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Kähler submanifolds of a quaternion projective space

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In this note, I want to give an exposition of some recent developments on the Kähler submanifolds in a quaternionic Kähler manifold, due mainly to D.V.Alekseevsky and S.Marchiafava [1], [2] and N.Ejiri and the author [5], [11]. I expect interesting interplay of Kähler geometry and quaternionic Kähler geometry.

1 Basic definitions

Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{Q})$ be a quaternionic Kähler manifold with the quaternionic Kähler structure $(\tilde{g}, \tilde{Q})$, that is , $\tilde{g}$ is the Riemannian metric on $\tilde{M}$ and $\tilde{Q}$ is a rank 3 subbundle of End $T\tilde{M}$ which satisfies the following conditions:

(a) For each $p \in \tilde{M}$, there is a neighborhood U of p over which there exists a local frame field $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of $\tilde{Q}$ satisfying

$$\tilde{I}^2 = \tilde{J}^2 = \tilde{K}^2 = -\mathrm{id}, \quad \tilde{I}\tilde{J} = -\tilde{J}\tilde{I} = \tilde{K},$$
$$\tilde{J}\tilde{K} = -\tilde{K}\tilde{J} = \tilde{I}, \quad \tilde{K}\tilde{I} = -\tilde{I}\tilde{K} = \tilde{J}.$$

(b) For any element $L \in \tilde{Q}_p$, $\tilde{g}_p$ is invariant by L, i.e., $\tilde{g}_p(Lu, v) + \tilde{g}_p(u, Lv) = 0$ for $u, v \in T_p\tilde{M}$, $p \in \tilde{M}$.

(c) The vector bundle $\tilde{Q}$ is parallel in End $T\tilde{M}$ with respect to the Riemannian connection $\tilde{\nabla}$ associated with $\tilde{g}$.

In this note we assume that the dimension of $\tilde{M}^{4n}$ is not less than 8 and that $\tilde{M}^{4n}$ has nonvanishing scalar curvature. A submanifold M^{2m} of $\tilde{M}$ is said to be *almost Hermitian* if there exists a section $\tilde{I}$ of the bundle $\tilde{Q}|_M$ such that (1) $\tilde{I}^2 = -\mathrm{id}$, (2) $\tilde{I}TM = TM$ (cf. D.V.Alekseevsky and

S.Marchiafava [1]). We denote by I the almost complex structure on M induced from $\tilde{I}$. Evidently (M, I) with the induced metric g is an almost Hermitian manifold. If (M, g, I) is Kähler, we call it a *Kähler submanifold* of a quaternionic Kähler manifold $\tilde{M}$. An almost Hermitian submanifold M together with a section $\tilde{I}$ of $\tilde{Q}|_M$ is said to be *totally complex* if at each point $p \in M$ we have $LT_pM \perp T_pM$, for each $L \in \tilde{Q}_p$ with $\tilde{g}(L, \tilde{I}_p) = 0$ (cf. S.Funabashi [6]). Alekseevsky and Marchiafava studied the integrability and the Kählerity conditions for almost Hermitian submanifolds. In particular they proved the following.

Theorem 1.1 ([1] Theorem 1.12) *In a quaternionic Kähler manifold* $(\tilde{M}^{4n}, \tilde{g}, \tilde{Q})$ *with nonvanishing scalar curvature, a* $2m(m \geq 2)$*-dimensional almost Hermitian submanifold* M^{2m} *is Kähler if and only if it is totally complex.*

Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{Q})$ be a quaternionic Kähler manifold with nonvanishing scalar curvature and M^{2m} be a $2m(m \geq 2)$-dimensional Kähler submanifold of $\tilde{M}$ together with a section $\tilde{I}$ of $\tilde{Q}|_M$. By the above theorem it is totally complex. Then the bundle $\tilde{Q}|_M$ has the following decomposition:

$$\tilde{Q}|_M = \mathbb{R}\tilde{I} + Q', \tag{1.1}$$

where Q' is defined by $Q'_p = \{L \in \tilde{Q}_p | \tilde{g}(L, \tilde{I}_p) = 0\}$ at each point $p \in M$. The following is a key fact.

Proposition 1.2 ([10] Lemma 2.10) *Under the assumption above, the section* $\tilde{I}$ *of* $\tilde{Q}|_M$ *and the vector subbundle* Q' *are parallel with respect to the induced connection* $\tilde{\nabla}$ *on* $\tilde{Q}|_M$.

At each point $p \in M$, we define a complex structure I on the fibre Q'_p by $IL = \tilde{I}L$ for $L \in Q'_p$. Hence Q' becomes a complex line bundle over M. Moreover the induced connection $\tilde{\nabla}$ is complex linear on Q' . The curvature form R' of the connection $\tilde{\nabla}$ on Q' is given by

$$R'(x, y) = -\frac{\tilde{\tau}}{4n(n+2)}\Omega(x, y)I, \tag{1.2}$$

where $\tilde{\tau}$ is the scalar curvature of $\tilde{M}$ and $\Omega(x, y) = g(Ix, y)$ for $x, y \in T_pM$. In particular the curvature R' is of degree (1.1) . Then there is a unique holomorphic line bundle structure in Q' such that a (local) holomorphic section L is defined by $\tilde{\nabla}_{IX}L = I\tilde{\nabla}_X L$ for any vector field X.

The normal bundle $T^{\perp}M$ is a complex vector bundle with the complex strucutre I induced from $\tilde{I}$ which satisfies $\nabla^{\perp}_X I = 0$, where $\nabla^{\perp}$ denotes the connection of $T^{\perp}M$. Let σ be the second fundamental form of M in $\tilde{M}$. By Proposition 2.11 and Lemma 2.13 in [10], we have the following.

Lemma 1.3 *At each point $p \in M$, we have*
(1) $\sigma(Ix,y) = \sigma(x,Iy) = I\sigma(x,y)$ *for* $x,y \in T_pM$,
(2) $\tilde{g}(\sigma(x,y),Lz) = \tilde{g}(\sigma(x,z),Ly)$ *for* $L \in Q'_p, x,y,z \in T_pM$.

2 Natural lifts to the twistor space.

We recall the theory of twistor spaces of quaternionic Kähler manifolds, which is an important ingrudient for the study of quaternionic Kähler manifolds. The *twistor space* $\tilde{\mathcal{Z}}$ of a quaternionic Kähler manifold $(\tilde{M}^{4n},\tilde{g},\tilde{Q})$ is defined by $\tilde{\mathcal{Z}} = \{\tilde{I} \in \tilde{Q} | \tilde{I}^2 = -\mathrm{id}\}$. We normalize the fibre metric $\langle , \rangle$ of the bundle $\tilde{Q}$ such that a local canonical basis $\{\tilde{I},\tilde{J},\tilde{K}\}$ is an orthonormal basis, putting $\langle , \rangle = \frac{1}{4n}\tilde{g}$. Then the fibre $\tilde{\mathcal{Z}}_p$ of $\tilde{\mathcal{Z}}$ at $p \in \tilde{M}$ is given by

$$\tilde{\mathcal{Z}}_p = \{\tilde{I} \in \tilde{Q}_p | \tilde{I}^2 = -\mathrm{id}\} = \{\tilde{I} \in \tilde{Q}_p | \langle \tilde{I},\tilde{I}\rangle = 1\}.$$

Hence the natural projection $\tilde{\pi} : \tilde{\mathcal{Z}} \to \tilde{M}$ is an S^2-bundle over $\tilde{M}$. Since $\tilde{\mathcal{Z}}$ is a parallel fibre subbundle in $\tilde{Q}$ with respect to the Riemannian connection $\tilde{\nabla}$, the tangent bundle $T\tilde{\mathcal{Z}}$ is decomposed to the direct sum

$$T\tilde{\mathcal{Z}} = \mathcal{V} + \mathcal{H}, \tag{2.1}$$

where $\mathcal{V}$ is the vertical distribution tangent to the fibres of $\tilde{\pi}$ and $\mathcal{H}$ is the supplementary horizontal distribution defined by the Riemannian connection. The twistor space $\tilde{\mathcal{Z}}$ has a natural complex structure such that the distribution $\mathcal{H}$ is a holomporphic contact structure (S.Salamon [8],see also Besse Chapter 14 [3]). Moreover $\tilde{\mathcal{Z}}$ of a quaternionic Kähler manifold $\tilde{M}$ of positive scalar curvature admits a Einstein-Kähler metric.

From now on we assume that a quaternionic Kähler manifold $(\tilde{M}^{4n},\tilde{g},\tilde{Q})$ is of postive scalar curvature. Let M^{2m} be an almost Hermitian submanifold of $\tilde{M}$ together with a section $\tilde{I}$ of $\tilde{Q}|_M$. Then the map $M \ni p \mapsto \tilde{I}_p \in \tilde{\mathcal{Z}}_p$ is a section of the bundle $\tilde{\mathcal{Z}}|_M$ over M. The submanifold $\tilde{I}(M)$ of $\tilde{\mathcal{Z}}$ is called the *natural lift* of an almost Hermitian submanifold (D.V.Alekseevsky and S.Marchiafava [2]).

Theorem 2.1 ([2]) *A $2m(m \geq 2)$-dimensional almost Hermitian submanifold M^{2m} of $\tilde{M}$ is Kähler if and only if its natural lift $\tilde{I}(M)$ is a complex submanifold of $\tilde{\mathcal{Z}}$ which is an integral submanifold of the holomorphic contact structure $\mathcal{H}$. In particular the natural lift $\tilde{I}(M^{2n})$ of a half dimensional Kähler submanifold M^{2n} of $\tilde{M}^{4n}$ is a Legendrian submanifold of the twistor space $\tilde{\mathcal{Z}}$. Conversely, any Legendrian submanifold N of $\tilde{\mathcal{Z}}$ defines a half dimensional Kähler submanifold $M = \tilde{\pi}(N)$ of $\tilde{M}$.*

Legendrian submanifolds of $(\tilde{\mathcal{Z}}, \mathcal{H})$ are constructed locally as follows: By the holomorphic Darboux theorem, there exist local complex coordinates $u, p_1, \cdots, p_n, q^1, \cdots, q^n$ such that the holomorphic contact structure $\mathcal{H}$ is given by the kernel of a holomorphic 1-form $du - \sum_{i=1}^{n} p_i dq^i$. In term of these coordinates, a Legendrian submanifold locally has the form:

$$u = f(q^1, \cdots, q^n), p_i = \frac{\partial f}{\partial q^i}(q^1, \cdots, q^n), \quad (i = 1, \cdots, n),$$

where f is a holomorphic function called a *generating function* of the Legendrian submanifold. These Legendrian submanifolds project onto half dimensional Kähler submanifolds in a quaternionic Kähler manifold $\tilde{M}$. This is a natural generalization of the Bryant's famous construction of superminimal surfaces in $S^4 = \mathbf{H}P^1$ ([4]).

We consider another natural lift. For a $2m (m \geq 2)$-dimensional Kähler submanifold M^{2m} of $\tilde{M}$, we recall the orthogonal decomposition $\tilde{Q}|_M = \mathbb{R}\tilde{I} + Q'$. We put $\mathcal{Z} = Q' \cap \tilde{\mathcal{Z}}|_M$.Then the natural projection $\pi : \mathcal{Z} \to M$ is an S^1−bundle over M. It may be viewed as a kind of tube along the natural lift $\tilde{I}(M)$. Let $\hat{f} : \mathcal{Z} \to \tilde{\mathcal{Z}}$ and $f : M \to \tilde{M}$ be inclusion maps. Then We have a commutative diagram:

$$\begin{array}{ccc} (\mathcal{Z}, k) & \xrightarrow{\hat{f}} & (\tilde{\mathcal{Z}}, \tilde{k}) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ (M, g) & \xrightarrow{f} & (\tilde{M}, \tilde{g}) \end{array} \tag{2.2}$$

Our observation is the following.

Theorem 2.2 ([5]) *The space $\mathcal{Z}$ is a totally real and minimal submanifold of the twistor space $\tilde{\mathcal{Z}}$. In particular the space $\mathcal{Z}$ of a half dimensional Kähler submanifold M^{2n} of $\tilde{M}^{4n}$ is a minimal Lagrangian submanifold of $\tilde{\mathcal{Z}}$.*

We remark that Proposition 1.2 is a key fact for Theorems 2.1 and 2.2.

We denote by σ and $\hat{\sigma}$ the second fundamental forms of the submanifolds M in $\tilde{M}$ and $\mathcal{Z}$ in $\tilde{\mathcal{Z}}$, respectively. Then the following holds.

Proposition 2.3 *For each $z \in \mathcal{Z}$, the image of $\hat{\sigma}$ is contained in the horizontal subspace $\mathcal{H}_z$. Moreover we have*

$$\tilde{\pi}_* \hat{\sigma}(X, Y) = \sigma(\pi_* X, \pi_* Y) \qquad \textit{for} \quad X, Y \in T_z \mathcal{Z}.$$

This Proposition implies that if M^{2m} is a $2m(m \geq 2)$-dimensional totally geodesic Kähler submanifold of $\tilde{M}$, then $\mathcal{Z}$ is a totally geodesic submanifold of $\tilde{\mathcal{Z}}$.

Example 2.1 M.Takeuchi [9] studied a complete totally complex totally geodesic submanifold M^{2n} of a quaternionic symmetric space $\tilde{M}^{4n}$ of compact type or non-compact type. He called such a pair $(\tilde{M}, M)$ a *TCG-pair* and classified TCG-pairs. He also studied the twistor space $\tilde{\mathcal{Z}}$ of $\tilde{M}$ and constructed the diagram as in (2.2) for a TCG-pair $(\tilde{M}, M)$. In this case, he showed that a natural lift $\mathcal{Z}$ of M is given by the set of fixed points of an anti-holomorphic involution of $\tilde{\mathcal{Z}}$. We give here the table for classical TCG-pairs of compact type due to [9].

M	$\tilde{M}$
$\mathbf{C}P^n$	$\mathbf{H}P^n$
$(Q_p(\mathbf{C}) \times Q_q(\mathbf{C}))/\mathbf{Z}_2$	$\tilde{G}_{4,p+q}(\mathbf{R})$
$G_{2,m}(\mathbf{C})$	$\tilde{G}_{4,2m}(\mathbf{R})$
$\mathbf{C}P^p \times \mathbf{C}P^q$	$G_{2,p+q}(\mathbf{C})$
$G_{2,n}(\mathbf{R})$	$G_{2,n}(\mathbf{C})$

Notations in the table.
$\mathbf{C}P^n$: n- dimensional complex projective space
$\mathbf{H}P^n$: n- dimensional quaternion projective space
$Q_p(\mathbf{C})$: Complex hyperquadric of dimension p
$\tilde{G}_{p,q}(\mathbf{R})$: Grassmann manifold of oriented p-subspaces in $\mathbf{R}^{p+q}$
$G_{p,q}(\mathbf{F})$: Grassmann manifold of p-subspaces in $\mathbf{F}^{p+q}$

Here we remark that the definition of a totally complex submanifold in Takeuchi [9] is slightly different from our one in this note. Due to his definition, the section $\tilde{I}$ of $\tilde{Q}|_M$ is locally defined on a neighborhood of each point not necessarily globally defined (see also [10]).

3 Parallel Kähler submanifolds.

This section is devoted to a Kähler (and hence totally complex) submanifold M^{2n} of a quaternionic Kähler manifold $\tilde{M}^{4n}$ with parallel second fundamental form. Shortly we call it a *parallel Kähler submanifold.* First we show examples.

Example 3.1 The author [10] studied totally complex submanifolds with parallel second fundamental form in a quaternion projective space $\mathbf{H}P^n$ and classified them.They are locally congruent to one of the following:

$$
\begin{array}{ll}
(1) & \mathbf{C}P^n \hookrightarrow \mathbf{H}P^n \quad \text{(totally geodesic)} \\
(2) & Sp(3)/U(3) \hookrightarrow \mathbf{H}P^6 \\
(3) & SU(6)/S(U(3)\times U(3)) \hookrightarrow \mathbf{H}P^9 \\
(4) & SO(12)/U(6) \hookrightarrow \mathbf{H}P^{15} \\
(5) & E_7/E_6\cdot T^1 \hookrightarrow \mathbf{H}P^{27} \\
(6) & \mathbf{C}P^1(\tilde{c})\times \mathbf{C}P^1(\tilde{c}/2) \hookrightarrow \mathbf{H}P^2 \\
(7) & \mathbf{C}P^1(\tilde{c})\times \mathbf{C}P^1(\tilde{c})\times \mathbf{C}P^1(\tilde{c}) \hookrightarrow \mathbf{H}P^3 \\
(8) & \mathbf{C}P^1(\tilde{c})\times SO(n+1)/SO(2)\cdot SO(n-1) \hookrightarrow \mathbf{H}P^n \quad (n\geq 4),
\end{array}
$$

where $\mathbf{H}P^n$ has the scalar curvature $4n(n+2)\tilde{c}$ and $\mathbf{C}P^1(\tilde{c})$ is of constant curvature $\tilde{c}$. Their immersions in the above are given in [10].

We show a classification of Kähler submanifolds M^{2n} of a quaternionic Kähler manifold $\tilde{M}^{4n}$ with parallel non zero second fundamental form σ due to Alekseevsky and Marchiafava [1].

Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{Q})$ be a quaternionic Kähler manifold with nonvanishing scalar curvature and M^{2n} be a $2n$-dimensional Kähler (and hence totally complex) submanifold of $\tilde{M}^{4n}$. We use the notations in section 1. At each point $p \in M$, for non zero $L \in Q'_p$ L is a complex anti-linear isomorphism of T_pM to $T_p^{\perp}M$ since $LI = -IL$. For $L \in Q'_p$, we define a trilinear form $\psi(L)$ on T_pM by putting

$$\psi(L)(x,y,z) = \tilde{g}(\sigma(x,y), Lz) \qquad \text{for } x,y,z \in T_pM.$$

Then by Lemma 1.3 (2), $\psi(L)$ is a symmetric trilinear form. Thus we obtain a bundle homomorphism $\psi : Q' \to S^3(T^*M)$ of real vector bundles. We calculate the covariant derivative $(\bar{\nabla}_V\psi)(L) = \nabla_V(\psi(L)) - \psi(\tilde{\nabla}_V L)$, where ∇ and $\tilde{\nabla}$ denote the Riemannian connection of the Kähler manifold (M,g) and the induced connection on Q'. Then we have

$$(3.1) \qquad (\bar{\nabla}_v\psi)(L)(x,y,z) = \tilde{g}((\bar{\nabla}_v\sigma)(x,y), Lz).$$

We denote by $TM^{\mathbf{C}} = TM^+ + TM^-$ the decomposition of the complexified tangent bundle into $\pm\sqrt{-1}$-eigenspaces with respect to the complex structure I on M and by $T^*M^{\mathbf{C}} = T^*M^+ + T^*M^-$ the dual decomposition of the cotangent bundle. We extend $\psi(L)$ complex linearly to $S^3(T^*M^{\mathbf{C}})$. By Lemma 1.3 (1), there exists a $\phi(L) \in S^3(T^*M^+)$ such that

$$\psi(L) = \phi(L) + \overline{\phi(L)} \in S^3(T^*M^+) + S^3(T^*M^-).$$

The bundle homomorphism $\phi : Q' \to S^3(T^*M^+)$ is complex anti-linear, that is, $\phi(IL) = -\sqrt{-1}\phi(L)$. Let us denote by $\bar{Q}'$ the complex line bundle obtained from Q' by taking the opposite complex structure $\bar{I}$,i.e., $\bar{I} = -I$. Then $\phi : \bar{Q}' \to S^3(T^*M^+)$ is a bundle homomorphism of complex vector bundles.

The induced connection $\tilde{\nabla}$ is complex linear on $\bar{Q}'$,too and the curvature form R' of the connection $\tilde{\nabla}$ on $\bar{Q}'$ is given by

$$R'(x,y) = \frac{\tilde{\tau}}{4n(n+2)}\Omega(x,y)\bar{I}. \tag{3.2}$$

We can see this formula comparing with (1.2). Similarly to Q', there exists a unique holomorphic line bundle structure on $\bar{Q}'$ compatible with the connection $\tilde{\nabla}$. If the submanifold M satisfies the equation of Codazzi type, that is, $(\bar{\nabla}_x\sigma)(y,z) = (\bar{\nabla}_y\sigma)(x,z)$, then $\phi : \bar{Q}' \to S^3(T^*M^+)$ is a holomorphic bundle homomorphism of holomorphic vector bundles. This is proved by using (3.1). Now we assume that the Kähler submanifold M^{2n} of $\tilde{M}^{4n}$ has parallel non zero second fundamental form σ. Then ϕ vanishes nowhere and by (3.1) we have $\bar{\nabla}\phi = 0$. From this, it follows that $Q = \phi(\bar{Q}')$ is a holomorphic line subbundle of $S^3(T^*M^+)$ which is parallel with respect to the Riemannian connection ∇. Moreover the curvature form R^Q of the connection ∇^Q on Q induced by the the Riemannian connection ∇ is given by

$$R^Q = i\frac{\tilde{\tau}}{4n(n+2)}\Omega(x,y). \tag{3.3}$$

Consequently we obtain a nice observation due to Alekseevsky and Marchiafava ([1] Proposition 3.1).

Proposition 3.1 *Let M^{2n} be a parallel Kähler submanifold of a quaternionic Kähler manifold $\tilde{M}^{4n}$ with scalar curvature $\tilde{\tau} \neq 0$. If it is not totally geodesic, then on M there is a parallel holomorphic line subbundle Q of $S^3(T^*M^+)$ such that the curvature form of the connection ∇^Q on Q induced by the the Riemannian connection ∇ is given by* (3.3).

Alekseevsky and Marchiafava called a parallel holomorphic line subbundle $Q \subset S^3(T^*M^+)$ with curvature form (3.3) a *parallel cubic line subbundle* of type ν, $\nu = \frac{\tilde{\tau}}{4n(n+2)}$. They tried to classify Kähler manifolds which admit parallel cubic line subbundles and proved the following surprising result.

Theorem 3.2 ([1] Theorem 3.14) *Let M^{2n} $(n \geq 2)$ be a simply connected complete Kähler manifold which admits a parallel cubic line subbundle of type ν. If $\nu > 0$, then M^{2n} is one of compact Hermitian symmetric spaces described in Example 3.1 (2) $\sim$ (8). If $\nu < 0$, then M^{2n} is one of the noncompact dual spaces of the symmetric spaces in the case of $\nu > 0$.*

As we have already shown in Example 3.1, all of compact Hermitian symmetric spaces M^{2n} which appeared in the classification above admit realization as non totally geodesic parallel Kähler submanifolds of the quaternion projective space $\mathbf{H}P^n$. Alekseevsky and Marchiafava posed the similar problem of realization of M^{2n} as parallel Kähler submanifolds of the other quaternionic Kähler manifolds.

4 Einstein-Kähler submanifolds

In this section we characterize Kähler submanifolds in Example 3.1 under some curvature conditions. We obtained the following results ([11]).

Theorem 4.1 *Let M be a $2n$-dimensional Einstein-Kähler submanifold in $\mathbf{H}P^n$ $(n \geq 2)$. Then it has parallel second fundamental form and in particular is locally congruent to one of (1),(2),(3),(4),(5), and (7) in Example 3.1.*

Theorem 4.2 *Let M be a $2n$-dimensional locally reducible Kähler submanifold in $\mathbf{H}P^n$ $(n \geq 2)$. Then it has parallel second fundamental form and in particular is locally congruent to one of (6),(7), and (8) in the Example 3.1.*

Corollary 4.3 *Let M be a $2n$-dimensional Kähler submanifold with parallel Ricci tensor in $\mathbf{H}P^n$ $(n \geq 2)$. Then it has parallel second fundamental form and in particular is locally congruent to one of Kähler submanifolds in the Example 3.1.*

Can we replace our assumptions in the above by a weaker one, for example , a Kähler submanifold with constant scalar curvature?

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