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Pregeometric Shells of a Rational Quartic Curve and of a Veronese surface

姫工大・理　遊佐　毅 (Takeshi Usa)
Dept. of Math. Himeji Institute of Technology *

Abstract

This is a resume of the author's talk given at R.I.M.S. on June 18 2003 and is a partial revision of a recent paper [23] with additional results. The theme of that talk was to give a supporting evidence for general conjectures raised in [20] (cf. Conjecture 0.1). We classify all the pregeometric shells of a rational normal quartic curve X (resp. of a Veronese surface X), namely the closed subschemes in $\mathbb{P}^4(\mathbb{C})$ (resp. in $\mathbb{P}^5(\mathbb{C})$) which include X and whose homogeneous coordinate rings satisfy the Tor injectivity condition. This Tor injectivity condition is the same as to impose that for every non-negative integer q, minimal generators of q-th syzygy of its homogeneous coordinate ring form a part of minimal generators of q-th syzygy of the homogeneous coordinate ring R_X of X. We see that all those pregeometric shells turn out to be reduced and irreducible, and moreover the varieties of Δ-genus zero (embedded by their complete linear systems, cf. Remark 1.3), namely the varieties of minimal degree, as predicted from these general conjectures.

Keywords: pregeometric shell, rational quartic curve, Veronese surface, Δ-genus, variety of minimal degree

§0 Introduction.

Dreaming to construct a theory of projective embeddings modeled after the classical Galois theory, we presented several conjectures and problems on the geometric structures of projective embeddings in our previous paper [20]. One of the key concepts appeared in these conjectures and problems is "pregeometric shell" (abbr. PG-shell; cf. Definition 2.4), which was first introduced in [19] with expectation that it may play a similar role as a concept of "intermediate extension field" in the Galois theory and may cut our way to the dream.

We already saw in [20] and [22] that the (pre)geometric shells inherit many excellent properties (cf. Proposition 2.5, Corollary 3.7) from their (pre)geometric core (abbr. (P)G-core ; cf. Definition 2.4), which reflect the structure of higher syzygies of the homogeneous coordinate ring of the (pre)geometric core. "Pregeometric shells" had appeared implicitly in many classical works (cf. [13], [15], [5], [7], [16], [6],[3] etc.) as actual examples. However, there are still deep mysteries on pregeometric shells left (cf. [20]). For example, the following conjectures are still open with slight modifications (precisely, only the first one is added after publishing of [20]).

*2167 Shosha, Himeji, 671-2201 Japan.
E-mail address : usa@sci.himeji-tech.ac.jp
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Conjecture 0.1 *We fix the total space $P = \mathbb{P}^N(\mathbb{C})$ with the tautological ample line bundle $O_P(1) = O_P(H)$ and consider its closed subschemes defined over $\mathbb{C}$ with induced polarizations.*

(0.1.1) *Assume that a closed subscheme W is a pregeometric shell of an arithmetically Cohen-Macaulay closed subscheme V. Then the subscheme W is also an arithmetically Cohen-Macaulay subscheme. (N.B. If $dim(W) = dim(V)$, then this is obviously true by Auslander-Buchsbaum formula on the depths and the homological dimensions. Thus, in this case, e.g. if $dim(V) > 0$, then the case: $W = V \amalg \{1\ pt.\}$ never occurs.)*

(0.1.2) *Assume that a closed subscheme W is a pregeometric shell of a closed subvariety $V \subseteq P$. Then the subscheme W is also a variety, namely reduced and irreducible. (N.B. When the subscheme W is a scheme of codimension one, this is true. cf. Lemma 3.2, Proposition 2.5 (2.5.1))*

(0.1.3) *Take closed subvarieties V and W of positive dimension. Assume that the subvariety V satisfies the arithmetic D_2 (depth ≥ 2) condition, namely the natural map $H^0(P, O_P(m)) \to H^0(V, O_V(m))$ is surjective for every integer m. If W is a pregeometric shell of V, then on their Δ-genera (e.g. $\Delta(V, O_V(1)) := dim(V) + deg(O_V(1)) - h^0(V, O_V(1))$;* (cf. [6]), *the inequality:*

$$\Delta(V, O_V(1)) \geq \Delta(W, O_W(1))$$

holds. (e.g. As a typical case, if the polarized manifold $(V, O_V(1))$ is arithmetically normal and is a hypersurface cut of the polarized manifold $(W, O_W(1))$, then W is a PG-shell of V and this inequality is obviously true. On the other hand, if we assume that V is non-degenerate arithmetically Buchsbaum, and W is a hypersurface, then the result [17] on special cases of Eisenbud-Goto conjecture shows that this claim is also true.)

(0.1.4) *Take a closed subvariety W, a vector bundle E on W, a section $\sigma \in \Gamma(W, E)$, and put: $V = Z(\sigma)$. Assume :(a) the subscheme V is a variety and satisfies the arithmetic D_2 condition; (b) the restricted section $\sigma|_{Reg(W)} \in \Gamma(Reg(W), E|_{Reg(W)})$ is transverse to the zero section on the open set $Reg(W)$; (c) W is a pregeometric shell of V and $Reg(V) \subseteq Reg(W)$. Then the bundle E is a nef bundle (N.B. If we do not assume the arithmetic D_2 condition on V, we have a counter-example. cf. [21]. On the other hand, if $W = P$ and $rank(E) = 2$, this claim is true).*

Once we fix a closed scheme V, its PG-shells generally exist infinitely many, but by the elementary properties of PG-shells (cf. Proposition 2.5 (2.5.4)), they are bounded by an algebraic family of finite components, which suggests the theoretical possibility of classifying its PG-shells completely. Thus, to find evidences for these conjectures, what we should do first is to classify all the PG-shells of the variety V of Δ-genus zero.

Applying an elementary property (cf. Lemma 2.11) of PG-shells, we can easily reduced this problem to the case that the variety V is a non-degenerate rational normal curve of degree $d \geq 3$. Then the easiest but non-trivial case is the case : $d = 4$. This case also relates with a Veronese surface which is the only heretical (and the most interesting) case in the list of the varieties with Δ-genus zero.

In this article, with a help of a classical result on varieties of minimal degree (cf. [12]), or of Fujita's modern theory on polarized varieties by using Δ-genera (cf. [6]), using rather primitive and classical methods, we study these two cases as a first step toward those conjectures. After the classifications of PG-shells, we see that all the claims above are affirmative in these two cases.

The author would like to express his deep gratitude to Prof. M. Green for inviting him to UCLA, which brought him calm days and a nice environment for research, to Prof. T. Ashikaga for much incentive for this classification (cf. [1]), and to Prof. K. Konno for giving a chance of the talk to the author.

§1 Main Results.

To give an overview, let us summarize our main results as two main theorems. We should emphasize again here that our "schemes" of course may have a non-equidimensional component or a non-reduced structure. For precise definitions on terminology, see the next section §2.

The first main theorem is on the classification of PG-shells of a rational normal quartic curve. We will give its full proof in the latter part of this article.

Main Theorem 1.1 *Let X be a non-degenerate non-singular rational quartic curve in a 4-th projective space: $P = \mathbb{P}^4(\mathbb{C})$ with the tautological line bundle $O_P(1) = O_P(H)$, and a closed subscheme $W \subsetneq P$ a pregeometric shell of X with $codim(W, P) > 0$. Then the scheme W is one of the following cases.*

(1.1.1) *If codim(W)=1, then W be a divisor of P and an irreducible and reduced quadric hypersurface of rank* 3, 4, *or* 5*(i.e. non-singular). None of these quadric hypersurfaces is a FG-shell(cf. Definition2.4) of X.*

(1.1.2) *If codim(W)=2, then the scheme W is irreducible and reduced. The polarized variety $(W, O_W(1))$ is of Δ-genus zero (of minimal degree). More precisely, the surface W is a projective cone of a non-singular twisted cubic curve or a one point blow-up of a projective plane $\mathbb{P}^2(\mathbb{C})$. In the former case, the curve X passes through the vertex of the cone, and is not a Cartier divisor. Hence there is no shell frame of X in W. In the latter case, the variety W is embedded by a linear system coming from the conics passing the center of blow-up, and the curve X is a nef (Cartier) divisor on W (thus a unique shell frame $(O_W(X), \sigma_X)$ exists), which comes from a singular irreducible and reduced cubic plane curve passing through the center of blow-up doubly or from an irreducible conic which does not pass through the center of blow-up (more precisely, cf. Proposition 3.6 and §4.).*

(1.1.3) *If codim(W)=3, then the scheme W exactly coincides with X.*

Including the trivial case: codim(W) = 0, i.e. $W = P$, a polarized scheme $(W, O_W(1))$ which is a pregeometric shell of the curve X is always a variety of Δ-genus zero (of minimal degree), and therefore an arithmetically Cohen-Macaulay variety.

Based on Main Theorem1.1, applying Lemma2.11, we can get the next main theorem on a classification of PG-shells of a Veronese surface which is the only heretical case in the classification on the varieties of Δ-genus zero. In this article, we omit its proof because of the page limit. For its full proof, see [24].

Main Theorem 1.2 *Let X be a Veronese surface, namely the image of a projective plain: $\mathbb{P}^2(\mathbb{C})$ by a Veronesean embedding of degree 2 given by a complete linear system : $|O_{\mathbb{P}^2(\mathbb{C})}(2)|$, in a 5-th projective space: $P = \mathbb{P}^5(\mathbb{C})$ with the tautological line bundle $O_P(1) = O_P(H)$, and a closed subscheme $W \subsetneq P$ a pregeometric shell of X with $codim(W, P) > 0$. Then the scheme W is one of the following cases.*

(1.2.1) *If codim(W)=1, then W be a divisor of P and an irreducible and reduced quadric hypersurface. None of these quadric hypersurfaces is a FG-shell of V.*

(1.2.2) *If codim(W)=2, then the scheme W is irreducible and reduced. The polarized variety $(W, O_W(1))$ is of Δ-genus zero (of minimal degree). More precisely, The singular locus $Sing(W)$ of W is only one point v_0, which is included by the surface V. The one point chordal variety(cf. Lemma2.15): $Cd(v_0, V)$ of the surface V coincides with the variety W. Take a hyperplain H which does not pass the point v_0 and intersects the variety W transversely. Then, the hyperplain cut $W \cap H$ is isomorphic to a one point blow up of $\mathbb{P}^2(\mathbb{C})$. Thus, the variety W is also a projective cone of a one point blow up of $\mathbb{P}^2(\mathbb{C})$.*

(1.2.3) *If codim(W)=3, then the scheme W exactly coincides with X.*

Including the trivial case: codim(W) = 0, i.e. $W = P$, a polarized scheme $(W, O_W(1))$ which is a pregeometric shell of the curve X is always a variety of Δ-genus zero (of minimal degree), and therefore an arithmetically Cohen-Macaulay variety.

Remark 1.3 *With respect to a polarized variety (V, L), namely a pair of a projective variety V and an ample line bundle L on V, the main concern of Fujita's theory or of the theory of Δ-genus is in the geometric analysis of the embedding into a weighted projective space associated to the line bundle L instead of the embedding into a usual projective space by the linear system $|L|$. However, if we restrict ourselves to the case that the line bundle L is simply generated, or the variety V is embedded into a usual projective space with the "arithmetic D_2"condition, Fujita's theory can be applied directly to our problems. Thus, in this article, if we say that the subvariety $V \subset P = \mathbb{P}^N(\mathbb{C})$ is a variety of Δ-genus zero, it means that the pair $(V, O_P(1)|_V)$ is a variety of Δ-genus zero in the sense of Fujita's theory and the variety V is embedded into P by the comlete linear system $|O_P(1)|_V|$ (and therefore the variety V is non-degenerate in P), where the simple generation of the (very) ample line bundle $O_P(1)|_V$ is always guaranteed by Fujita's theory for the variety of Δ-genus zero (cf. [6]).*

§2 Preliminaries.

To avoid needless confusions, let us confirm our notation used in this article.

Notation and Conventions 2.1 *In this paper, we use the terminology of [8] without mentioned, and always admit the conventions and use the notation below for simplicity.*

(2.1.1) *Every object under consideration is defined over the field of complex numbers $\mathbb{C}$. We will work in the category of algebraic schemes over $\mathbb{C}$ and algebraically holomorphic morphisms (or rational maps) or in the categories of coherent sheaves and their ($\mathcal{O}$-linear) homomorphisms otherwise mentioned.*

(2.1.2) *Let us take a complex projective scheme X of dimension n and one of its embeddings $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$. The sheaf of ideals defining $j(X)$ in P and the conormal sheaf are denoted by I_X and $N^\vee_{X/P} = I_X/I_X^2$, respectively. Taking a $\mathbb{C}$-basis $\{Z_0, \dots, Z_N\}$ of $H^0(P, O_P(1))$. Then we put:*

$$\begin{aligned}
S &:= \bigoplus_{m\geq 0} H^0(P, O_P(m)) \cong \mathbb{C}[Z_0, \dots, Z_N] \\
S_+ &:= = \bigoplus_{m>0} H^0(P, O_P(m)) \cong (Z_0, \dots, Z_N)\mathbb{C}[Z_0, \dots, Z_N] \\
\widetilde{R_X} &:= \bigoplus_{m\geq 0} H^0(X, O_X(m)) \\
\mathbb{I}_X &:= \bigoplus_{m\geq 0} H^0(P, I_X(m)) \\
R_X &:= Im[S \to \widetilde{R_X}] \cong S/\mathbb{I}_X.
\end{aligned}$$

In this case, the induced ample line bundle $j^*O_P(1) = j^*O_P(H)$ is denoted by $O_X(1)$ or $O_X(H)$. If the scheme X itself is another projective space $\mathbb{P}^n(\mathbb{C})$, then its tautological ample line bundle $O_{\mathbb{P}^n(\mathbb{C})}(1)$ is denoted simply by $O(1)$ without any letter to specify the space, otherwise mentioned explicitly. Thus, if we consider the d-th Veronesean embedding $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$ for $X = \mathbb{P}^n(\mathbb{C})$, then there is an identification of the line bundles $O_X(H) = O_X(1) = O(d)$.

(2.1.3) *For a graded S-module E, the symbol* $E_{(m)}$ *denotes the degree m part of E, namely* $E = \bigoplus_{m\in\mathbb{Z}} E_{(m)}$. *The degree shift:* $E(d)$ *of the module E means that* $(E(d))_{(m)} := E_{(m+d)}$. *Affine sheafication of the S-module E (i.e. a canonically constructed* $O_{Spec(S)}$*-module from a S-module) is denoted by* $E^{\sim}$ *and projective sheafication of the graded S-module E (i.e. a canonically constructed* $O_{Proj(S)}$*-module from a graded S-module) is denoted by* $E^{(\sim)}$*, respectively.*

(2.1.4) *For a coherent sheaf F on a closed subscheme* $V \subseteq P$*, we set the Hilbert polynomial of the sheaf F to be:*

$$A_F(m) := \chi(F(m)) = \sum_{q=0}^{N}(-1)^q \dim(H^q(V, F(m))).$$

In case of $F = O_V$*, we denote its Hilbert polynomial as* $A_V(m)$ *instead of* $A_{O_V}(m)$*. Moreover, if* $(V, O_V(1)) = (\mathbb{P}^k(\mathbb{C}), O_{\mathbb{P}^k(\mathbb{C})}(1))$*, we simply write* $A_k(m)$ *for* $A_{\mathbb{P}^k(\mathbb{C})}(m)$*. (Precise properties of Hilbert polynomials are referred to [11]).*

(2.1.5) *For a real valued function* $f(x)$ *defined on the field of real numbers* $\mathbb{R}$ *(or on the ring of rational integers* $\mathbb{Z}$*), we define the (first) (backward) difference function to be:* $(\nabla f)(x) := f(x) - f(x-1)$ *and the k-th difference function to be* $(\nabla^k f)(x) := (\nabla(\nabla^{k-1} f))(x)$ *for a positive integer k (N.B.* $(\nabla^0 f)(x) := f(x)$*). The operator* ∇ *is called* (backward) difference operator.

(2.1.6) *Let B be a projective line* $\mathbb{P}^1(\mathbb{C})$*. For a non-negative integer e (dare to include* $e = 1$*), we set :* $\pi : \Sigma_e := \mathbb{P}(O_{\mathbb{P}^1(\mathbb{C})} \oplus O_{\mathbb{P}^1(\mathbb{C})}(-e)) \to B$*, (i.e. a rational ruled surface of degree e), a curve* C_e *in the surface* Σ_e *to be a tautological* π*-ample divisor determined by the vector bundle* $O_{\mathbb{P}^1(\mathbb{C})} \oplus O_{\mathbb{P}^1(\mathbb{C})}(-e)$ *and a curve f to be the fibre of the morphism* π*.*

We will need a classical and well-known result on the Picard group of a rational ruled surface (cf. [8]).

Lemma 2.2 *Under the circumstances of (2.1.6), let us consider a rational ruled surface* Σ_e*. Then the curves:* $\{f,\ C_e\}$ *form* $\mathbb{Z}$*-free basis of the Picard group* $Pic(\Sigma_e)$ *of the surface* Σ_e *and have the following properties.*

(2.2.1) *The intersection numbers are :* $C_e^2 = -e \leq 0$*;* $f^2 = 0$*; and* $f.C_e = 1$*.*

(2.2.2) *For integers u and v,*

the divisor $uf + vC_e$ *is very ample* $\Leftrightarrow$ *the divisor* $uf + vC_e$ *is ample* $\Leftrightarrow$ *"* $v > 0,\ u > ve$ *".*

(2.2.3) *For integers u and v,*

the linear system $|uf + vC_e|$ *contains an irreducible non-singular curve* $\Leftrightarrow$ *it contains an irreducible curve* $\Leftrightarrow$ *"*$(u, v) = (1, 0)$ *or* $(0, 1)$*; or* $v > 0,\ u > ve$ *; or* $e > 0,\ v > 0,\ u = ve$*".*

(2.2.4) *On the canonical divisor of the surface* Σ_e*, we have:*

$$K_{\Sigma_e} = (-2 - e)f + (-2)C_e.$$

Since in our classification, we have to consider non-reduced and non-equidimensional schemes, need an accurate handling on their nilpotent structures, and can not reduce our problems to the case of reduced structures, we have to pay special attention to "primary (or irreducible)" decompositions of schemes.

Definition 2.3 *For a Noetherian scheme W and finite number of its closed subschemes $\{V_s\}_{s=1}^k$, scheme theoretic union $Y = \cup_{s=1}^k V_s$ is defined by an ideal sheaf $I_Y := \cap_{s=1}^k I_{V_s}$, namely the kernel sheaf: $Ker[O_P \to \oplus_{s=1}^k O_{V_s}]$. On the other hand, since there is no uniqueness of primary decomposition of ideals in Noetherian rings, once if an arbitrary Noetherian scheme Y which may have a nilpotent structure is given first, then it is not so trivial to find its closed subschemes $\{V_s\}_{s=1}^k$ such that each topological space $|V_s|$ of the subscheme V_s is irreducible and the scheme theoretic union $\cup_{s=1}^k V_s$ coincides with Y. Thus we will restrict ourselves to the case that the scheme Y is a closed subscheme of $P = \mathbb{P}^N(\mathbb{C}) = Proj(S)$ and will define its "primary decomposition" and "irreducible decomposition" (not uniquely). For the homogeneous ideal $\mathbb{I}_Y$ of Y, we have a homogeneous primary decomposition in the shortest representation: $\mathbb{I}_Y = \cap_{s=1}^k \mathbb{J}_s$, and we set $J_s := (\mathbb{J}_s)^{(\sim)}$, $V_s := (Supp(O_P/J_s), O_P/J_s)$ for $s = 1, \ldots, k$. It is easy to see that their scheme theoretic union $\cup_{s=1}^k V_s$ coincides with Y. We call this finite set of closed subschemes $\{V_s\}_{s=1}^k$ as* a primary decomposition *of Y. Next we pick up all the minimal prime ideals of $\mathbb{I}_Y$. Then we may assume that $\{\mathbb{J}_i\}_{i=1}^t$ $(t \leq k)$ are the primary ideals associated to these minimal prime ideals, respectively. As is well-known, only these primary ideals: $\{\mathbb{J}_i\}_{i=1}^t$ (resp. only these "maximal" primary component subschemes :$\{V_i\}_{i=1}^t$) are determined uniquely by the ideal $\mathbb{I}_Y$ (resp. by the subscheme Y). Now, for $i = 1, \ldots, t$, we set a subscheme U_i to be the scheme theoretic union:*

$$U_i := \bigcup_{V_s \subseteq V_i} V_s.$$

Then we call the finite set of closed subschemes $\{U_i\}_{i=1}^t$ as an irreducible decomposition *of Y. We should make a remark that the homogeneous ideal $\mathbb{I}_Y$ does not have the irrelevant maximal ideal S_+ as an associated prime by its definition, the homogeneous ideal $\mathbb{I}_{V_s}$ coincides with the ideal $\mathbb{J}_s$, and that the homogeneous ideal $\mathbb{I}_{U_i}$ does with the ideal:*

$$\bigcap_{\sqrt{\mathbb{J}_i} \subseteq \sqrt{\mathbb{J}_s}} \mathbb{J}_s.$$

If $dim(Y) = n$, then a primary component V_s of dimension n (resp. an irreducible component U_i of dimension n) is called a main primary component *(resp.* a main irreducible component*).*

Now let us recall our key concepts for studying the geometric structures of projective embeddings, the first two : ("PG-shell" and "G-shell") of them were introduced first in [19]. On G-shells, we have slightly modified its definition from $V \subseteq Reg(W)$ in [19] to $Reg(V) \subseteq Reg(W)$. By reconsidering classical examples in Complex Projective Geometry including *varieties of minimal degree*, with standing on this new point of view, we can find many good actual examples for these concepts in a number of works such as [13], [15], [5], [7], [16], [6], [3] and so on.

Definition 2.4 (shells and cores) *Let V and W be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$ which satisfy $V \subseteq W$ (namely the inclusion of the defining ideal sheaves: $I_V \supseteq I_W$ in the structure sheaf O_P of P; In this case, the subscheme W is called simply an* intermediate ambient scheme *of V). If the natural map:*

$$\mu_q : Tor_q^S(R_W, S/S_+) \to Tor_q^S(R_V, S/S_+)$$

is injective for every integer $q \geq 0$ (abbr. "Tor injectivity condition"*), we say that W is a* pregeometric shell *(abbr.* PG-shell*) of V and that V is a* pregeometric core *(abbr.* PG-core*) of W. Moreover, if V and W are closed subvarieties and the regular locus $Reg(W)$ of W contains $Reg(V)$, we say that W is a* geometric shell *(abbr.* G-shell*) of V and V is a* geometric core *(abbr.* G-core*) of W. Furthermore, we assume that (a) there exist a vector bundle E on W and a global section $\sigma \in \Gamma(W, E)$; (b) the zero locus $Z(\sigma)$ (the scheme structure of $Z(\sigma)$ is defined by an ideal sheaf: $I_{Z(\sigma)} := Im[\sigma : E^\vee \to O_P]$) coincides* *with V including their scheme structures; (c) the restricted section $\sigma|_{Reg(W)} \in \Gamma(Reg(W), E|_{Reg(W)})$ is transverse to the zero section on $Reg(W)$, then we say that W is a* framed geometric shell *(abbr.* FG-shell*) of V and V is a* framed geometric core *(abbr.* FG-core*) of W. In this case, the pair (E,σ) is called a* shell frame *of V in W. For the subscheme V, the total space P and V itself are called* trivial PG-shells *(or* trivial G-shells *if V is a variety).*

Let us recall a proposition on several elementary properties of PG-shells from [20]. The outline of proof is referred to [20] (on the properties of G-shells related to their restricted syzygy bundles and infinitesimal syzygy bundles, which are not presented here, see [22]).

Proposition 2.5 *Let V and W be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$ which satisfy $V \subseteq W$.*

(2.5.1) *If W is a hypersurface (i.e. a divisor of P; cf. also Lemma 3.2), then W is a pregeometric shell of V if and only if the equation of W is a member of minimal generators of the homogeneous ideal $\mathbb{I}_V$ of V.*

(2.5.2) *Assume that the subscheme V is a complete intersection. Then the scheme W is a pregeometric shell of V if and only if the subscheme W is defined by a part of minimal generators of $\mathbb{I}_V$.*

(2.5.3) *Take a closed scheme Y such that $V \subseteq Y \subseteq W$. Assume that W is a pregeometric shell of V. Then W is also a pregeometric shell of Y. In particular, the subscheme W is also a pregeometric shell of the m-th infinitesimal neighborhood $Y = (V/W)_{(m)}$ of V in W, where $(V/W)_{(m)} = (|V|, O_W/I_{V/W}^{m+1})$.*

(2.5.4) *Fix the subscheme V of $codim(V,P) \geq 2$. Then all non-trivial pregeometric shells of V form non empty algebraic family of finite components (N.B. The family of all non-trivial G-shells of V may be empty even if V itself is a smooth variety).*

(2.5.5) *If W is a pregeometric shell of V, then we have an inequality: $arith.depth(V) \leq arith.depth(W)$ on their arithmetic depths. In particular, if the natural restriction map $H^0(P, O_P(m)) \to H^0(V, O_V(m))$ is surjective for all integers m (i.e. $R_V = \widetilde{R_V}$), then the natural restriction map $H^0(P, O_P(m)) \to H^0(W, O_W(m))$ is also surjective for all integers m (i.e.$R_W = \widetilde{R_W}$). In other words, the arithmetic D_2 condition is inherited from pregeometric cores to their pregeometric shells.*

(2.5.6) *If the subscheme W is a pregeometric shell of the subscheme V with $arith.depth(V) \geq 2$, then we have an inequality on their Castelnuovo-Mumford regularity(cf. [4]): $reg^{CaM}(V) \geq reg^{CaM}(W)$.*

(2.5.7) *Assume that there exist r hypersurfaces $D_1, \ldots, D_r$ in P with homogeneous equations $F_1, \cdots, F_r$ of degree $m_1, \cdots, m_r$, respectively, and satisfying the conditions : (a) $V = W \cap D_1 \cap \ldots \cap D_r$; (b) $H^0(W_t, O_{W_t}) = \mathbb{C}$ ($t = 0, \cdots, r$), where $W_0 := W$ and $W_t := W \cap D_1 \cap \ldots \cap D_t$ ($t = 1, \cdots, r$) ; (c) the homogeneous equations $F_1, \cdots F_r$ form an O_W-regular sequence, namely the sequence :*

$$0 \longrightarrow O_{W_{t-1}}(-m_t) \xrightarrow{\times F_t} O_{W_{t-1}},$$

is exact for $t = 1, \cdots, r$. If $arith.depth(V) \geq 2$, then W is a pregeometric shell.

(2.5.8) *Assume that the subscheme V is non-degenerate, namely no hyperplane contains V. If W has a 2-linear resolution, i.e. the homogeneous coordinate ring R_W of W has a minimal S-free resolution of the form : $0 \leftarrow R_W \leftarrow S \leftarrow F_1(-2) \leftarrow F_2(-3) \leftarrow \cdots \leftarrow F_p(-p-1) \leftarrow \cdots$, where $F_u(v)$ denotes $\oplus S(v)$: a direct sum of several copies of S with degree v shift, then W is a pregeometric shell of V. (cf. also Corollary 3.7)*

Remark 2.6 *Related to Proposition 2.5, there were two minor mistakes in the claims of [20]. The first one was in the old version of the claim (2.5.6). The author had failed to attach the condition: arith.depth$(V) \geq 2$ by a misprint, which is corrected in the improved version of [20]:math.AG/0001004. The second one was a failure to attach the condition: $H^0(W_t, O_{W_t}) = \mathbb{C}$ $(t = 0, \cdots, r)$ of the claim (2.5.7). Without this condition, the claim (2.5.7) is not true in general. A counterexample is given by the next example.*

Example 2.7 *In $P = \mathbb{P}^N(\mathbb{C})$ $(N \geq 4)$, take a smooth hypersuface $D_0 \subset P$ of degree $m_0 \geq 2$, a line L_0 which intersects the hypersuface D_0 transversely, and define a closed subscheme W to be $D_0 \cup L_0$, namely by a sheaf of ideals : $I_W := I_{D_0} \cap I_{L_0} \subset O_P$. Now we take a smooth hypersuface $D_1 \subset P$ of degree $m_1 \geq 2$ which intersects the hypersuface D_0 and the line L_0 transversely, and satisfies $D_1 \cap D_0 \cap L_0 = \phi$. Then we see that $W_1 = W \cap D_1 = (D_0 \cap D_1) \cup \{p_1, \cdots, p_{m_1}\}$, where the set $\{p_1, \cdots, p_{m_1}\}$ is a set of finite points : $D_1 \cap L_0$. Last we choose a smooth hypersurface $D_2 \subset P$ of degree $m_2 \geq 2$ with two conditions: $D_2 \cap \{p_1, \cdots, p_{m_1}\} = \phi$; D_2 intersects $D_0 \cap D_1$ transversely, and put $V := W \cap D_1 \cap D_2$. The scheme V coincides with $D_0 \cap D_1 \cap D_2$. Then we see easily that arith.depth$(V) \geq 2$, the two hypersurfaces D_1 and D_2 form an O_W-regular sequence, $H^0(O_W) \cong H^0(O_V) \cong \mathbb{C}$, but $H^0(O_{W_1}) \cong \mathbb{C}^{\oplus m_1+1}$. Since the variety V is a complete intersection, its pregeometric shells are only complete intersections. However, the scheme W is obviously not a complete intersection, and therefore is not a pregeometric shell of V. In this case, it is easy to check that arith.depth$(W) \leq 2$ by seeing $h^1(O_W) = m_0 - 1 \geq 1$.*

Relating to Remark(2.6), we summarize an easy result on homogeneous coordinate rings of closed subschemes, which is well-known in the case of closed subvarieties.

Proposition 2.8 *Let $W \subseteq P = \mathbb{P}^N(\mathbb{C})$ $(N \geq 2)$ be a closed subscheme satisfying $H^0(O_W) = \mathbb{C}$, $D \subset P$ a hypersurface with a homogeneous equation F of degree m_0. Assume that the equation F of D is an O_W-regular element, namely the multiplication : $\times F : O_W(-m_0) \to O_W$ is injective, and that arith.depth$(W \cap D) \geq 2$. Then, $R_{W\cap D} = R_W / F.R_W$, and arith.depth$(W) =$ arith.depth$(W \cap D) + 1$.*

Before starting its proof, let us recall an easy fact from (5.2)Lemma of [18].

Lemma 2.9 *Let $W \subseteq P = \mathbb{P}^N(\mathbb{C})$ be a closed subscheme satisfying* $\dim(W) \geq 1$ *and $H^0(O_W) = \mathbb{C}$, and $O_W(1)$ the restriction bundle of the tautological ample line bundle $O_P(1)$. Then $H^0(O_W(-m)) = 0$ for any positive integer m.*

Proof of Proposition (2.8) Since we assume *arith.depth*$(W \cap D) \geq 2$, we have the surjectivity of the restriction map: $H^0(P, O_P(m)) \to H^0(W \cap D, O_{W\cap D}(m))$ for any integer $m \in \mathbb{Z}$. Let us show first that *arith.depth*$(W) \geq 2$, namely the surjectivity of the restriction map : $H^0(P, O_P(m)) \to H^0(W, O_W(m))$ for any integer $m \in \mathbb{Z}$ by induction on m. By the assumtion, we can apply Lemma(2.9) to the scheme W and may assume that the restriction map: $H^0(P, O_P(m')) \to H^0(W, O_W(m'))$ is surjective for any integer $m' \in \mathbb{Z}$ with $m' < m$ as an induction hypothesis. We apply the snake lemma to the diagram:

$$\begin{tikzcd}
 & 0 & & 0 & \\
0 \arrow[r] & H^0(W, O_W(m-m_0)) \arrow[u] \arrow[r, "\times F"] & H^0(W, O_W(m)) \arrow[r] & H^0(W\cap D, O_{W\cap D}(m)) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & H^0(P, O_P(m-m_0)) \arrow[u] \arrow[r, "\times F"] & H^0(P, O_P(m)) \arrow[u] \arrow[r] & H^0(D, O_D(m)) \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

and see the surjectivity of the restriction map: $H^0(P, O_P(m)) \to H^0(W, O_W(m))$ (This argument is the same to apply Nakayama's lemma to the S-module $\oplus_m H^0(W, O_W(m))$ based on the finite generation of this module asserted by Lemma(2.9)). Then, we again apply the snake lemma to the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & H^0(W, O_W(m-m_0)) & \xrightarrow{\times F} & H^0(W, O_W(m)) & \longrightarrow & H^0(W\cap D, O_{W\cap D}(m)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (R_W)_{(m-m_0)} & \xrightarrow{\times F} & (R_W)_{(m)} & \longrightarrow & (R_W/F.R_W)_{(m)} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and see that $R_{W\cap D} = R_W/F.R_W$, which implies that $arith.depth(W) = arith.depth(W \cap D) + 1$. ∎

Corollary 2.10 *Let V and W be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$ which satisfy $V \subseteq W$. Assume that there exist r hypersurfaces $D_1, \ldots, D_r$ in P with homogeneous equations $F_1, \cdots, F_r$ of degree $m_1, \cdots, m_r$, respectively, and satisfying the conditions : (a) $V = W \cap D_1 \cap \ldots \cap D_r$; (b) $H^0(W_t, O_{W_t}) = \mathbb{C}$ ($t = 0, \cdots, r$), where $W_0 := W$ and $W_t := W \cap D_1 \cap \ldots \cap D_t$ ($t = 1, \cdots, r$) ; (c) the homogeneous equations $F_1, \cdots F_r$ form an O_W-regular sequence. If $arith.depth(V) \geq 2$, then $R_{W_t} = R_W/(F_1, \cdots, F_t)R_W$, the sequence $F_1, \cdots, F_r$ form R_W-regular sequence, and $arith.depth(W) = arith.depth(V) + r$.*

By the similar argument, we can obtain the next helpful lemma for studying PG-shells of arithmetically Cohen-Macaulay PG-cores. In these cases, this lemma guarantees that we can apply the hyperplain cut (or "Apollonius") method to reduce the problems on higher dimensional PG-cores into those on PG-core curves.

Lemma 2.11 *Let V and W be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$ which satisfy $V \subseteq W$.*

(2.11.1) *Assume that $arith.depth(V) \geq 2$ and the scheme W is a pregeometric shell of V in P. If we take a hyperplain $H \subset P$ with a linear equation F which is an O_V and O_W-regular element, then the scheme $W \cap H$ is a pregeometric shell of $V \cap H$ in the projective space $H \cong \mathbb{P}^{N-1}(\mathbb{C})$ (or in P).*

(2.11.2) *Suppose that $H^0(O_V) = H^0(O_W) = \mathbb{C}$. Take a hyperplain $H \subset P$ with a linear equation F which is an O_V and O_W-regular element. Assume that $arith.depth(V \cap H) \geq 2$ and the scheme $W \cap H$ is a pregeometric shell of $V \cap H$ in the projective space H (or in P). Then the scheme W is a pregeometric shell of V in P.*

Proof. Let us suppose the assumption of the claim (2.11.1). By the definition, the rings R_V and R_W can be regarded as subrings of $\oplus_m H^0(V, O_V(m))$ and of $\oplus_m H^0(W, O_W(m))$, respectively, which implies that the equation F is a regular element for R_V and for R_W. The claim (2.5.5) shows that $arith.depth(W) \geq arith.depth(V) \geq 2$, which implies that $depth_{S_+}(R_W/F.R_W) \geq depth_{S_+}(R_V/F.R_V) \geq 1$. Thus we see that $R_{V\cap H} = R_V/F.R_V$ and $R_{W\cap H} = R_W/F.R_W$. By tensoring S/S_+ to the exact sequence:

$$0 \longrightarrow R_V(-1) \xrightarrow{\times F} R_V \longrightarrow R_{V\cap H} \longrightarrow 0,$$

we have an exact sequence:

$$Tor_q^S(R_V(-1), S/S_+) \xrightarrow{\times F} Tor_q^S(R_V, S/S_+) \longrightarrow Tor_q^S(R_{V\cap H}, S/S_+)$$

$$\longrightarrow Tor_{q-1}^S(R_V(-1), S/S_+) \xrightarrow{\times F} Tor_{q-1}^S(R_V, S/S_+).$$

Since the modules: $Tor_q^S(R_V(-1), S/S_+)$ $(q \geq 0)$ are S/S_+-modules, the multiplication by F annihilates the modules: $Tor_q^S(R_V(-1), S/S_+)$ $(q \geq 0)$. After the similar argument on R_W, we have an exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Tor_q^S(R_W, S/S_+) & \longrightarrow & Tor_q^S(R_{W\cap H}, S/S_+) & \longrightarrow & Tor_{q-1}^S(R_W(-1), S/S_+) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_q} & & \downarrow{\scriptstyle \overline{\mu_q}} & & \downarrow{\scriptstyle \mu_{q-1}} & & \\ 0 & \longrightarrow & Tor_q^S(R_V, S/S_+) & \longrightarrow & Tor_q^S(R_{V\cap H}, S/S_+) & \longrightarrow & Tor_{q-1}^S(R_V(-1), S/S_+) & \longrightarrow & 0. \end{array}$$

Then the assumption asserts the injectivity of the map μ_q and of the map μ_{q-1}, which brings the injectivity of the map $\overline{\mu_q}$, or equivalently the scheme $W \cap H$ is a PG-shell of $V \cap H$ in the projective space P.

Next let us suppose the assumption of the claim (2.11.2). Then, the claim (2.5.5) shows that $arith.depth(W \cap H) \geq arith.depth(V \cap H) \geq 2$. By Proposition (2.8), we see that $R_{V\cap H} = R_V/F.R_V$, $R_{W\cap H} = R_W/F.R_W$ and the equation F is a regular element for R_V and for R_W. By the similar argument in the proof of the claim (2.11.1) above, we get an exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Tor_q^S(R_W, S/S_+) & \longrightarrow & Tor_q^S(R_{W\cap H}, S/S_+) & \longrightarrow & Tor_{q-1}^S(R_W(-1), S/S_+) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_q} & & \downarrow{\scriptstyle \overline{\mu_q}} & & \downarrow{\scriptstyle \mu_{q-1}} & & \\ 0 & \longrightarrow & Tor_q^S(R_V, S/S_+) & \longrightarrow & Tor_q^S(R_{V\cap H}, S/S_+) & \longrightarrow & Tor_{q-1}^S(R_V(-1), S/S_+) & \longrightarrow & 0. \end{array}$$

In this case, the assumption asserts the injectivity of the map:$\overline{\mu_q}$, which induces the injectivity of the map: μ_q. Thus we see that the scheme W is a PG-shell of V in the projective space P.

Now let us show that the scheme $Y := W \cap H$ is a PG-shell of the scheme $X := V \cap H$ in H if and only if the scheme $Y := W \cap H$ is a PG-shell of the scheme $X := V \cap H$ in P. We may assume that $H = V_+(Z_N) = Proj(T)$, where $T = S/Z_N.S \cong \mathbb{C}[Z_0, \cdots, Z_{N-1}]$. Take a point p_0 in P which is not contained in the hyperplain H. Then the space P can be considered as the projective cone $C_{p_0}(H)$ with the vertex p_0, namely the ring S can be considered as the polynomial ring $T[Z_N]$, which is faithfully flat over the ring T. Consider the projective cones $\widehat{X} := C_{p_0}(X)$ of X and $\widehat{Y} := C_{p_0}(Y)$ of Y with the vertex p_0. Then $R_{\widehat{X}} = R_X \otimes_T S$, $R_{\widehat{Y}} = R_Y \otimes_T S$, $X = \widehat{X} \cap H$, $Y = \widehat{Y} \cap H$, and the equation Z_N is a regular element for the ring $R_{\widehat{X}}$ and for the ring $R_{\widehat{Y}}$. Thus, starting from the minimal T-free resolutions of R_X and of R_Y, we can construct the minimal S-free resolutions of $R_{\widehat{X}}$ and of $R_{\widehat{Y}}$ and the minimal S-free resolutions of $R_{\widehat{X}\cap H} = R_X$ and of $R_{\widehat{Y}\cap H} = R_Y$, successively. We can also construct all the induced homomorphisms of complexes for those minimal resolutions compatibly. Through these constructions, we see that:

$$\begin{array}{ccccc} Tor_q^S(R_W, S/S_+) & \cong & Tor_q^T(R_W, T/T_+) & \oplus & Tor_{q-1}^T(R_W, T/T_+) \\ \downarrow{\scriptstyle \mu_{q,S}} & & \downarrow{\scriptstyle \mu_{q,T}} & & \downarrow{\scriptstyle \mu_{q-1,T}} \\ Tor_q^S(R_V, S/S_+) & \cong & Tor_q^T(R_V, T/T_+) & \oplus & Tor_{q-1}^T(R_V, T/T_+), \end{array}$$

which shows that all the maps: $\{\mu_{q,S}\}_{q\geq 0}$ are injective if and only if all the maps: $\{\mu_{q,T}\}_{q\geq 0}$ are injective, which is what we wanted. ∎

Related to Conjecture 0.1(0.1.1), we have a very easy result which smplifies our argument later.

Proposition 2.12 *Let $V \subset P = \mathbb{P}^N(\mathbb{C})$ be a variety of Δ-genus zero (cf. Remark 1.3) whose dimension is n, and a scheme $W \subset P$ a pregeometric shell of V. Assume that the scheme W is arithmetically Cohen-Macaulay and of (pure) dimension m. Then, the scheme W is also a variety of Δ-genus zero.*

Proof. By the results of [6] and [5], our assumption implies that the homogeneous coordinate ring R_V is Cohen-Macaulay and has a 2-linear minimal S-free resolution. The minimal S-free resolution of R_V is of the form: $\mathbb{F}_{V,\bullet} : 0 \leftarrow R_V \leftarrow \mathbb{F}_{V,0} = S \leftarrow \mathbb{F}_{V,1} = \oplus S(-2) \leftarrow \mathbb{F}_{V,2} = \oplus S(-3) \leftarrow \cdots \leftarrow \mathbb{F}_{V,p} = \oplus S(-p-1) \leftarrow \cdots \leftarrow \mathbb{F}_{V,r} = \oplus S(-r-1)$, where $r = N - n$. Since we assume that the scheme W is arithmetically Cohen-Macaulay and is a PG-shell of V, the homogeneous coordinate ring R_W has a minimal S-free resolution of the form: $\mathbb{F}_{W,\bullet} : 0 \leftarrow R_W \leftarrow \mathbb{F}_{W,0} = S \leftarrow \mathbb{F}_{W,1} = \oplus S(-2) \leftarrow \mathbb{F}_{W,2} = \oplus S(-3) \leftarrow \cdots \leftarrow \mathbb{F}_{W,t} = \oplus S(-t-1)$, where $t = N - m = r - (m - n)$. Then we apply Theorem 4.1.15 of [2] and see that $e := deg(V) = (1/r!)\Pi_{k=1}^{r}(k+1) = r+1$ and $e' := deg(W) = (1/t!)\Pi_{k=1}^{t}(k+1) = t+1$. Let us show that the scheme W is a variety. Take a main primary component (cf.Definition 2.3): W_0 of W which includes V, give a reduced structure on W_0, and put $\overline{e_0} := deg((W_0)_{red})$. Since the scheme W is arithmetically Cohen-Macaulay, the scheme W is locally Cohen-Macaulay and equidimensional. The scheme W is a variety if and only if $\overline{e_0} = e'$. Now we assume $\overline{e_0} < e'$. Then $\overline{e_0} < t + 1 = N - m + 1 = N - dim((W_0)_{red}) + 1$, which implies that the variety $(W_0)_{red}$ in degenerate, namely there is a hyperplain H including the variety $(W_0)_{red}$. Then $V \subset (W_0)_{red}$ implies that the variety V is also degenerate, which contradicts our assumption. ∎

Corollary 2.13 *Let $V \subset P = \mathbb{P}^N(\mathbb{C})$ be a variety of Δ-genus zero and a scheme $W \subset P$ a pregeometric shell of V. Assume that* $\dim(V) = \dim(W)$. *Then, $V = W$.*

Proof. By the assumption, using the claim (2.5.5), $\dim(R_W) = \dim(R_V) = depth_{S_+}(R_V) \leq depth_{S_+}(R_W)$, which shows the ring R_W is Cohen-Macauilay. Then, applying Proposition 2.12, we see that the scheme W is a variety, which implies $V = W$. ∎

To handle Hilbert polynomials efficiently, we prepare the following lemma.

Lemma 2.14 (Finite factorial series expansion of Taylor type) *Let us consider a polynomial of real coefficients $f(x) \in \mathbb{R}[x]$ of degree r, in other words, a real valued function $f(x)$ defined on the field of real numbers $\mathbb{R}$ (or on the ring of rational integers $\mathbb{Z}$) which has an expression by factorial monomials $x^{[k]}$ $(k = 0, 1, \ldots, r)$:*

$$\begin{aligned} f(x) &= c_0 + \left(\frac{c_1}{1!}\right) x^{[1]} + \left(\frac{c_2}{2!}\right) x^{[2]} + \cdots + \left(\frac{c_{r-1}}{(r-1)!}\right) x^{[r-1]} + \left(\frac{c_r}{r!}\right) x^{[r]} \\ &= c_r A_r(x) + c_{r-1} A_{r-1}(x) + \cdots + c_1 A_1(x) + c_0, \end{aligned}$$

where the coefficient c_k is also a real number, and the k-th factorial monomial $x^{[k]}$ means $(x+k)(x+k-1)(x+k-2)\cdots(x+1)$ and the Hilbert function $A_k(x)$ of $\mathbb{P}^k(\mathbb{C})$ is

$$A_k(x) = \frac{x^{[k]}}{k!} = \binom{x+k}{k} = \frac{(x+k)(x+k-1)\cdots(x+1)}{k!}$$

Then the coefficient c_k can be computed by using the (backward) difference operator ∇ as follows.

$$c_k = (\nabla^k f)(-1)$$

Proof. Analogously to the usual Taylor expansion, it is easy to see the formula on c_k holding if we take a notice on the facts that $\nabla x^{[0]} = \nabla 1 = 0$, for a positive integer k, $\nabla x^{[k]} = k \cdot x^{[k-1]}$, namely $\nabla A_k(x) = A_{k-1}(x)$ and that $A_k(a) = 0$ if and only if $a = -1, -2, \ldots, -k$. ∎

We should also recall a classical and well-known result for later use (cf. [8], [6]).

Lemma 2.15 (One point Chordal Variety) *Let $V \subseteq P = \mathbb{P}^N(\mathbb{C})$ be a non-degenerate closed subvariety of degree ≥ 2. If we take a sufficiently general smooth point $x_0 \in V$, then there is a projective line L satisfying the conditions: (a) this line L contains the point x_0 and another point $y \in V$; (b) this line L itself is not contained in V. Now we put:*

$$Cd(x_0, V) := \overline{\bigcup_{x \in V - \{x_0\}} line(x_0, x)},$$

namely the Zariski closure of the union of all the lines joining any point of $V - \{x_0\}$ and the point x_0.

Then this variety $Cd(x_0, V)$ coincides with the projective cone with the vertex x_0 of the Zariski closure of the image of $V - \{x_0\}$ projected from the center x_0, and therefore $dim(Cd(x_0, V)) = dim(V) + 1$. Moreover, we have: $deg(Cd(x_0, V)) = deg(V) - 1$. (In case of emphasizing this construction process, we will call this variety $Cd(x_0, V)$ as "one point chordal variety" with vertex x_0 *instead of* "the cone" *).*

We will use the next proposition including singular cases. For its proof, we refer to [10] and [8].

Proposition 2.16 *Let Y be a complete intersection closed subscheme of type $(m_1, \ldots, m_r)$ in $P = \mathbb{P}^N(\mathbb{C})$ with $dim(Y) = N - r \geq 3$. Then $Pic(Y) \cong \mathbb{Z}O_Y(1)$. Moreover, if the scheme Y is a variety, then the canonical bundle $K_U := det(\Omega_U^1)$ of the regular locus $U := Reg(Y)$ of Y can be extended to the dualizing line bundle $K_Y^\circ \cong O_Y(-N - 1 + \sum_{i=1}^r m_i)$ of Y.*

§3 Classification on PG-shells of $(\mathbb{P}^1, O_{\mathbb{P}^1}(4))$.

Since a non-degenerate rational normal quartic curve $X \subset P = \mathbb{P}^4(\mathbb{C})$ is uniquely determined up to the action of $PGL(5, \mathbb{C})$, we set up the circumstances as follows.

Let X be a projective line $\mathbb{P}^1(\mathbb{C}) = Proj(\mathbb{C}[T_0, T_1])$ and embedded into the 4-th projective space: $P = \mathbb{P}^4(\mathbb{C}) = Proj(\mathbb{C}[Z_0, \cdots, Z_4])$ which has the tautological line bundle $O_P(1) = O_P(H)$ by using the ample line bundle $O(4) = O_{\mathbb{P}^1(\mathbb{C})}(4)$, or more precisely, by using the morphism in which the homogeneous coordinates $[Z_0 : Z_1 : Z_2 : Z_3 : Z_4]$ correspond to $[T_0^4 : T_0^3T_1 : T_0^2T_1^2 : T_0T_1^3 : T_1^4]$. Then the homogeneous ideal $\mathbb{I}_X$ coincides with $(Z_3^2 - Z_2Z_4, Z_2Z_3 - Z_1Z_4, Z_1Z_3 - Z_0Z_4, Z_2^2 - Z_0Z_4, Z_1Z_2 - Z_0Z_3, Z_1^2 - Z_0Z_2) \cdot S$, where $S = \mathbb{C}[Z_0, \cdots, Z_4]$. Since the sheaf O_X has the Castelnuovo-Mumford regularity 1 with respect the line bundle $O_P(H)$, it is easy to see that the homogeneous coordinate ring $R_X = S/\mathbb{I}_X$ has a minimal graded S-free resolution of 2-linear type:

$$\mathbb{F}_{X\bullet} \quad 0 \longleftarrow S \xleftarrow{\varphi_1} \oplus^6 S(-2) \xleftarrow{\varphi_2} \oplus^8 S(-3) \xleftarrow{\varphi_3} \oplus^3 S(-4) \longleftarrow 0,$$

where the maps φ_i $(i = 1, 2, 3)$ are described by the matrices:

$$\varphi_1 = [Z_1^2 - Z_0Z_2, \quad Z_1Z_2 - Z_0Z_3, \quad Z_2^2 - Z_0Z_4, \quad Z_1Z_3 - Z_0Z_4, \quad Z_2Z_3 - Z_1Z_4, \quad Z_3^2 - Z_2Z_4]$$

$$\varphi_2 = \begin{bmatrix} -Z_2 & 0 & -Z_3 & 0 & Z_4 & 0 & 0 & 0 \\ Z_1 & -Z_2 & 0 & -Z_3 & 0 & Z_4 & Z_4 & 0 \\ -Z_0 & Z_1 & 0 & 0 & 0 & -Z_3 & 0 & Z_4 \\ Z_0 & 0 & Z_1 & Z_2 & -Z_2 & 0 & -Z_3 & -Z_4 \\ 0 & -Z_0 & -Z_0 & 0 & Z_1 & Z_2 & 0 & -Z_3 \\ 0 & 0 & 0 & -Z_0 & 0 & 0 & Z_1 & Z_2 \end{bmatrix}$$

$$\varphi_3 = \begin{bmatrix} Z_3 & -Z_4 & 0 \\ 0 & Z_3 & -Z_4 \\ -Z_2 & 0 & Z_4 \\ Z_1 & -Z_2 & 0 \\ 0 & -Z_2 & Z_3 \\ -Z_0 & Z_1 & 0 \\ Z_0 & 0 & -Z_2 \\ 0 & -Z_0 & Z_1 \end{bmatrix}.$$

Next we take a PG-shell W of the rational normal quartic curve $X \subseteq P$ and consider a minimal graded S-free resolution $\mathbb{F}_{W\bullet}$ of the homogeneous coordinate ring R_W of W. Then by the definition of PG-shell, we see easily that this complex (resolution) has a form:

$$\mathbb{F}_{W\bullet} \qquad 0 \longleftarrow S \xleftarrow{\psi_1} \oplus^a S(-2) \xleftarrow{\psi_2} \oplus^b S(-3) \xleftarrow{\psi_3} \oplus^c S(-4) \longleftarrow 0,$$

and is a subcomplex of the complex $\mathbb{F}_{X\bullet}$. Then we can compute the Hilbert polynomial $A_W(x)$ of W as follows.

Lemma 3.1 *Under the circumstances, using the integers $a \in \{0, 1, \ldots 6\}$, $b \in \{0, 1, \ldots 8\}$, $c \in \{0, 1, \ldots 3\}$, the Hilbert polynomial $A_W(x)$ is written in the form:*

$$A_W(x) = (1 - a + b - c)A_4(x) + (2a - 3b + 4c)A_3(x) + (-a + 3b - 6c)A_2(x) + (-b + 4c)A_1(x) + (-c).$$

Proof. From the projective sheafication of the minimal graded S-free resolution $\mathbb{F}_{W\bullet}$ of the ring R_W, we have the exact complex of sheaves:

$$0 \longleftarrow O_W \longleftarrow O_P \xleftarrow{\psi_1} \oplus^a O_P(-2) \xleftarrow{\psi_2} \oplus^b O_P(-3) \xleftarrow{\psi_3} \oplus^c O_P(-4) \longleftarrow 0,$$

which shows that $A_W(x) = A_4(x) - a \cdot A_4(x-2) + b \cdot A_4(x-3) - c \cdot A_4(x-4)$. By applying Lemma 2.14, we have the result. ∎

Now let us classify all the PG-shells of the rational normal quartic curve $X \subseteq P$ depending on their codimensions. Here we should pay attention to the fact that the curve X is a non-degenerate arithmetically normal non-singular projective variety of Δ-genus zero and satisfies all the assumptions of Conjecture 0.1 except the last one (i.e. the existence of a shell frame (E, σ)).

- **The Case of** $codim(W) = 1$

First we handle the case: $codim(W) = 1$. Before we apply Lemma 2.5 directly, we should take care the fact that for an arbitrary closed scheme $V \subseteq P$, the condition that $codim(W) = 1$ and the scheme W is a PG-shell of the scheme V does not in general imply that the scheme W is a divisor of P since we do not assume, for example, the scheme W is equidimensional and so on. It may happen that the scheme W has a primary component of codimension 1 and has another component of codimension more than 1 or an embedded component. Thus, to consider this case, we need the following lemma.

Lemma 3.2 *Let V be a reduced and irreducible closed subscheme of $P = \mathbb{P}^N(\mathbb{C})$, a closed subscheme W is of codimension 1 in the total space P and a pregeometric shell of the variety V. Then W is an irreducible and reduced divisor of P.*

Proof. We have only to show that $dim_{(S/S_+)}Tor_1^S(R_W, S/S+) = 1$. Now assume that

$$dim_{(S/S_+)}Tor_1^S(R_W, S/S+) \geq 2.$$

Then there are at least two equations $\{G_1, G_2\}$ of W which is linearly independent in the (S/S_+)-vector space $Tor_1^S(R_W, S/S+)$. By the Tor injectivity condition of PG-shells, we see that the equations $\{G_1, G_2\}$ form a part of a minimal generators of the homogeneous ideal $\mathbb{I}_V$ of V. Since the ideal $\mathbb{I}_V$ is a prime ideal, both the equations G_1 and G_2 are irreducible polynomials, otherwise they can not be a part of minimal generators of the ideal $\mathbb{I}_V$. Since the equations : $\{G_1, G_2\}$ are linearly independent, they form S-regular sequence. Thus the closed subscheme $Y = \{G_1 = G_2 = 0\}$ of P is of pure codimension 2. On the other hand, by its construction, $W \subseteq Y$ and $codim(Y) = 2$, which is a contradiction. ∎

Corollary 3.3 *Let W be a pregeometric shell of the rational quartic curve $X \subseteq P$ and of codimension 1 in P. Then the scheme W is a reduced and irreducible quadric hypersurface of rank 3,4, or 5, which means that $\Delta(W, O_W(1)) = 0$ and the variety W is arithmetically Cohen-Macaulay. Moreover, if the variety W be a G-shell of X, then there is no shell frame (E, σ) of X in W.*

Proof. For the claim on the rank of quadric equations, we see that irreducibility of minimal generators implies : rank ≥ 3. To see that every case occurs, we give examples of the quadric equations of the rational quartic curve X in Table 1.

rank	quadric hypersurface $\supset X$
3	$Z_1^2 - Z_0Z_2$
4	$Z_1Z_3 - Z_0Z_4$
5 (non-sing)	$Z_2^2 - 2Z_0Z_4 + Z_1Z_3$

Table 1: Examples of Quadrics

All the claims except the last one have been already proved in Lemma 3.2. Now we assume that there exists a shell frame (E, σ) of X in W as in Definition 2.4. From the homomorphism $\sigma : E^\vee \to I_{X/W} \subseteq O_W$, we see that the normal bundle $N_{X/W}$ of X in W is isomorphic to $E \otimes O_X$. Putting $U := Reg(W)$ and the dualizing line bundle of W to be K_W° (cf. Proposition 2.16), we have $O(-2) \cong K_X \cong K_U|_X \otimes detN_{X/W} \cong (K_W^\circ \otimes detE) \otimes O_X$, namely the line bundle $O(-2)$ can be extended to a line bundle $L := K_W^\circ \otimes detE$ on W. By Proposition 2.16 again, we have an isomorphism $Pic(W) \cong \mathbb{Z}O_W(1)$. Then the line bundle L and therefore the line bundle $O(-2)$ can be extended to a multiple of the tautological line bundle $O_P(1)$ of P, which contradicts to the fact : $O_X(1) \cong O(4)$. ∎

• The Case of $codim(W) = 3$

Next, skipping the most bothersome case of $codim(W) = 2$, we proceed to the case of $codim(W) = 3$. Then we easily get the following result from Corollary 2.13.

Lemma 3.4 *Let W be a pregeometric shell of the rational quartic curve $X \subseteq P$ and of codimension 3 in P. Then the scheme W coincides with the curve X.*

• The Case of $codim(W) = 2$

Now we has come to the remaining case: $codim(W) = 2$. Let us list up all the cases of the triplet (a, b, c) in the order of handling in the sequel.

Lemma 3.5 *Let W be a pregeometric shell of the rational quartic curve $X \subseteq P$ and of codimension 2 in P. Then all the cases of the Tor-Betti numbers (a, b, c) of the minimal graded S-free resolution $\mathbb{F}_{W,\bullet}$ and of the Hilbert polynomials are listed in Table 2 below.*

Case No.	(a,b,c)	$A_W(x)$
(1)	$(4,4,1)$	$2A_2(x) - 1$
(2)	$(5,6,2)$	$A_2(x) + 2A_1(x) - 2$
(3)	$(3,2,0)$	$3A_2(x) - 2A_1(x)$

Table 2: Betti Numbers and Hilbert Polynomials

Proof. Since the degree of the Hilbert polynomial $A_W(x)$ is 2, applying Lemma 3.1, we have the equations below ($(a, b, c) \in \{0, 1, \ldots 6\} \times \{0, 1, \ldots 8\} \times \{0, 1, \ldots 3\}$), which are easily solved and bring Table 2.

$$\begin{cases} 1 - a + b - c &= 0 \\ 2a - 3b + 4c &= 0 \\ -a + 3b - 6c &\neq 0 \end{cases}$$

∎

Now we will follow the order in Table 2 and handle each case, respectively.

•• (1) The Case:$(a, b, c) = (4, 4, 1)$

It is easy to see that this case never happens.

Since the Hilbert polynomial $A_W(x)$ is $2A_2(x) - 1$, we see that $deg(W) = 2$. On the other hand, we have $a = 4$, which means that there exit four $\mathbb{C}$-linearly independent quadric equations $\{G_1, G_2, G_3, G_4\}$ generating the homogeneous ideal $\mathbb{I}_W$. Now, using the condition that W is a PG-shell of the curve X, we recall the proof of Lemma 3.2 and see that $\{G_1, G_2\}$ forms a S-regular sequence and $Y = \{G_1 = G_2 = 0\}$ is an arithmetically Cohen-Macaulay closed subscheme of pure codimension two. Obviously, $X \subset W \subseteq Y$, $dim(W) = dim(Y) = 2$, and $deg(Y) = 4$. Now we take a primary component Y_0 of Y containing X and consider $(Y_0)_{red}$ after putting the reduced structure on the space $|Y_0|$. If $deg((Y_0)_{red}) \leq 2$, then after the process of Lemma 2.15, the one point chordal variety $Cd(x_0, (Y_0)_{red})$ turns out to be a linear variety of dimension 3. Since $X \subseteq (Y_0)_{red} \subseteq Cd(x_0, (Y_0)_{red})$, we see that the curve X is degenerate, which is a contradiction. Hence we have $deg((Y_0)_{red}) \geq 3$ and the component Y_0 can not contain any main component of W. Now we have $W \cup (Y_0)_{red} \subseteq Y$ and therefore $deg(Y) \geq 5$, which is absurd.

•• **(2) The Case:**$(a, b, c) = (5, 6, 2)$

Using a rather delicate argument than the case above, we show that this case never occurs.

On the Hilbert polynomial, we know that $A_W(x) = A_2(x) + 2A_1(x) - 2 = (1/2)x^2 + (7/2)x + 1$, which implies : $deg(W) = 1$. This shows us that the main component F is only one and its structure sheaf as a primary component of W is already reduced and isomorphic to the projective plane $\mathbb{P}^2(\mathbb{C})$. Since $X \subseteq W$ and X is non-degenerate, we have $X \not\subseteq F$ and $Y = X \cup F \subseteq W$, where the closed subscheme Y is a scheme theoretic union of the closed subschemes X and F. At the generic point ζ of the main component of W, $O_{W,\zeta} \cong O_{Y,\zeta}$, which implies that the support of the ideal sheaf $I_{Y/W}$ does not contain the generic point ζ, and therefore $dim(I_{Y/W}) \leq 1$. Then, the Hilbert polynomial of the ideal sheaf $I_{Y/W}$ is the form :$\chi(I_{Y/W}(x)) = px + q$ ($p, r \in \mathbb{Z}$ and $p \geq 0$) (cf. [11]). On the other hand, $X \cap F$ is finite number of points, which implies that the Hilbert polynomial is a constant:$\chi(O_{X \cap F}(x)) = k$, where $k = length O_{X \cap F}$. Now let us consider the following two exact sequences:

$$0 \longrightarrow I_{Y/W} \longrightarrow O_W \longrightarrow O_Y \longrightarrow 0$$

$$0 \longrightarrow O_Y \longrightarrow O_X \oplus O_F \longrightarrow O_{X \cap F} \longrightarrow 0.$$

Let us take their Hilbert polynomials, bind them up and get:

$$\begin{aligned} A_W(x) &= \chi(O_W(x)) = \chi(O_Y(x)) + \chi(I_{Y/W}(x)) \\ &= \chi(O_X(x)) + \chi(O_F(x)) - \chi(O_{X \cap F}(x)) + \chi(I_{Y/W}(x)) \\ &= A_X(m) + A_2(x) - k + px + q \\ &= 4x + 1 + (1/2)(x+2)(x+1) + px + q - k \\ &= (1/2)x^2 + ((11/2) + p)x + (2 + q - k). \end{aligned}$$

Comparing the coefficient of the second term in this Hilbert polynomial with that of $A_W(x)$ previously obtained, we see that $(7/2) = ((11/2) + p) \geq (11/7)$, which is a contradiction.

•• **(3) The Case:**$(a, b, c) = (3, 2, 0)$

This case really occurs and is the most interesting from our view point.

Let us recall: $A_W(x) = 3A_2(x) - 2A_1(x)$, which implies $deg(W) = 3$. Moreover, the length of the minimal graded S-free resolution $\mathbb{F}_{W,\bullet}$ is 2, and therefore $arith.depth(W) = depth(R_W) = 5 - hd_S(R_W) = 5 - 2 = 3 = dim(R_W)$. Thus the homogeneous coordinate ring R_W is an arithmetically Cohen-Macaulay ring. Applying Proposition 2.12, we see that the scheme W is a variety of $\Delta(W, O_W(1)) = 0$ and of degree 3.

By the structure theorem on the projective varieties of Δ-genus zero (cf. [6] or more classically [12]), the singular locus $Sing(W)$ of W is a linear space and the variety W is the generalized projective cone over a non-singular projective variety M (this can be obtained by a generic linear space section of W which does not meet the linear space $Sing(W)$) of Δ-genus zero with the vertex at their singular locus. Since $dim(W) = 2$, we have only two cases : (3-1) $dim(Sing(W)) = -1$ (namely W is non-singular) and the non-singular variety M is W itself; or (3-2) $dim(Sing(W)) = 0$ (namely $Sing(W) = \{p_0\}$) and the non-singular variety M is a rational normal cubic curve.

Moreover, the structure theorem says that in the case of (3-1) above, the polarized variety $(W, O_W(1))$ is a rational scroll $(\mathbb{P}(E), O_{\mathbb{P}(E)}(1))$, where the vector bundle E is the one over a rational curve $B = \mathbb{P}^1(\mathbb{C})$ and of the form: $E \cong O_{\mathbb{P}^1(\mathbb{C})}(2) \oplus O_{\mathbb{P}^1(\mathbb{C})}(1)$, the ample line bundle $O_{\mathbb{P}(E)}(1)$ is the relative tautological line bundle of the projective bundle $\mathbb{P}(E) \to B$ determined from the ample vector bundle E. In terms of rational ruled surfaces, this variety W is isomorphic to Σ_1, a rational ruled surface of degree 1, and is embedded by a linear system $|2f + C_1|$.

On the other hand, in the case of (3-2), the blow-up $q : \widetilde{W} \to W \subseteq P$ of the variety W at the vertex p_0 is obtained by $\widetilde{W} \cong \mathbb{P}(O_{\mathbb{P}^1(\mathbb{C})}(3) \oplus O_{\mathbb{P}^1(\mathbb{C})})$ and by a natural homomorphism $\oplus^5 O_{\mathbb{P}^1(\mathbb{C})} \to O_{\mathbb{P}^1(\mathbb{C})}(3) \oplus O_{\mathbb{P}^1(\mathbb{C})}$ on a rational curve $B = \mathbb{P}^1(\mathbb{C})$. This variety $\widetilde{W}$ is isomorphic to Σ_3, a rational ruled surface of degree 3 and the morphism $q : \widetilde{W} \to W \subseteq P$ is given by a linear system $|3f + C_3|$.

Let us summarize these two cases in the following Table 3 and proceed to study how the curve X is embedded in W in each case.

Case No.	$dim(Sing(W))$	$\widetilde{W}$ or W	linear system (morph.)
(3-1)	-1 (non-sing)	$W \cong \Sigma_1$	$\lvert 2f + C_1 \rvert$ (embedding)
(3-2)	0	$\widetilde{W} \cong \Sigma_3$	$\lvert 3f + C_3 \rvert$

Table 3: Cases of $(W, O_W(1))$

••• The Case (3-1)

Now we assume that the variety W is isomorphic to the ruled surface Σ_1 and embedded into P by the linear system $|2f + C_1|$. Since $X \in |uf + vC_1|$ for some integer u v, the fact: $deg(X) = 4$ and the Lemma 2.2 (2.2.3), we have : $(u, v) = (3, 1)$ or $(2, 2)$. Let us take a blow-down morphism $b : W \to \mathbb{P}^2(\mathbb{C}) = Y$ of contracting the exceptional curve C_1 to a point $p \in Y$. Then the pull-back : $b^*O_{\mathbb{P}^2(\mathbb{C})}(1)$ of the tautological ample line bundle of the projective plane Y corresponds to the linear system $|f + C_1|$. Thus, if the curve $X \in |3f + C_1|$, applying the projection formula to the curve X with respect to the morphism b, and the computation : $X.(f + C_1) = 3$, $X.C_1 = 2$ show that the the curve X comes from a singular irreducible and reduced cubic plane curve passing through the point p. Considering that $3f + C_1 = (2f + C_1) + f$, namely a sum of the ample divisor $2f + C_1$ and an (effective) nef divisor f, we see that the curve X is a nef divisor (an ample divisor).

By the similar argument, if the curve $X \in |2f + 2C_1|$, we see that the curve X comes from a non-singular conic which does not pass through the point p. To see the curve X is a nef divisor (N.B. not ample! e.g. $X.C_1 = 0$), we have only to apply the projection formula to a test curve with respect to the morphism b since $X \in |b^*O_{\mathbb{P}^2(\mathbb{C})}(2)|$.

••• The Case (3-2)

Next we consider the case : the blow-up $\widetilde{W}$ of the variety W at the vertex p_0 is isomorphic to the ruled surface Σ_3 and the morphism $q : \widetilde{W} \to W \subset P$ is given by the linear system $|3f + C_3|$. Now, taking the strict transform $q^{-1}(X)$ of the curve X via the morphism q, we seek the integers u and v such that $q^{-1}(X) \in |uf + vC_3|$. Then the condition $deg(X) = 4$ means $q^{-1}(X).(3f + C_3) = 4$. Since the curve $q^{-1}(X)$ is irreducible, we can apply Lemma 2.2 (2.2.3) to this case and get $(u, v) = (4, 1)$. Then $X.C_3 = (4f + C_3).C_3 = 4 - 3 = 1$, which shows the curve X passes simply through the vertex p_0.

♠ Existence

Now let us show the existence of the three cases above and summarize the classification of the case $codim(W) = 2$ we have already done.

Proposition 3.6 *Let X be a non-degenerate quartic normal curve in $P = \mathbb{P}^4(\mathbb{C}) = Proj(S)$ and W a pregeometric shell of X with $codim(W) = 2$. Then W is reduced and irreducible and a variety of Δ-genus zero. The homogeneous coordinate ring R_W of W has a minimal graded S-free resolution $\mathbb{F}_{W,\bullet}$ with the Tor-Betti numbers $(3, 2, 0)$. There are three possible cases satisfying these properties as in Table 4.*

Case No.	*dim(Sing(W))*	$\widetilde{W}$ *or* W	*linear system (morph.)*	X *or* $q^{-1}(X) \in$ *linear system*
(3-1-1)	-1 *(non-sing)*	$W \cong \Sigma_1$	$\|2f + C_1\|$ *(embedding)*	$X \in \|3f + C_1\|$ *(ample)*
(3-1-2)	-1 *(non-sing)*	$W \cong \Sigma_1$	$\|2f + C_1\|$ *(embedding)*	$X \in \|2f + 2C_1\|$ *(nef, not ample)*
(3-2)	0	$\widetilde{W} \cong \Sigma_3$	$\|3f + C_3\|$	$q^{-1}(X) \in \|4f + C_3\|$

Table 4: Cases of a surface W and a curve X

Conversely, there exist a non-singular projective surface W and a non-singular projective curve X on W which satisfy the conditions in Table 4 above. Moreover, once if the surface W and the curve X on W are given as in Table 4 above, then the surface W is always a pregeometric shell of the curve X (cf. also §4.).

Proof. It is enough to show the existence part of the claim above. If we do not assume the condition of "pregeometric shells", Lemma 2.2 (2.2.3) shows the existences of pairs of a non-singular projective curve and a non-singular projective surface: $X \subset W$ in $P = \mathbb{P}^4(\mathbb{C})$ as in Table 4. Then, the curve X is a curve of degree 4 and the surface W is of degree 3. By this construction, the surface W is a linearly normal, i.e. a natural map $H^0(P, O_P(1)) \to H^0(W, O_W(1))$ is surjective. On the non-degeneracy of X, using Table 4, we can check it easily by computing $H^0(W, I_{X/W}(H)) \cong H^0(W, O_W(-X + H)) = 0$ (if $W \cong \Sigma_1$) or $H^0(\widetilde{W}, I_{q^{-1}(X)/\widetilde{W}}(q^*H)) \cong H^0(\widetilde{W}, O_{\widetilde{W}}(-q^{-1}(X) + q^*H)) = 0$ (if $\widetilde{W} \cong \Sigma_3$). This also shows the linear normality of the rational curve X (adjunction formula and Lemma 2.2 (2.2.4)).

Let us show that the surface W is a pregeometric shell of X in these three cases. In every case, both the surface W and the curve X are varieties of Δ-genera zero. Now we refer to the book [6] and apply its results: (4.12) Corollary and the argument of (5.1), which imply that both varieties X and W are arithmetically Cohen-Macaulay. Then the result of [5] shows that the homogeneous coordinate rings R_W has 2-linear minimal graded S-free resolution. Applying Proposition 2.5 (2.5.8), we see that the surface W is a pregeometric shell of X. ∎

The argument in the proof of Proposition 3.6 brings also the following useful result.

Corollary 3.7 *Let $V \subset W$ be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$. Assume that the scheme V is non-degenerate and the scheme W is linearly normal and of a variety of Δ-genus zero. Then the variety W is a pregeometric shell of V.*

Corollary 3.8 *Let $V \subset P = \mathbb{P}^N(\mathbb{C})$ be a non-degenerate, linearly normal closed subvariety of codimension r and of Δ-genus zero. Then, there is a chain of varieties:*

$$V = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_k \subset \cdots \subset W_r = P,$$

where the variety W_k $(k = 0, 1, \cdots, r)$ is a pregeometric shell of V with $codim(W_k, P) = r - k$.

Proof. Starting from the variety V, we first construct inductively a chain of varieties $\{W_k\}_{k=0}^{r}$ with Δ-genus zero. As an induction hypothesis, we assume that we have a chain of varieties :

$$V = W_0 \subset W_1 \subset \cdots \subset W_{k_0},$$

where all the varieties W_k $(k = 0, 1, \cdots, k_0)$ are varieties of Δ-genus zero. Then we take a nonsingular point x of the variety W_{k_0} and set W_{k_0+1} to be a one point chordal variety $Cd(x, W_{k_0})$ with the vertex x. Then it is easy to see that the variety W_{k_0+1} is also of Δ-genus zero. Thus we obtain a chain of the varieties $\{W_k\}_{k=0}^{r}$ with Δ-genus zero. Then we apply Corollary3.7 to each pair $V \subset W_k$ and get what we wanted.

∎

§4 Explicit Examples.

In this section, using homogeneous coordinates explicitly, we give an example of a pregeometric shell of codimension two for the rational quartic curve X, which appeared as one of the three cases in the list of Proposition3.6. On explicit examples of the rest two cases, see [23]. The equations for X and so on are the same as in the first part of the previous section.

Example 4.1 (The Case:(3-1-1)) *Taking $Y = \mathbb{P}^2(\mathbb{C}) = Proj(\mathbb{C}[U_0, U_1, U_2])$, we get a surface W as the blow-up of Y at the center $p_1 = [1:0:0]$. The embedding of W by the linear system $|2f + C_1|$, namely the linear system coming from the conics passing through the point p_1, is given by $[Z_0 : Z_1 : Z_2 : Z_3 : Z_4] = [U_0U_1 : U_0U_2 : U_1^2 : U_1U_2 : U_2^2]$. The equations of W are : $Z_1Z_2 - Z_0Z_3 = 0$, $Z_1Z_3 - Z_0Z_4 = 0$, $Z_3^2 - Z_2Z_4 = 0$. Then a minimal graded S-free resolution of the homogeneous coordinate ring R_W of W is :*

$$\mathbb{F}_{W\bullet} \quad 0 \longleftarrow S \xleftarrow{\psi_1} \oplus^3 S(-2) \xleftarrow{\psi_2} \oplus^2 S(-3) \longleftarrow 0,$$

where the maps ψ_i $(i = 1, 2)$ are described by the matrices:

$$\psi_1 = \begin{bmatrix} Z_1Z_2 - Z_0Z_3, & Z_1Z_3 - Z_0Z_4, & Z_3^2 - Z_2Z_4 \end{bmatrix}$$

$$\psi_2 = \begin{bmatrix} -Z_3 & Z_4 \\ Z_2 & -Z_3 \\ -Z_0 & Z_1 \end{bmatrix}.$$

The curve X comes from a curve $[U_0 : U_1 : U_2] = [T_0^3 : T_0T_1^2 : T_1^3]$ in Y, namely a singular cubic $U_1^3 - U_0U_2^2 = 0$ with a cusp at the point $p_1 = [1:0:0]$.

The Tor injectivity condition can be confirmed by using a homomorphism of complexes:

$$\begin{array}{cccccccccc} \mathbb{F}_{X\bullet} & 0 \longleftarrow & S & \xleftarrow{\varphi_1} & \oplus^6 S(-2) & \xleftarrow{\varphi_2} & \oplus^8 S(-3) & \xleftarrow{\varphi_3} & \oplus^3 S(-4) & \longleftarrow 0 \\ & & \| & & \uparrow{\scriptstyle \nu_1} & & \uparrow{\scriptstyle \nu_2} & & & \\ \mathbb{F}_{W\bullet} & 0 \longleftarrow & S & \xleftarrow{\psi_1} & \oplus^3 S(-2) & \xleftarrow{\psi_2} & \oplus^2 S(-3) & \longleftarrow & & 0, \end{array}$$

where the matrices for the maps ν_i $(i = 1, 2)$ are:

$$\nu_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \nu_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

References

[1] T. Ashikaga : private correspondence on "singular ambient filtrations for a rational normal curve" (will be published in somewhere).

[2] W. Bruns and J. Herzog : Cohen-Macaulay Rings, Cambridge studies in advanced mathematics 39, Cambridge University Press, (1993).

[3] D. Eisenbud : Green's conjecture, in "Free Resolutions in Commutative Algebra and Algebraic Geometry", (Editors) D. Eisenbud et al. Boston, MA (1992).

[4] D. Eisenbud and S. Goto : Linear free resolutions and minimal multiplicity, J. of Algebra 88, pp. 89-133 (1984).

[5] D. Eisenbud, O. Riemenschneider and F. O. Schreyer : Resolutions of Cohen-Macaulay algebras, Math.Ann.257, pp. 85-98 (1981).

[6] T. Fujita : Classification theories of polarized varieties, London Math. Soc. Lect. Note Ser. 155, Cambridge Univ. Press (1990).

[7] M. Green : Koszul cohomology and geometry of projective varieties, J. Differ. Geom. 19, pp. 125-171 (1984).

[8] R. Hartshorne : Algebraic Geometry, GTM52, Springer-Verlag, (1977).

[9] H. Hironaka : A note on algebraic geometry over ground rings, The invariance of Hilbert characteristic functions under the specialization process, Ill. J. Math. 2, pp.355-366, (1958).

[10] R. Hartshorne: Ample subvarieties of algebraic varieties, Springer Lecture Notes 156, Springer-Verlag, (1970).

[11] S. L. Kleiman : Toward a numerical theory of ampleness, Ann. of Math., 84 pp. 293-344 (1966).

[12] M. Nagata: On rational surfaces I, II, Mem. Coll. Sci. Kyoto (A) 32 pp. 351-370 (1960), and 33 pp.271-293 (1960).

[13] K. Petri : Über die invariante Darstellung algebraischer Funktionen einer Variablen, Math. Ann. 88, pp. 243-289 (1923).

[14] Z. Ran : Absence of the Veronese from Smooth Threefolds in $\mathbb{P}^5$, Contemp. Math. 116, pp.125-128 (1991).

[15] B. Saint-Donat : On Petri's analysis of the linear system of quadrics through a canonical curve, Math. Ann. 206 pp. 157-175 (1973).

[16] F. O. Schreyer : Syzygies of canonical curves and special linear series, Math. Ann. 275, pp. 105-137 (1986).

[17] J.Stückrad and W. Vogel : Castelnuovo bounds for locally Cohen-Macaulay schemes. Math. Nachr. 136, pp. 307-320 (1988).

[18] T. Usa : Obstructions of infinitesimal lifting, Comm. Algebra, 17(10), pp. 2469-2519 (1989).

[19] T. Usa : An example of filtrations on syzygies induced by meta-Lefschetz operators, Report Sci. H. I. T., No.8, pp. 12-25 (1997).

[20] T. Usa : Problems on geometric structures of projective embeddings, Report Sci. H. I. T. , No.9, pp. 12-29 (1998) (Improved version : Duke E-print math.AG/0001004).

[21] T. Usa : 2-bundles constructed from projected canonical curves in $\mathbb{P}^3(\mathbb{C})$, Report Sci. H. I. T. , No.11, pp. 1-10 (2000).

[22] T. Usa : Infinitesimal Syzygy Bundles, Report Sci. H. I. T. , No.12, pp. 1-17 (2001).

[23] T. Usa : A Classification of Pregeometric Shells of a Rational Quartic Curve, Report Sci. H. I. T. , No.13, pp. 1-17 (2002).

[24] T. Usa : A Classification of Pregeometric Shells of a Veronese surface (in preparation).