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On the construction and classification of almost complex curves in a nearly Kähler 6-sphere

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This report is based on a joint work with Prof H. Hashimoto(Meijo University), Dr. T. Taniguchi(Tohoku University) (cf. [HTU]).

It is well known that a standard 6-dimensional sphere S^6 has a nearly Kähler structure. We denote it by J. Although a submanifold whose tangent space is invariant under the action of J is 2-dimensional or 4-dimensional, there is no 4-dimensional J-invariant submanifold by the result of A. Gray([Gr]). Therefore, the only possible case of J-invariant submanifold is that of immersed surface. We express the J-invariant surface as the image of almost complex conformal immersion of some Riemann surface M. In this case, we denote it by $f : M \longrightarrow S^6$. On the other hand, (totally real or CR) 3-dimensional submanifold can be often constructed as a tube of some radius over some almost complex curve(cf. Mashimo's article in this volume). For example, Ejiri immersion : $S^3(\frac{1}{16}) \longrightarrow S^6$ can be realized as a tube of radius $\frac{\pi}{2}$ in the direction of second normal space over almost complex curve $S^2(\frac{1}{6}) \longrightarrow S^6$. Almost complex curves of S^6 is divided into the four types of the following : (I) linearly full and superminimal in S^6, (II) linearly full and non-superminimal in S^6, (III) linearly fully immersed in some totally geodesic 5-dimensional sphere S^5 (which is necessarily non-superminimal), (IV) totally geodesic almost complex 2-sphere.

Since the automorphism group of the nearly Kähler structure is the exceptional Lie group G_2, S^6 can be expressed as a homogeneous space $S^6 = G_2/SU(3)$, which is a 3-symmetric space. For Type (I), it can be lifted to a horizontal holomorphic curve in $Q^5 = G_2/U(2)$ which is the twister space over $S^6 = G_2/SU(3)$. Bryant([Br]) gave the representation formula for almost complex curve of type (I) using this twister space. For types (II) and (III), Bolton-Pedit-Woodward([BPW]) showed that f has a Toda-framing into a 6-symmetric space $\widetilde{f} : M \longrightarrow G_2/\mathcal{T}^2$, where $\mathcal{T}^2$ is the maximal torus of $SU(3)$. From these points of views, we may consider the following problem : "Construct and classify the cases of type (II) and (III)". For the classification of type (II) and (III), there are some pioneering works by Bolton-Vrancken-Woodward([BVW]). In this note, we present some construction and classification of almost complex 2-tori of type (III).

1. Primitive map of finite type into 6-symmetric space

Theorem 1.1([BPW]). *Any almost complex 2-torus of type (II) or (III) f :*

$T^2 \longrightarrow S^6$ can be lifted to a primitive map of finite type into 6-symmetric space G_2/T^2.

This theorem, together with the results of McIntosh([Mc1], [Mc2]), says that any almost complex 2-torus can be constructed from the spectral data. However, since the spectral data describes the moduli space of primitive map of finite type from 2-tori, it is not so easy to pick up only the data for almost complex 2-tori.

In the following, we consider only the almost complex curves of type (III). We denote by V_6 the hyperplane of $\mathbf{R}^7$ whose sixth coordinate is identically zero. We then consider the following correspondence.

$$(*) \qquad V_6 \ni \mathbf{x_R} = {}^t(x_1, x_2, x_3, x_4, x_5, 0, x_7) \longleftrightarrow \mathbf{x_C} = \begin{pmatrix} x_1 + ix_7 \\ x_2 + ix_4 \\ -x_3 + ix_5 \end{pmatrix} \in \mathbf{C}^3.$$

By this correspondence $(*)$, we identify an almost complex curve of type (III) $f : M \longrightarrow S^6 \cap V_6 (= S^5_N)$ with a conformal immersion $f_{\mathbf{C}} : M \longrightarrow S^5 \subset \mathbf{C}^3$. We have the following theorem.

Theorem 1.2. *Let $f_{\mathbf{C}} : T^2 \longrightarrow S^5 \subset \mathbf{C}^3$ be the one corresponding to an almost complex 2-torus $f : T^2 \longrightarrow S^5_N$ by $(*)$. Then, $f_{\mathbf{C}}$ may be lifted to a primitive map of finite type into 6-symmetric space $SU(3)/T^1$. Moreover, its homogeneous projection into S^6, $f_{\mathbf{C}} : T^2 \longrightarrow S^6 = G_2/SU(3)$, is a harmonic map of finite type.*

Remark. (1) If a harmonic map into a symmetric space has a lift $\widetilde{f}$ into a generalized flag manifold so that $\widetilde{f}$ is a primitive map of finite type, then its homogeneous projection into some symmetric space is a harmonic map of finite type in very many cases (cf. [OU]).
(2) Arbitrary non-isotropic harmonic map of 2-torus $\varphi : T^2 \longrightarrow S^6 = SO(7)/SO(6)$ can be lifted to a primitive map of finite type $\widetilde{\varphi} : T^2 \longrightarrow SO(7)/T^2$, where $SO(7)/T^2$ is a 6-symmetric space and the above G_2/T^2 is a 6-symmetric submanifold of $SO(7)/T^2$.

We then have an diagram :

$$\begin{array}{ccccc} SO(7)/T^2 & \supset & G_2/T^2 & \supset & SU(3)/T^1 \\ \downarrow & & \downarrow & & \downarrow \\ S^6 \cong SO(7)/SO(6) & \cong & G_2/SU(3) & \supset & S^5 \cong SU(3)/SU(2) \end{array}$$

The last inclusion is due to the correspondence $(*)$.

2. Kähler angle of conformal immersion into S^5 and examples

Since an almost complex curve of type (III) $f_{\mathbf{C}} : M \longrightarrow S^5$ is a horizontal curve with respect to the Hopf fibration $S^5 \longrightarrow \mathbf{C}P^2$, $f_{\mathbf{C}}$ can be realized as a horizontal lift of a totally real minimal surface in $\mathbf{C}P^2$. However, to construct an almost complex curve as a horizontal lift of a totally real minimal surface in $\mathbf{C}P^2$, we need to know the Kähler angle of the lift with respect to the nearly Kähler structure J. Once we come to know the Kähler angle, we may make the lifted horizontal curve into an almost complex curve by rotating it in $\mathbf{C}^3$. The latter fact is due to [BVW]. Hence, we need to know the Kähler angle of a horizontal surface in S^5 and the following proposition gives the answer.

Proposition 2.1. *Suppose that $f : M \longrightarrow S^5_N \subset S^6$ is a conformal immersion. Let θ be the Kähler angle of f with respect to the nearly Kähler structure J on S^6. If $f_{\mathbf{C}} : M \longrightarrow S^5 \subset \mathbf{C}^3$ is horizontal with respect to the Hopf fibration $S^5 \longrightarrow \mathbf{C}P^2$, then*

$$\cos\theta = \mathrm{Re}\left\{ i \ \det\left(f_{\mathbf{C}} \ e^{-\frac{\omega}{2}} \frac{\partial f_{\mathbf{C}}}{\partial z} \ \ e^{-\frac{\omega}{2}} \frac{\partial f_{\mathbf{C}}}{\partial \bar{z}} \right) \right\} \tag{2.2}$$

Proposition 2.1 yields the machinary method of constructing almost complex curves of type (III) as follows.

Theorem 2.3. *Let $s_0 : M \longrightarrow \mathbf{C}^3$ be a smooth map and $\omega : M \longrightarrow \mathbf{R}$ a smooth function. Set $s_1 = e^{-\frac{\omega}{2}} \frac{\partial s_0}{\partial z}, s_2 = e^{-\frac{\omega}{2}} \frac{\partial s_0}{\partial \bar{z}}$. If $S = (s_0 \ s_1 \ s_2)$ has values in $U(3)$ and satisfies $\det S = -i$, then $f : M \longrightarrow S^5_N \subset S^6$ corresponding to $f_{\mathbf{C}} : M \longrightarrow S^5 \subset \mathbf{C}^3$ defined by $f_{\mathbf{C}} = s_0$ is a conformal immersion and an almost complex curve with respect to J. The converse is also true.*

Using Theorem 2.3, we may construct a 1-parameter family of almost complex curve of type (III) in terms of Jacobi elliptic functions based on the known example of totally real minimal surface due to Castro-Urbano([CU]).

[Example] Define $f : \mathbf{R}^2 \longrightarrow S^6$ by

$$\begin{aligned} f = {}^t\Bigg(&\sqrt{\frac{r_2}{r_1+r_2}} \cos\left(r_1 y + \frac{\pi}{6}\right) \mathrm{dn}(rx,p), \sqrt{\frac{r_1}{r_1+r_2}} \cos\left(\frac{\pi}{6} - r_2 y\right) \mathrm{cn}(rx,p), \\ &- \sqrt{\frac{r_2 p^2 + r_1}{r_1+r_2}} \cos\left(\frac{\pi}{6} - r_3 y\right) \mathrm{sn}(rx,p), \sqrt{\frac{r_1}{r_1+r_2}} \sin\left(\frac{\pi}{6} - r_2 y\right) \mathrm{cn}(rx,p), \\ &\sqrt{\frac{r_2 p^2 + r_1}{r_1+r_2}} \sin\left(\frac{\pi}{6} - r_3 y\right) \mathrm{sn}(rx,p), 0, \sqrt{\frac{r_2}{r_1+r_2}} \sin\left(r_1 y + \frac{\pi}{6}\right) \mathrm{dn}(rx,p) \Bigg), \end{aligned} \tag{2.4}$$

where

$$\begin{cases} r^2 = \dfrac{\alpha^3 - 2 + 2\sqrt{\alpha^3+1}}{\alpha^2} \quad , \quad p^2 = \dfrac{\alpha^3 - 2 - 2\sqrt{\alpha^3+1}}{\alpha^3 - 2 + 2\sqrt{\alpha^3+1}}, \\ r_1 = \dfrac{\sqrt{\alpha^3+1}+1}{\alpha} \quad , \quad r_2 = \dfrac{\sqrt{\alpha^3+1}-1}{\alpha} \quad , \quad r_3 = \dfrac{2}{\alpha}, \end{cases} \tag{2.5}$$

and $\alpha(\geqq 2)$ is a real number. Then, it follows from the correspondence $(*)$ and Theorem 2.3 that f gives a 1-parameter family of almost complex curves of type (III).

3. Spectral data and representation formula in term of Prym-theta function

The spectral data for constructing almost complex 2-torus of type (III), f_C : $T^2 \longrightarrow S^5$ is given by the triplet $\left(\hat{C}, \hat{\mathcal{L}}, \pi\right)$ which satisfies the following four conditions : For $d \equiv 1 \bmod 6$

(1) $\hat{C}$: compact Riemann surface of genus $\hat{g} = 2d$ admitting an anti-holomorphic involution ρ and a holomorphic involution σ, which satisfy $\rho\sigma = \sigma\rho$.

(2) $\pi : \hat{C} \longrightarrow \mathbf{C}P^1$ is a three-fold holomorphic covering map with $\pi \circ \rho = \overline{\pi}^{-1}$. Moreover, the divisor (π) and the ramification divisor R of π are given by

$$(\pi) = 3P_0 - 3P_\infty, \quad R = 2P_0 + 2P_\infty + D_0 + \rho D_0,$$

where two points P_0 and P_∞ satisfy $\sigma(P_0) = P_0, \sigma(P_\infty) = P_\infty, \rho(P_0) = P_\infty$ and D_0 is defined by

$$\begin{cases} D_0 = \{P_1, \cdots, P_g, P_{g+1}, \cdots, P_{\hat{g}}\}, \\ P_{g+j} = \sigma\rho(P_j) \qquad \text{for } j = 1, \cdots, g, \end{cases}$$

and arbitrary two points of D_0 are distinct each other.

(3) A complex line bundle $\hat{\mathcal{L}} = \mathcal{O}_{\hat{C}}(2P_0 + D_0)$ of degree $(\hat{g} + 2)$ over $\hat{C}$.

(4) π has no branch points over $S^1_\lambda = \{\lambda \in P^1(\mathbf{C}) : |\lambda| = 1\}$ and ρ fixes each point of $\pi^{-1}(S^1_\lambda)$.

If we define a complex line bundle $\hat{\mathcal{L}}_0$ over $\hat{C}$ by $\hat{\mathcal{L}}_0 = \mathcal{O}_{\hat{C}}(2P_0 - 2P_\infty + D_0)$, then we have $\deg\left(\hat{\mathcal{L}}_0\right) = \hat{g}$. It is known ([Mc1]) that $\hat{\mathcal{L}}_0$ is non-special, hence $H^0(\hat{C}, \hat{\mathcal{L}}_0)$ is of 1-dimension. Therefore, if we express a non-trivial section ψ of $\hat{\mathcal{L}}_0$ explicitly, then we can write down $f_C : T^2 \longrightarrow S^5$ explicitly. The section ψ is given in terms of Prym-theta function η (cf. [Fa], [MaMa], [HTU]) as follows :

(3.1)

$$\psi(P) = \frac{\eta(\mathcal{B}(P) + iUz + iV\bar{z} - \mathbf{e})}{\eta(\mathcal{B}(P) - \mathbf{e})}\nu^{-2} \exp\left(\int_{P_0}^{P} iz\Omega_\infty + \int_{P_\infty}^{P} i\bar{z}\Omega_0\right),$$

where Ω_0 and Ω_∞ are normalized(= "zero $\mathcal{A}$-periods") Abelian differentials of second kind. They satisfy the relations $\rho^*\Omega_\infty = \overline{\Omega_0}, \sigma^*\Omega_0 = -\Omega_0, \sigma^*\Omega_\infty = -\Omega_\infty$ and have the asymptotic behaviors as follows :

$$\begin{cases} \displaystyle\int_{P_\infty}^{P} \Omega_0 = c\nu^{-1} + o(\nu^{-2}) \\ \displaystyle\int_{P_0}^{P} \Omega_\infty = \nu + o(\nu^{-2}) \quad \text{around } P_\infty, \end{cases} \qquad \begin{cases} \displaystyle\int_{P_\infty}^{P} \Omega_0 = \nu^{-1} + o(\nu^{2}) \\ \displaystyle\int_{P_0}^{P} \Omega_\infty = c\nu + o(\nu^{2}) \quad \text{around } P_0, \end{cases}$$

where $c \in \mathbf{R}$ and $U = (U_1, \cdots, U_g), V = (V_1, \cdots, V_g)$ are $\mathcal{B}$-periods given by $U_j = \int_{B_j} \Omega_\infty, V_j = \int_{B_j} \Omega_0$. Let e^ω be the one define by

$$e^\omega = 2\partial_z \partial_{\bar{z}} \log \eta(iUz + iV\bar{z} - \mathbf{e}) + c. \tag{3.2}$$

This gives a finite gap solution for Tzitzéica equation, which is the integrability condition for almost complex curve of type (III). The induced metric on T^2 may be expressed as $2e^\omega dz d\bar{z}$. That e^ω is $\mathbf{R}$-valued follows from the relation $\psi(\sigma(P)) = \nu^{-4}\overline{\psi(\rho(P))}$. If we define $\hat{\psi}_0, \hat{\psi}_1$ and $\hat{\psi}_2$ by

$$\hat{\psi}_0 = \psi, \quad \hat{\psi}_1 = \partial_z \psi, \quad \hat{\psi}_2 = i\lambda e^{-\omega} \partial_{\bar{z}} \psi \tag{3.3}$$

then the expression (3.1) of ψ follows from the integrability condition of the system of differential equations formed by $\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2$.

Now, we obtain the following theorem.

Theorem 3.4. *Given the spectral data* $\left(\hat{C}, \hat{\mathcal{L}}, \pi\right)$, *we define* $\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2$ *by (3.3). Moreover, define* $\hat{S}$ *by*

$$\hat{S} = \frac{1}{\sqrt{3}} \begin{pmatrix} \hat{\psi}_0(Q_1) & e^{-\frac{\omega}{2}}\hat{\psi}_1(Q_1) & e^{\frac{\omega}{2}}\hat{\psi}_2(Q_1) \\ \hat{\psi}_0(Q_2) & e^{-\frac{\omega}{2}}\hat{\psi}_1(Q_2) & e^{\frac{\omega}{2}}\hat{\psi}_2(Q_2) \\ \hat{\psi}_0(Q_3) & e^{-\frac{\omega}{2}}\hat{\psi}_1(Q_3) & e^{\frac{\omega}{2}}\hat{\psi}_2(Q_3) \end{pmatrix}. \tag{3.5}$$

Then, $S = \exp(\frac{\pi}{2}i + \frac{2n}{3}\pi i)\left(\det(\hat{S})\right)^{-\frac{1}{3}} \hat{S}$, $(n = 0, 1, 2)$, *gives a Toda-framing for almost complex curve of type (III). Thus, the first column vector of S gives an almost complex curve of type (III),* $f : \mathbf{R}^2 \longrightarrow S^5_N$. *Moreover, the necessary condition for f be doubly-periodic is that there are two complez numbers* c_1, c_2 *satisfying* $c_1\overline{c_2} \neq \overline{c_1}c_2$ *such that*

$$\mathrm{Re}\,(c_k U_j)\,, \mathrm{Re}\left(c_k \int_{P_0}^{Q_l} \Omega_\infty\right) \in \pi\mathbf{Z} \quad \text{for } j = 1, \cdots, g\,; k = 1, 2\,; l = 1, 2, 3.$$

hold.

Remark. There are some overlapps in fundamental concepts of the works between [Mc3] and [HTU]. Although the paper [Mc3] has been already published, our work had been announced in [U].

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