\lambda, \rho_\sigma \in \mathrm{Ker}(F^1)$, then $\mathbf{H}[\rho_\lambda, \rho_\sigma] = 0$.*

**(a) The case $n = 2$.**
We can choose a basis $\{\rho_1, \rho_2, \ldots, \rho_{43-2r}\}$ satisfying

$$F^1(\rho_1) \neq 0,\ F^1(\rho_2) = 0,\ \zeta_*(\rho_2) \neq 0,\ \zeta_*(\rho_\lambda) = 0 \quad (\lambda \geq 3)$$

By Lemma 4.1, we have

$$\mathbf{H}[\varphi(t), \varphi(t)] = \sum_{\lambda=1}^{43-2r} a_\lambda t_1 t_\lambda + O(t^3)$$

We can show the following as in [1], Lemma 31.

**Lemma 4.2.** *If $\mathbf{H}[\rho_\lambda, \rho_\sigma] = 0$, then $\gamma(\rho_\lambda)\gamma(\rho_\sigma)$ is the restriction of an element of $H^0(S, 2K_S)$.*

Therefore, we have only to show the following to see that $a_2 \neq 0$.

**Lemma 4.3.** *$\gamma(\rho_1)\gamma(\rho_2)$ is not the restriction of an element of $H^0(S, 2K_S)$.*

*Proof.* Assume the contrary. It follows from Lemma 3.3 that $\gamma(\rho_1)\gamma(\rho_2)$ is either zero or its divisor is of the form $2(e_1 + \cdots + e_r)$. Both cases are inadequate. □

**Theorem 4.4.** *Assume that $n = 2$, and let $p : \mathscr{S} \to M$ be the Kuranishi family of deformations of $S$. Then*

(1) *$M = M_1 \cup M_2$, where the $M_i$'s are complex manifolds of dimension $42 - 2r$.*

(2) *$N = M_1 \cap M_2$ is a complex manifold of dimension $41 - 2r$.*

(3) *For $t \in M_1$, $S_t = p^{-1}(t)$ has a genus two fibration. For $t \in M_2 \setminus N$, $S_t$ does not have a genus two fibration.*

*Proof.* Let $p_1 : \mathscr{S}_1 \to M_1$ be the family of surfaces with genus two fibrations obtained before. By the completeness of the Kuranishi family, we have a holomorphic map $s : M_1 \to M$ such that

$\mathscr{S}_1$ is induced from $\mathscr{S}$. We can assume that the tangent map $ds : T_0M_1 \to T_0M$ is a bijection onto the linear subspace spanned by $\partial/\partial t_2, \ldots, \partial/\partial t_{43-2r}$. Hence $M_1$ can be identified with the submanifold defined by $r(t) := t_1 - \psi(t_2, \cdots, t_{43-2r}) = 0$, $(\partial\psi/\partial t_\lambda)|_{t=0} = 0$ for $\lambda \geq 2$. Now, since we have $\mathbf{H}[\varphi(t), \varphi(t)] = 0$ on $M_1$, we can write $\mathbf{H}[\varphi(t), \varphi(t)] = q(t)r(t)$. In view of the expansion of $\mathbf{H}[\varphi(t), \varphi(t)]$, we see that $q(t)$ is of the form

$$q(t) = \sum_{\lambda=1}^{43-2r} a_\lambda t_\lambda + O(t^2), \quad a_2 \neq 0.$$

Therefore, $M_2 = \{t \in M : q(t) = 0\}$ is a submanifold of $M$. □

**(b) The case $n \geq 3$.**

We take a basis $\{\rho_\lambda\}$ of $H^1(S, \Theta_S)$ satisfying

$$F^1(\rho_1) \neq 0,\ F^1(\rho_2) = \cdots = F^1(\rho_n) = 0,\ \zeta_*(\rho_\lambda) = 0 \quad (\forall\lambda > n)$$

and $\{\zeta_*(\rho_1), \zeta_*(\rho_2), \ldots, \zeta_*(\rho_n)\}$ forms a basis for $H^1(G, \mathscr{N}_G)$. If we put $Z_\lambda = Z(\rho_\lambda)$, then we can assume that $Z_1 = 0$ and $\{Z_2, \ldots, Z_n\}$ forms a basis for $H^1(S, [G])$. We put

$$x_i^{n-2-k} y_i^k Z_{\lambda ij} = A_\lambda^{(k)} \xi_{ij} + d_{ij}^{n-2} \zeta_{ij} w_{\lambda j}^{(k)} - w_{\lambda i}^{(k)}$$

Since $\{\rho_{\lambda ij} \cdot \phi_i^k - \nu_{\lambda ij}\phi_i^k\}$ gives us a cohomology class in $H^1(S, K_S) = 0$, we can write

$$\rho_{\lambda ij} \cdot \phi_i^k - \nu_{\lambda ij}\phi_i^k = \kappa_{ij}\tau_{\lambda j}^k - \tau_{\lambda i}^k.$$

We put

$$\Phi_i^k = \phi_i^k + \sum_\lambda \tau_{\lambda i}^k t_\lambda$$

Then we have

$$\begin{aligned}(\Phi_i^k)^2 - \Phi_i^{k-1}\Phi_i^{k+1} = \;& (\phi_i^k)^2 - \phi_i^{k-1}\phi_i^{k+1} + \textstyle\sum_\lambda (2\tau_{\lambda i}^k \phi_i^k - \tau_{\lambda i}^{k-1}\phi_i^{k+1} - \tau_{\lambda i}^{k+1}\phi_i^{k-1}) t_\lambda \\ & + (\textstyle\sum_\lambda \tau_{\lambda i}^k t_\lambda)^2 - (\sum_\lambda \tau_{\lambda i}^{k-1} t_\lambda)(\sum_\lambda \tau_{\lambda i}^{k+1} t_\lambda)\end{aligned}$$

The proof of the following lemma will be given in Appendix.

**Lemma 4.5.** *Fix a pair of indices $\mu, \nu$ such that $\mathbf{H}[\rho_\mu, \rho_\nu] = 0$. Then, for $0 < m < n$, the following congruence holds modulo $(t^3)$:*

$$\begin{aligned}& (\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1} \\ \equiv\;& (\kappa_{ij})^2[(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}] + 2\phi_i^m \Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\ & - \textstyle\sum_\lambda \{\rho_{\lambda ij} \cdot [(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}] - 2\nu_{\lambda ij}[(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}]\} t_\lambda\end{aligned}$$

*If $\hat{\Gamma}_{ij}^m$ denotes the coefficient of $t_\mu t_\nu$ in $\Gamma_{ij}^m$, then $\{\hat{\Gamma}_{ij}^m\}$ forms a 1-cocycle with coefficients in $\mathscr{O}_S(K_S)$.*

**Lemma 4.6.** *Let $\rho \in \mathrm{Span}(\rho_2, \cdots, \rho_n)$ be a non-zero element. Then $\mathbf{H}[\rho_1, \rho] \neq 0$.*

*Proof.* Without loosing generality, we can assume that $\rho = \rho_2$. Then we have $x^{n-1-m}y^{m-1}Z_2 \neq 0$ in $H^1(S, [G + (n-2)D])$ for some integer $m$ with $1 \leq m \leq n-1$, that is, $A_2^{(m-1)} \neq 0$. We fix such $m$.

By using the first, second and fourth expressions in (3.5), respectively, we can write

$$\begin{aligned}
\tau_{\lambda i}^{m-1} &= A_\lambda x_i u_i + \tfrac{n-m}{n} x_i^2 \tilde{\alpha}_{\lambda i} + \tfrac{m}{n} x_i^2 \tilde{\beta}_{\lambda i} + \tfrac{1}{n} x_i^2 (\tilde{\alpha}_{\lambda i} - \tilde{\beta}_{\lambda i}) \\
\tau_{\lambda i}^{m} &= A_\lambda y_i u_i + \tfrac{n-m}{n} x_i y_i \tilde{\alpha}_{\lambda i} + \tfrac{m}{n} x_i y_i \tilde{\beta}_{\lambda i} \\
\tau_{\lambda i}^{m+1} &= A_\lambda y_i v_i + \tfrac{n-m}{n} y_i^2 \tilde{\alpha}_{\lambda i} + \tfrac{m}{n} y_i^2 \tilde{\beta}_{\lambda i} - \tfrac{1}{n} y_i^2 (\tilde{\alpha}_{\lambda i} - \tilde{\beta}_{\lambda i}),
\end{aligned}$$

where we put $A_\lambda = A_\lambda^{(m-1)}$ and

$$\tilde{\alpha}_{\lambda i} = w_{\lambda i}^{(m-1)} + x_i^{n-2-m} y_i^{m-1} \alpha_{\lambda i}, \quad \tilde{\beta}_{\lambda i} = w_{\lambda i}^{(m-1)} + x_i^{n-1-m} y_i^{m-2} \beta_{\lambda i}.$$

We have mod $(t^2)$,

$$\begin{aligned}
(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1} &\equiv x_i^{n-m-1} y_i^{m-1} \zeta_i \textstyle\sum_\lambda (2x_i y_i \tau_{\lambda i}^m - x_i^2 \tau_{\lambda i}^{m+1} - y_i^2 \tau_{\lambda i}^{m-1}) t_\lambda \\
&= x_i^{n-m-1} y_i^{m-1} \zeta_i \textstyle\sum_\lambda A_\lambda (2x_i y_i^2 u_i - x_i^2 y_i v_i - x_i y_i^2 u_i) t_\lambda \\
&= x_i^{n-m-1} y_i^{m-1} \zeta_i \textstyle\sum_\lambda A_\lambda x_i y_i (y_i u_i - x_i v_i) t_\lambda \\
&= x_i^{n-m} y_i^m \zeta_i \textstyle\sum_\lambda A_\lambda \delta_i t_\lambda \\
&= \phi_i^m \delta_i \textstyle\sum_\lambda A_\lambda t_\lambda
\end{aligned}$$

We next consider its second order terms. Since $\tau_{\lambda i}^{m-1} = (x_i/y_i)\tau_{\lambda i}^m + (1/n)x_i^2(\tilde{\alpha}_{\lambda i} - \tilde{\beta}_{\lambda i})$ and $\tau_{\lambda i}^{m+1} = (y_i/x_i)(\tau_{\lambda i}^m - A_\lambda \delta_i) - (1/n)y_i^2(\tilde{\alpha}_{\lambda i} - \tilde{\beta}_{\lambda i})$, we get for two indices $\mu$, $\nu$

$$\begin{aligned}
&2\tau_{\mu i}^m \tau_{\nu i}^m - \tau_{\mu i}^{m-1}\tau_{\nu i}^{m+1} - \tau_{\nu i}^{m-1}\tau_{\mu i}^{m+1} \\
= {}& A_\mu \delta_i (\tau_{\nu i}^m + \tfrac{x_i y_i}{n}(\tilde{\alpha}_{\nu i} - \tilde{\beta}_{\nu i})) + A_\nu \delta_i (\tau_{\mu i}^m + \tfrac{x_i y_i}{n}(\tilde{\alpha}_{\mu i} - \tilde{\beta}_{\mu i})) + 2\tfrac{x_i^2 y_i^2}{n^2}(\tilde{\alpha}_{\mu i} - \tilde{\beta}_{\mu i})(\tilde{\alpha}_{\nu i} - \tilde{\beta}_{\nu i})
\end{aligned}$$

which is the coefficient of $t_\mu t_\nu$ in $(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}$. On the other hand, by Lemma 4.5, we have the following congruence modulo $(t^3)$:

$$\begin{aligned}
&(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1} \\
\equiv {}& (\kappa_{ij})^2[(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}] + 2\phi_i^m \Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
&- \textstyle\sum_\lambda \{\rho_{\lambda ij} \cdot [(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}] - 2\nu_{\lambda ij}[(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}]\} t_\lambda
\end{aligned} \tag{3.6}$$

We have

$$\begin{aligned}
&\rho_{\lambda ij} \cdot [(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}] - 2\nu_{\lambda ij}[(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}] \\
= {}& \rho_{\lambda ij} \cdot (\phi_i^m \delta_i \textstyle\sum_\sigma A_\sigma t_\sigma) - 2\nu_{\lambda ij}(\phi_i^m \delta_i \textstyle\sum_\sigma A_\sigma t_\sigma) \\
= {}& \textstyle\sum_\sigma A_\sigma t_\sigma (\delta_i \rho_{\lambda ij} \cdot \phi_i^m - \nu_{\lambda ij}\phi_i^m \delta_i + \phi_i^m \rho_{\lambda ij} \cdot \delta_i - \nu_{\lambda ij}\phi_i^m \delta_i) \\
= {}& \textstyle\sum_\sigma A_\sigma t_\sigma (\delta_i(\kappa_{ij}\tau_{\lambda j}^m - \tau_{\lambda i}^m) + \phi_i^m(\kappa_{ij}\tau_{\lambda j}^{n+1} - \tau_{\lambda i}^{n+1})) \\
= {}& \textstyle\sum_\sigma A_\sigma t_\sigma (\kappa_{ij}^2(\delta_j \tau_{\lambda j}^m + \phi_j^m \tau_{\lambda j}^{n+1}) - (\delta_i \tau_{\lambda i}^m + \phi_i^m \tau_{\lambda i}^{n+1}))
\end{aligned}$$

Therefore, comparing the coefficient of $t_\mu t_\nu$ in both sides of (3.6), we get

$$\begin{aligned}
&A_\mu \delta_i (\tau_{\nu i}^m + \tfrac{x_i y_i}{n}(\tilde{\alpha}_{\nu i} - \tilde{\beta}_{\nu i})) + A_\nu \delta_i (\tau_{\mu i}^m + \tfrac{x_i y_i}{n}(\tilde{\alpha}_{\mu i} - \tilde{\beta}_{\mu i})) + 2\tfrac{x_i^2 y_i^2}{n^2}(\tilde{\alpha}_{\mu i} - \tilde{\beta}_{\mu i})(\tilde{\alpha}_{\nu i} - \tilde{\beta}_{\nu i}) \\
= {}& \kappa_{ij}^2 (A_\mu \delta_j (\tau_{\nu j}^m + \tfrac{x_j y_j}{n}(\tilde{\alpha}_{\nu j} - \tilde{\beta}_{\nu j})) + A_\nu \delta_j (\tau_{\mu j}^m + \tfrac{x_j y_j}{n}(\tilde{\alpha}_{\mu j} - \tilde{\beta}_{\mu j})) + 2\tfrac{x_j^2 y_j^2}{n^2}(\tilde{\alpha}_{\mu j} - \tilde{\beta}_{\mu j})(\tilde{\alpha}_{\nu j} - \tilde{\beta}_{\nu j})) \\
&+ 2\phi_i^m \hat{\Gamma}_{ij}^m - \phi_i^{m-1}\hat{\Gamma}_{ij}^{m+1} - \phi_i^{m+1}\hat{\Gamma}_{ij}^{m-1} \\
&- A_\mu(\kappa_{ij}^2(\delta_j \tau_{\nu j}^m + \phi_j^m \tau_{\nu j}^{n+1}) - (\delta_i \tau_{\nu i}^m + \phi_i^m \tau_{\nu i}^{n+1})) - A_\sigma(\kappa_{ij}^2(\delta_j \tau_{\mu j}^m + \phi_j^m \tau_{\mu j}^{n+1}) - (\delta_i \tau_{\mu i}^m + \phi_i^m \tau_{\mu i}^{n+1}))
\end{aligned}$$

where $\hat{\Gamma}^k_{ij}$ denotes the coefficient of $t_\mu t_\nu$ in $\Gamma^k_{ij}$. It follows that

$$2x_iy_i\zeta_i\hat{\Gamma}^m_{ij} - x_i^2\zeta_i\hat{\Gamma}^{m+1}_{ij} - y_i^2\zeta_i\hat{\Gamma}^{m-1}_{ij} = d^{n+2}_{ij}\zeta^2_{ij}\epsilon_j - \epsilon_i$$

where

$$\begin{aligned}\epsilon_i = \ & A_\mu(x_iy_i\zeta_i\tau^{n+1}_{\nu i} - \tfrac{\delta_i}{n}(y_i\alpha_{\nu i} - x_i\beta_{\nu i})) + A_\nu(x_iy_i\zeta_i\tau^{n+1}_{\mu i} - \tfrac{\delta_i}{n}(y_i\alpha_{\mu i} - x_i\beta_{\mu i})) \\ & -\tfrac{2}{n^2}x_i^{n-m-1}y_i^{m-1}(y_i\alpha_{\mu i} - x_i\beta_{\mu i})(y_i\alpha_{\nu i} - x_i\beta_{\nu i})\end{aligned}$$

Now, we think $\mu = 1$ and $\nu = 2$. Then $A_\mu = 0$ and $\gamma(\rho_\nu) = 0$, that is, we can write $y_i\alpha_{\nu i} - x_i\beta_{\nu i} = \zeta_i w_{\nu i}$. Then

$$\epsilon_i = A_\nu(x_iy_i\zeta_i\tau^{n+1}_{\mu i} - \frac{\delta_i}{n}(y_i\alpha_{\mu i} - x_i\beta_{\mu i})) - \frac{2}{n^2}x_i^{n-m-1}y_i^{m-1}\zeta_i w_{\nu i}(y_i\alpha_{\mu i} - x_i\beta_{\mu i}).$$

If $\mathbf{H}[\rho_\mu, \rho_\nu] = 0$, then $\{\hat{\Gamma}^m_{ij}\}$ forms a 1-cocycle with coefficients in $\mathscr{O}_S(K_S)$ and hence cohomologous to zero. By restricting to $G$, we get

$$0 = d^{n+2}_{ij}\zeta^2_{ij}(-\frac{1}{n}A_\nu\delta_j\gamma_j(\rho_\mu)|_G) - (-\frac{1}{n}A_\nu\delta_i\gamma_i(\rho_\mu)|_G).$$

Since $A_\nu \neq 0$ by assumption, this implies that $\gamma(\rho_\mu)\delta|_G$ can be obtained by restricting an element of $H^0(S, [2G + (n+2)D])$ to $G$, which is impossible by Lemma 3.9. Hence $\mathbf{H}[\rho_\mu, \rho_\nu] \neq 0$. □

Recall that we have $h^2(S, \Theta_S) = n - 1$. We have

$$\mathbf{H}[\varphi(t), \varphi(t)] = \sum_{\lambda=1}^{7n+29-2r} \mathbf{H}[\rho_1, \rho_\lambda]t_1t_\lambda + O(t^3).$$

Since $\eta_\lambda := \mathbf{H}[\rho_1, \rho_\lambda]$, $2 \le \lambda \le n$, are linearly independent by Lemma 4.5, we can assume that they form a basis for $H^2(S, \Theta_S)$. We write

$$\mathbf{H}[\varphi(t), \varphi(t)] = \sum_{\lambda=2}^{n}\mathbf{H}[\varphi(t), \varphi(t)]_\lambda\eta_\lambda, \quad \mathbf{H}[\varphi(t), \varphi(t)]_\lambda = t_1\sum_{\mu=1}^{7n+29-2r} a_{\lambda\mu}t_\lambda + O(t^3),$$

with $a_{\lambda\lambda} = 1$ and $a_{\lambda\mu} = 0$ when $\mu \neq \lambda$ for $2 \le \lambda, \mu \le n$.

**Theorem 4.7.** *Let $M$ be the Kuranishi space of deformations of $S$. Then*

(1) *$M = M_1 \cup M_2$, where the $M_i$'s are complex manifolds with* $\dim M_1 = 7n + 28 - 2r$, $\dim M_2 = 6n + 30 - 2r$.

(2) *$N = M_1 \cap M_2$ is a complex manifold of dimension $6n + 29 - 2r$.*

(3) *For $t \in M_1$, $S_t$ has a genus two fibration. For $t \in M_2 \setminus N$, $S_t$ does not have a genus two fibration.*

*Proof.* Let $p_1 : \mathscr{S}_1 \to M_1$ be the family of surfaces with genus two fibrations obtained before. By the completeness of the Kuranishi family, we have a holomorphic map $s : M_1 \to M$ such that

$\mathscr{S}_1$ is induced from $\mathscr{S}$. We can assume that the tangent map $ds : T_0M_1 \to T_0M$ is a bijection onto the linear subspace spanned by $\partial/\partial t_2, \ldots, \partial/\partial t_{7n+29-2r}$. Hence $M_1$ can be identified with the submanifold of $M$ defined by $r(t) := t_1 - \psi(t_2, \cdots, t_{7n+29-2r}) = 0$, $(\partial\psi/\partial t_\lambda)|_{t=0} = 0$ for $\lambda \geq 2$. Now, since we have $\mathbf{H}[\varphi(t), \varphi(t)] = 0$ on $M_1$, we can write $\mathbf{H}[\varphi(t), \varphi(t)]_\lambda = q_\lambda(t)r(t)$ for $2 \leq \lambda \leq n$. In view of the expansion of $\mathbf{H}[\varphi(t), \varphi(t)]$, we see that $q(t)$ is of the form

$$q_\lambda(t) = t_\lambda + \sum_{\mu \neq \lambda} a_{\lambda\mu} t_\mu + O(t^2).$$

Therefore, $M_2 = \{t \in M : \ q_2(t) = \cdots = q_n(t) = 0\}$ is a submanifold of $M$. □

Now, Main Theorem in Introduction is nothing but the restatement of Theorems 4.4 and 4.7.

# 5 Appendix.

We show Lemma 4.5. We fix a couple of indices $(\mu, \nu)$ such that $\mathbf{H}[\rho_\mu, \rho_\nu] = 0$. Let $z_i = (z_i^1, z_i^2)$ be a system of coordinates on each $U_i$. Then there is a holomorphic function $b_{ij}^\alpha(z_j)$, $(\alpha = 1, 2)$ such that $z_i^\alpha = b_{ij}^\alpha(z_j)$ on $U_i \cap U_j$. Put

$$\rho_{\lambda ij} = \sum_\alpha \rho_{\lambda ij}^\alpha \frac{\partial}{\partial z_i^\alpha}$$

Then we can find holomorphic vector fields

$$\tilde{\rho}_{ij} = \sum_\alpha \tilde{\rho}_{ij}^\alpha \frac{\partial}{\partial z_i^\alpha}$$

on $U_i \cap U_j$ such that, if we set

$$\varphi_{ij}^\alpha = \varphi_{ij}^\alpha(z_j, t) = b_{ij}^\alpha(z_j) + \sum_\lambda \rho_{\lambda ij}^\alpha t_\lambda + \tilde{\rho}_{ij}^\alpha t_\mu t_\nu$$

and

$$\xi_{ijk}^\alpha = \varphi_{ik}^\alpha - \varphi_{ij}^\alpha(\varphi_{jk}, t),$$

then the coefficient of $t_\mu t_\nu$ in $\xi_{ijk}^\alpha$ vanishes. Put

$$\Psi_{ij}(z_j, t) = \det\left(\frac{\partial \varphi_{ji}^\beta}{\partial z_i^\alpha}(\varphi_{ij}(z_j, t), t)\right)$$

Then

$$\Psi_{ij} \equiv \kappa_{ij} - \sum_\lambda \kappa_{ij}\nu_{\lambda ij} t_\lambda \mod (t^2)$$

Assume that $\phi \in H^0(S, K_S)$ is represented by $\{\phi_i\}$, $\phi_i = \kappa_{ij}\phi_j$. Then there exists a cochain $\{\tau_{\lambda i}\}$ such that $\rho_{\lambda ij} \cdot \phi_i - \nu_{\lambda ij}\phi_i = \kappa_{ij}\tau_{\lambda j} - \tau_{\lambda i}$. We put

$$\Phi_i(z_i, t) = \phi_i(z_i) + \sum_\lambda \tau_{\lambda i} t_\lambda$$

and $\tilde{\Phi} = \Phi_i(\varphi_{ij}, t)$. Since $\phi_i(\varphi_{ij}) = \phi_i(b_{ij} + \sum_\lambda \rho_{\lambda ij} t_\lambda + \tilde{\rho}_{ij} t_\mu t_\nu)$, we get $[\phi_i(\varphi_{ij})]_0 = \phi_i(b_{ij}) = \phi_i$,

$$[\phi_i(\varphi_{ij})]_1 = \sum_\alpha \frac{\partial \phi_i}{\partial z_i^\alpha} \frac{\partial \varphi_{ij}^\alpha}{\partial t_\lambda}\bigg|_{t=0} = \sum \partial_\alpha \phi_i \rho_{\lambda ij}^\alpha t_\lambda = \sum_\lambda \rho_{\lambda ij} \cdot \phi_i t_\lambda$$

$$
\begin{aligned}
[\phi_i(\varphi_{ij})]_2 &= \tfrac{1}{2}\sum \frac{\partial^2\phi_i}{\partial z_i^\alpha \partial z_i^\beta}\frac{\partial\varphi_{ij}^\alpha}{\partial t_\lambda}\frac{\partial\varphi_{ij}^\beta}{\partial t_\sigma}\bigg|_{t=0} + \sum \frac{\partial f_i}{\partial z_i^\alpha}\frac{\partial^2\varphi_{ij}^\alpha}{\partial t_\lambda \partial t_\sigma}\bigg|_{t=0} t_\lambda t_\sigma \\
&= \tfrac{1}{2}\sum(\partial_\alpha\partial_\beta\phi_i)\rho_{\lambda ij}^\alpha\rho_{\sigma ij}^\beta t_\lambda t_\sigma + \textstyle\sum_\alpha \partial_\alpha\phi_i\tilde{\rho}_{ij}^\alpha(t_\mu t_\nu + t_\nu t_\mu) \\
&= \tfrac{1}{2}\sum(\partial_\alpha\partial_\beta\phi_i)\rho_{\lambda ij}^\alpha\rho_{\sigma ij}^\beta t_\lambda t_\sigma + 2(\tilde{\rho}_{ij}\cdot\phi_i)t_\mu t_\nu
\end{aligned}
$$

Similarly, we have $[\sum_\lambda \tau_{\lambda i}(\varphi_{ij})t_\lambda]_1 = \sum_\lambda \tau_{\lambda i}t_\lambda$ and

$$
[\sum_\lambda \tau_{\lambda i}(\varphi_{ij})t_\lambda]_2 = \sum \frac{\partial\tau_{\lambda i}}{\partial z_i^\alpha}\frac{\partial\varphi_{ij}^\alpha}{\partial t_\sigma}\bigg|_{t=0} t_\sigma t_\lambda = \sum \partial_\alpha\tau_{\lambda i}\rho_{\sigma ij}^\alpha t_\sigma t_\lambda = \sum \rho_{\sigma ij}\cdot\tau_{\lambda i}t_\lambda t_\sigma
$$

It follows that $[\tilde{\Phi}_i]_0 = \phi_i$, $[\tilde{\Phi}_i]_1 = \sum_\lambda \rho_{\lambda ij}\cdot\phi_i t_\lambda + \sum_\lambda \tau_{\lambda i}t_\lambda$ and

$$
[\tilde{\Phi}_i]_2 = \frac{1}{2}\sum_{\alpha,\beta,\lambda,\sigma}(\partial_\alpha\partial_\beta\phi_i)\rho_{\lambda ij}^\alpha\rho_{\sigma ij}^\beta t_\lambda t_\sigma + \sum_{\lambda,\sigma}\rho_{\lambda ij}\cdot\tau_{\sigma i}t_\lambda t_\sigma + 2(\tilde{\rho}_{ij}\cdot\phi_i)t_\mu t_\nu
$$

Therefore,

$$
\Phi_i \equiv \tilde{\Phi}_i - \sum_\lambda \rho_{\lambda ij}\cdot\phi_i t_\lambda - [\tilde{\Phi}_i]_2 \mod (t^3)
$$

On the other hand, we have $\tilde{\Phi}_i \equiv \Psi_{ij}\Phi_j \mod (t^2)$. Put $\Gamma_{ij} = [\tilde{\Phi}_i]_2 - [\Psi_{ij}\Phi_j]_2$. Then we get $\tilde{\Phi}_i \equiv \Psi_{ij}\Phi_j + \Gamma_{ij} \mod (t^3)$. It follows that

$$
\Phi_i \equiv \Psi_{ij}\Phi_j - \sum_\lambda \rho_{\lambda ij}\cdot\phi_i t_\lambda + \Gamma_{ij} - [\tilde{\Phi}_i]_2 \mod (t^3)
$$

We have the following congruence modulo $(t^3)$:

$$
\begin{aligned}
& (\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1} \\
\equiv\ & (\Psi_{ij})^2\left\{(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}\right\} + \Psi_{ij}(2\Phi_j^m\Gamma_{ij}^m - \Phi_j^{m-1}\Gamma_{ij}^{m+1} - \Phi_j^{m+1}\Gamma_{ij}^{m-1}) \\
& -\Psi_{ij}(2\Phi_j^m[\tilde{\Phi}_i^m]_2 - \Phi_j^{m-1}[\tilde{\Phi}_i^{m+1}]_2 - \Phi_j^{m+1}[\tilde{\Phi}_i^{m-1}]_2) \\
& -\Psi_{ij}\textstyle\sum_\lambda(2\Phi_j^m\rho_{\lambda ij}\cdot\phi_i^m - \Phi_j^{m-1}\rho_{\lambda ij}\cdot\phi_i^{m+1} - \Phi_j^{m+1}\rho_{\lambda ij}\cdot\phi_i^{m-1})t_\lambda \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)^2 - (\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)(\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda) \\
\equiv\ & (\Psi_{ij})^2\left\{(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}\right\} + \kappa_{ij}(2\phi_j^m\Gamma_{ij}^m - \phi_j^{m-1}\Gamma_{ij}^{m+1} - \phi_j^{m+1}\Gamma_{ij}^{m-1}) \\
& -\kappa_{ij}(2\phi_j^m[\tilde{\Phi}_i^m]_2 - \phi_j^{m-1}[\tilde{\Phi}_i^{m+1}]_2 - \phi_j^{m+1}[\tilde{\Phi}_i^{m-1}]_2) \\
& -2(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)\kappa_{ij}(\phi_j^m + \phi_j^m\sum_\lambda \nu_{\lambda ij}t_\lambda + \sum_\lambda \tau_{\lambda j}^m t_\lambda) \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)\kappa_{ij}(\phi_j^{m+1} + \phi_j^{m+1}\sum_\lambda \nu_{\lambda ij}t_\lambda + \sum_\lambda \tau_{\lambda j}^{m+1}t_\lambda) \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda)\kappa_{ij}(\phi_j^{m-1} + \phi_j^{m-1}\sum_\lambda \nu_{\lambda ij}t_\lambda + \sum_\lambda \tau_{\lambda j}^{m-1}t_\lambda) \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)^2 - (\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)(\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda) \\
\equiv\ & (\Psi_{ij})^2\left\{(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}\right\} + 2\phi_i^m\Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
& -(2\phi_i^m[\tilde{\Phi}_i^m]_2 - \phi_i^{m-1}[\tilde{\Phi}_i^{m+1}]_2 - \phi_i^{m+1}[\tilde{\Phi}_i^{m-1}]_2) \\
& -2(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)(\phi_i^m + \phi_i^m\sum_\lambda \nu_{\lambda ij}t_\lambda + \sum_\lambda \kappa_{ij}\tau_{\lambda j}^m t_\lambda) \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)(\phi_i^{m+1} + \phi_i^{m+1}\sum_\lambda \nu_{\lambda ij}t_\lambda + \sum_\lambda \kappa_{ij}\tau_{\lambda j}^{m+1}t_\lambda) \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda)(\phi_i^{m-1} + \phi_i^{m-1}\sum_\lambda \nu_{\lambda ij}t_\lambda + \sum_\lambda \kappa_{ij}\tau_{\lambda j}^{m-1}t_\lambda) \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)^2 - (\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)(\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda) \\
=\ & (\Psi_{ij})^2\left\{(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}\right\} + 2\phi_i^m\Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
& -(2\phi_i^m[\tilde{\Phi}_i^m]_2 - \phi_i^{m-1}[\tilde{\Phi}_i^{m+1}]_2 - \phi_i^{m+1}[\tilde{\Phi}_i^{m-1}]_2) \\
& -2(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)(\phi_i^m + \sum_\lambda(\tau_{\lambda i}^m + \rho_{\lambda ij}\cdot\phi_i^m)t_\lambda) \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)(\phi_i^{m+1} + \sum_\lambda(\tau_{\lambda i}^{m+1} + \rho_{\lambda ij}\cdot\phi_i^{m+1})t_\lambda) \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda)(\phi_i^{m-1} + \sum_\lambda(\tau_{\lambda i}^{m-1} + \rho_{\lambda ij}\cdot\phi_i^{m-1})t_\lambda) \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)^2 - (\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)(\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda)
\end{aligned}
$$

$$\begin{aligned}
= \quad & (\Psi_{ij})^2 \left\{(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}\right\} + 2\phi_i^m\Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
& -(2\phi_i^m[\tilde{\Phi}_i^m]_2 - \phi_i^{m-1}[\tilde{\Phi}_i^{m+1}]_2 - \phi_i^{m+1}[\tilde{\Phi}_i^{m-1}]_2) \\
& -2(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)\Phi_i^m + (\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)\Phi_i^{m+1} + (\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda)\Phi_i^{m-1} \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)^2 - (\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda) \\
= \quad & (\Psi_{ij})^2 \left\{(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}\right\} + 2\phi_i^m\Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
& -(2\phi_i^m[\tilde{\Phi}_i^m]_2 - \phi_i^{m-1}[\tilde{\Phi}_i^{m+1}]_2 - \phi_i^{m+1}[\tilde{\Phi}_i^{m-1}]_2) \\
& -\textstyle\sum_\lambda \rho_{\lambda ij}\cdot[(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}]t_\lambda \\
& +\textstyle\sum_{\lambda,\sigma}(2\phi_i^m\rho_{\lambda ij}\cdot\tau_{\sigma i}^m - \phi_i^{m-1}\rho_{\lambda ij}\cdot\tau_{\sigma i}^{m+1} - \phi_i^{m+1}\rho_{\lambda ij}\cdot\tau_{\sigma i}^{m-1})t_\lambda t_\sigma \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)^2 - (\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda) \\
= \quad & (\Psi_{ij})^2 \left\{(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}\right\} + 2\phi_i^m\Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
& -(2\phi_i^m[\tilde{\Phi}_i^m]_2 - \phi_i^{m-1}[\tilde{\Phi}_i^{m+1}]_2 - \phi_i^{m+1}[\tilde{\Phi}_i^{m-1}]_2) \\
& -2(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)\Phi_i^m + (\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)\Phi_i^{m+1} + (\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda)\Phi_i^{m-1} \\
& +(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)^2 - (\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda) \\
= \quad & (\Psi_{ij})^2 \left\{(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}\right\} + 2\phi_i^m\Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
& -\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\left\{(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}\right\}t_\lambda \\
& +\sum_{\lambda,\sigma}(\phi_i^m\sum_{\alpha,\beta}\partial_\alpha\partial_\beta\phi_i^m\rho_{\lambda ij}^\alpha\rho_{\sigma ij}^\beta - (\phi_i^{m-1}/2)\sum_{\alpha,\beta}\partial_\alpha\partial_\beta\phi_i^{m+1}\rho_{\lambda ij}^\alpha\rho_{\sigma ij}^\beta - (\phi_i^{m+1}/2)\sum_{\alpha,\beta}\partial_\alpha\partial_\beta\phi_i^{m-1}\rho_{\lambda ij}^\alpha\rho_{\sigma ij}^\beta)t_\lambda t_\sigma \\
& +\left\{(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^m t_\lambda)^2 - (\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m-1}t_\lambda)(\textstyle\sum_\lambda \rho_{\lambda ij}\cdot\phi_i^{m+1}t_\lambda)\right\} \\
& -2(2\phi_i^m\tilde{\rho}_{ij}\cdot\phi_i^m - \phi_i^{m-1}\tilde{\rho}_{ij}\cdot\phi_i^{m+1} - \phi_i^{m+1}\tilde{\rho}_{ij}\cdot\phi_i^{m-1})t_\mu t_\nu
\end{aligned}$$

In the last expression, we do not need the last term, because we have

$$2\phi_i^m\tilde{\rho}_{ij}\cdot\phi_i^m - \phi_i^{m-1}\tilde{\rho}_{ij}\cdot\phi_i^{m+1} - \phi_i^{m+1}\tilde{\rho}_{ij}\cdot\phi_i^{m-1} = \tilde{\rho}_{ij}\cdot\left\{(\phi_i^m)^2 - \phi_i^{m-1}\phi_i^{m+1}\right\} = 0$$

Furthermore, a calculation shows that the fourth and fifth terms are canceled out. To see this, we formally put $z = y_i/x_i$. Then we have $\phi_i^{m-1} = \phi_i^m/z$ and $\phi_i^{m+1} = z\phi_i^m$. Noting that $\partial_\alpha z = (x_i\partial_\alpha y_i - y_i\partial_\alpha x_i)/x_i^2$, we get

$$\frac{1}{z}\phi_i^m\partial_\alpha\partial_\beta(z\phi_i^m) = \frac{1}{z}(\phi_i^m)^2\partial_\alpha\partial_\beta z + \frac{1}{z}\phi_i^m(\partial_\alpha z\partial_\beta\phi_i^m + \partial_\beta z\partial_\alpha\phi_i^m) + \phi_i^m\partial_\alpha\partial_\beta\phi_i^m$$

$$z\phi_i^m\partial_\alpha\partial_\beta(\frac{\phi_i^m}{z}) = z(\phi_i^m)^2\partial_\alpha\partial_\beta\frac{1}{z} + z\phi_i^m(\partial_\alpha\frac{1}{z}\partial_\beta\phi_i^m + \partial_\beta\frac{1}{z}\partial_\alpha\phi_i^m) + \phi_i^m\partial_\alpha\partial_\beta\phi_i^m$$

Since $\frac{1}{z}\partial_\alpha z + z\partial_\alpha\frac{1}{z} = \partial_\alpha 1 = 0$, we have

$$\frac{1}{z}\partial_\alpha\partial_\beta z + z\partial_\alpha\partial_\beta\frac{1}{z} = \frac{1}{z}\partial_\alpha\partial_\beta z - z\partial_\alpha(\frac{1}{z^2}\partial_\beta z) = \frac{2}{z^2}\partial_\alpha z\partial_\beta z.$$

Therefore,

$$\begin{aligned}
& \phi_i^m\sum_{\alpha,\beta}\partial_\alpha\partial_\beta\phi_i^m\rho_{\lambda ij}^\alpha\rho_{\sigma ij}^\beta - (\phi_i^{m-1}/2)\sum_{\alpha,\beta}\partial_\alpha\partial_\beta\phi_i^{m+1}\rho_{\lambda ij}^\alpha\rho_{\sigma ij}^\beta - (\phi_i^{m+1}/2)\sum_{\alpha,\beta}\partial_\alpha\partial_\beta\phi_i^{m-1}\rho_{\lambda ij}^\alpha\rho_{\sigma ij}^\beta \\
& = \left(\frac{\phi_i^m}{x_i y_i}\right)^2 (x_i\rho_{\lambda ij}\cdot y_i - y_i\rho_{\lambda ij}\cdot x_i)(x_i\rho_{\sigma ij}\cdot y_i - y_i\rho_{\sigma ij}\cdot x_i)
\end{aligned}$$

To rewrite the fifth term, we note that

$$\rho_{\lambda ij}\cdot(\frac{1}{z}\phi_i^m) = \phi_i^m\rho_{\lambda ij}\cdot\frac{1}{z} + \frac{1}{z}\rho_{\lambda ij}\cdot\phi_i^m = -\frac{1}{z^2}\phi_i^m\rho_{\lambda ij}\cdot z + \frac{1}{z}\rho_{\lambda ij}\cdot\phi_i^m,$$

$$\rho_{\lambda ij}\cdot(z\phi_i^m) = \phi_i^m\rho_{\lambda ij}\cdot z + z\rho_{\lambda ij}\cdot\phi_i^m.$$

Then

$$
\begin{aligned}
& (\textstyle\sum_\lambda \rho_{\lambda ij} \cdot \phi_i^{m-1} t_\lambda)(\sum_\lambda \rho_{\lambda ij} \cdot \phi_i^{m+1} t_\lambda) \\
= \ & (-(\phi_i^m/z^2)\textstyle\sum_\lambda \rho_{\lambda ij} \cdot z t_\lambda + (1/z)\sum_\lambda \rho_{\lambda ij} \cdot \phi_i^m t_\lambda)(\phi_i^m \sum_\lambda \rho_{\lambda ij} \cdot z t_\lambda + z\sum_\lambda \rho_{\lambda ij} \cdot \phi_i^m t_\lambda) \\
= \ & (\textstyle\sum_\lambda \rho_{\lambda ij} \cdot \phi_i^m t_\lambda)^2 - (\frac{\phi_i^m}{z})^2 (\sum_\lambda \rho_{\lambda ij} \cdot z t_\lambda)^2 \\
= \ & (\textstyle\sum_\lambda \rho_{\lambda ij} \cdot \phi_i^m t_\lambda)^2 - (\frac{\phi_i^m}{x_i y_i})^2 (\sum_\lambda (x_i \rho_{\lambda ij} \cdot y_i - y_i \rho_{\lambda ij} \cdot x_i) t_\lambda)^2
\end{aligned}
$$

Hence the fourth and fifth terms are canceled. Now, we continue to calculate the congruence modulo $(t^3)$

$$
\begin{aligned}
& (\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1} \\
\equiv \ & (\Psi_{ij})^2 \left\{ (\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1} \right\} + 2\phi_i^m \Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
& - \textstyle\sum_\lambda \rho_{\lambda ij} \cdot \left\{ (\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1} \right\} t_\lambda \\
\equiv \ & \kappa_{ij}^2 \left\{ (\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1} \right\} + 2\phi_i^m \Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
& + 2\kappa_{ij}^2 \textstyle\sum_\lambda \nu_{\lambda ij} \left\{ (\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1} \right\} t_\lambda - \sum_\lambda \rho_{\lambda ij} \cdot \left\{ (\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1} \right\} t_\lambda \\
\equiv \ & (\Psi_{ij})^2 \left\{ (\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1} \right\} + 2\phi_i^m \Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
& - \textstyle\sum_\lambda \rho_{\lambda ij} \cdot \left\{ (\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1} \right\} t_\lambda \\
\equiv \ & \kappa_{ij}^2 [(\Phi_j^m)^2 - \Phi_j^{m-1}\Phi_j^{m+1}] + 2\phi_i^m \Gamma_{ij}^m - \phi_i^{m-1}\Gamma_{ij}^{m+1} - \phi_i^{m+1}\Gamma_{ij}^{m-1} \\
& - \textstyle\sum_\lambda \left\{ \rho_{\lambda ij} \cdot [(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}] - 2\nu_{\lambda ij}[(\Phi_i^m)^2 - \Phi_i^{m-1}\Phi_i^{m+1}] \right\} t_\lambda
\end{aligned}
$$

which is what we want. For the rest, see [1]. □

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