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EXISTENCE OF TIME-PERIODIC SOLUTIONS OF THE EQUATIONS OF MAGNETO-MICROPOLAR FLUID FLOW

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1. INTRODUCTION

We consider the time-periodic problem for the system of equations of magneto-micropolar fluid motion in a bounded domain.

Micropolar fluid was first introduced by Eringen [3], which gives a model of a viscous fluid consisting of randomly oriented (or spherical) particles. This model describes the behavior of various real fluids better than the classical Navier-Stokes model. For more information, we refer the reader to [6] and [7]. Ahmadi and Shahinpoor [1] derived the governing equations of magneto-micropolar fluids as the generalized incompressible MHD fluids with neutral fluid seedings in the form of rigid microinclusions.

Let $\Omega \subset \mathbb{R}^N$ ($N=2$ or 3) be a container with rigid superconducting wall which a magneto-micropolar fluid occupies. In the case where the space dimension is three, the motion of the fluid is described by the following system of equations:

$$(1)\quad \frac{\partial u}{\partial t} - (\mu+\chi)\Delta u + (u\cdot \operatorname{grad})u - (b\cdot\operatorname{grad})b + \operatorname{grad}\left(p+\frac{1}{2}b\cdot b\right) = f + 2\chi\operatorname{curl}\omega,$$

$$(2)\quad \frac{\partial \omega}{\partial t} - \alpha\Delta\omega - \beta\operatorname{grad}(\operatorname{div}\omega) + 4\chi\omega + (u\cdot\operatorname{grad})\omega = g + 2\chi\operatorname{curl}u,$$

$$(3)\quad \frac{\partial b}{\partial t} + \nu\operatorname{curl}(\operatorname{curl}b) - \operatorname{curl}(u\times b) = 0,$$

$$(4)\quad \operatorname{div}u = 0,\quad \operatorname{div}b = 0,$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ is the velocity field, $\omega = (\omega^1(x,t),\omega^2(x,t),\omega^3(x,t))$ the microrotation field, $b = (b^1(x,t), b^2(x,t), b^3(x,t))$ the magnetic field, $p = p(x,t)$ the pressure, $f = (f^1(x,t), f^2(x,t), f^3(x,t))$ the body force, $g = (g^1(x,t), g^2(x,t), g^3(x,t))$ the body couple and $\mu,\chi,\alpha,\beta,\nu$ are the physical constants. The physical constants are usually assumed to satisfy the condition: $\min(\mu,\chi,\alpha,\alpha+\beta,\nu) > 0$. Here, for simplicity, the density of the fluid, the squared radius of gyration and the permeability are all normalized to 1.

We here consider the system under the periodicity conditions

$$(5)\qquad u(\cdot,0) = u(\cdot,T),\quad \omega(\cdot,0) = \omega(\cdot,T),\quad b(\cdot,0) = b(\cdot,T),$$

where T is a given positive number, and the boundary conditions

$$(6)\qquad u|_{\partial\Omega} = 0,\quad \omega|_{\partial\Omega} = 0,\quad b\cdot n|_{\partial\Omega} = 0,\quad (\operatorname{curl}b)\times n|_{\partial\Omega} = 0,$$

where n denotes the unit outward normal on $\partial\Omega$.

In the case $N = 2$, the system (1)–(4) and the boundary conditions (6) should be slightly modified. We define the operators curl, $\widetilde{\text{curl}}$ and the exterior product $\widetilde{\times}$ by

$$\begin{aligned} \operatorname{curl} v &= \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} && \text{for all } v = (v^1(x_1,x_2), v^2(x_1,x_2)), \\ \widetilde{\text{curl}}\varphi &= \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}\right) && \text{for all } \varphi = \varphi(x_1,x_2), \\ a \,\widetilde{\times}\, b &= a^1 b^2 - a^2 b^1 && \text{for all } a = (a^1,a^2) \text{ and } b = (b^1,b^2). \end{aligned}$$

As for the unknown functions (u,ω,b), note that u and b are $\mathbb{R}^2$-valued functions in $\Omega \times [0,T]$ and ω is a scalar function. Thus we put in (2) $\beta \operatorname{grad}(\operatorname{div}\omega) = 0$. Furthermore $\operatorname{curl}\omega$ should be replaced by $\widetilde{\text{curl}}\,\omega$ in equation (1), $\operatorname{curl}(\operatorname{curl} b)$ and $\operatorname{curl}(u \times b)$ replaced by $\widetilde{\text{curl}}(\operatorname{curl} b)$ and $\widetilde{\text{curl}}(u \,\widetilde{\times}\, b)$ in equation (3) respectively. As for the boundary conditions for b, $(\operatorname{curl} b) \times n|_{\partial\Omega} = 0$ should be replaced by $\operatorname{curl} b|_{\partial\Omega} = 0$.

For the case $N = 3$, Łukaszewicz et al.[8] showed the existence and uniqueness of time-periodic solutions of the system. Their arguments are based on a modification of the Galerkin's approximation method for some abstract semilinear periodic problem due to Kato [4]. Hence they needed the rather strong regularity of the external forces such as $f \in C^1([0,T]; \boldsymbol{L}^2(\Omega))$. Our arguments rely on the nonmonotone perturbation theory for nonlinear evolution equations governed by subdifferential operators due to Ôtani [10]. In our framework, the external forces can be taken from a weaker and more natural spaces such as $f \in L^2(0,T; \boldsymbol{L}^2(\Omega))$. Furthermore, the advantage of our method lies in the fact that our framework can cover much wider class of nonlinear problems including some quasilinear parabolic systems in regions with moving boundaries.

2. Functional settings

In this section, we introduce some function spaces and operators.

2.1. Function spaces. Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N = 2,3$) with smooth boundary $\partial\Omega$ (say C^2). For simplicity, assume further that Ω is simply connected.

For any function space $X(\Omega)$ on Ω, we denote by $\boldsymbol{X}(\Omega) = (X(\Omega))^N$ the $\mathbb{R}^N$-valued function space whose each component belongs to $X(\Omega)$.

We need the following function spaces:

$$\begin{aligned} \boldsymbol{C}^\infty_n(\overline{\Omega}) &= \{v \in \boldsymbol{C}^\infty(\overline{\Omega}) |\ \operatorname{div} v = 0 \text{ in } \Omega,\ v \cdot n = 0 \text{ on } \partial\Omega\}, \\ \boldsymbol{C}^\infty_\sigma(\Omega) &= \{v \in \boldsymbol{C}^\infty(\Omega) |\ \operatorname{div} v = 0 \text{ in } \Omega,\ \operatorname{supp} v \subset \Omega\}, \\ \boldsymbol{L}^2_\sigma(\Omega) &= \text{ the closure of } \boldsymbol{C}^\infty_n(\overline{\Omega}) \text{ in } \boldsymbol{L}^2(\Omega) \\ &= \text{ the closure of } \boldsymbol{C}^\infty_\sigma(\Omega) \text{ in } \boldsymbol{L}^2(\Omega) \\ &= \{v \in \boldsymbol{L}^2(\Omega) |\ \operatorname{div} v = 0 \text{ in } \Omega,\ v \cdot n = 0 \text{ on } \partial\Omega\}, \\ \boldsymbol{H}^1_n(\Omega) &= \text{ the closure of } \boldsymbol{C}^\infty_n(\overline{\Omega}) \text{ in } \boldsymbol{H}^1(\Omega) \\ &= \{v \in \boldsymbol{H}^1(\Omega) |\ \operatorname{div} v = 0 \text{ in } \Omega,\ v \cdot n = 0 \text{ on } \partial\Omega\}, \\ \boldsymbol{H}^1_\sigma(\Omega) &= \text{ the closure of } \boldsymbol{C}^\infty_\sigma(\Omega) \text{ in } \boldsymbol{H}^1(\Omega) \\ &= \{v \in \boldsymbol{H}^1_0(\Omega) |\ \operatorname{div} v = 0 \text{ in } \Omega\}, \end{aligned}$$

$$H = \begin{cases} \boldsymbol{L}^2_\sigma(\Omega) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2_\sigma & \text{if } N = 3; \\ \boldsymbol{L}^2_\sigma(\Omega) \times L^2(\Omega) \times \boldsymbol{L}^2_\sigma & \text{if } N = 2, \end{cases}$$

$$V = \begin{cases} \boldsymbol{H}^1_\sigma(\Omega) \times \boldsymbol{H}^1_0(\Omega) \times \boldsymbol{H}^1_n & \text{if } N = 3; \\ \boldsymbol{H}^1_\sigma(\Omega) \times H^1_0(\Omega) \times \boldsymbol{H}^1_n & \text{if } N = 2. \end{cases}$$

We set

$$\begin{aligned} (u,v) &= \sum_{i=1}^N \int_\Omega u^i v^i, \quad \|u\| = (u,u)^{1/2} && \text{for } u,v \in \boldsymbol{L}^2(\Omega), \\ (u,v)_\sigma &= (u,v), \qquad \|u\|_\sigma = \|u\| && \text{for } u,v \in \boldsymbol{L}^2_\sigma(\Omega), \\ \|\nabla u\| &= \left(\sum_{i,j=1}^N \int_\Omega \left| \frac{\partial u^i}{\partial x_j} \right|^2 \right)^{1/2} && \text{for } u,v \in \boldsymbol{H}^1(\Omega), \\ (U_1,U_2)_H &= (u_1,u_2)_\sigma + (\omega_1,\omega_2) + (b_1,b_2)_\sigma && \text{for } U_i = (u_i,\omega_i,b_i) \in H \ (i=1,2), \\ |U|_H &= (U,U)_H^{1/2} && \text{for } U \in H, \end{aligned}$$

where $u = (u^1,u^2,u^3)$, $v = (v^1,v^2,v^3)$.

In order to define the norms of $\boldsymbol{H}^1_\sigma(\Omega)$, $\boldsymbol{H}^1_0(\Omega)$ and $\boldsymbol{H}^1_n(\Omega)$, we need the following lemma:

Lemma 1. *There exist positive constants $\lambda_1,\lambda_2,\lambda_3$ depending only on Ω such that*

$$\begin{aligned} &(i) && \lambda_1\|u\|^2_\sigma \le \|\nabla u\|^2 && \text{for all } u \in \boldsymbol{H}^1_\sigma(\Omega), \\ &(ii) && \lambda_2\|\omega\|^2 \le \|\nabla \omega\|^2 && \text{for all } \omega \in \boldsymbol{H}^1_0(\Omega), \\ &(iii) && \lambda_3\|b\|^2_\sigma \le \|\operatorname{curl} b\|^2 && \text{for all } b \in \boldsymbol{H}^1_n(\Omega). \end{aligned}$$

Proof. (i) and (ii) result from the Poincaré inequality. For (iii), see for example Appendix I in [12]. □

In view of Lemma 1, we equip $\boldsymbol{H}^1_\sigma(\Omega)$, $\boldsymbol{H}^1_0(\Omega)$, $\boldsymbol{H}^1_n(\Omega)$ with the norms $\|\nabla u\|$, $\|\nabla\omega\|$, $\|\operatorname{curl} b\|$ respectively.

For an arbitrary normed space X, we denote by $L^p(0,T;X)$ the set of all strongly measurable functions v on $[0,T]$ with values in X satisfying

$$\int_0^T \|v(t)\|^p_X dt < \infty \quad \text{if } p \in [1,\infty); \quad \operatorname*{ess\,sup}_{t\in[0,T]} \|v(t)\|_X < \infty \quad \text{if } p = \infty.$$

The norm of $L^p(0,T;X)$ is defined by

$$\|v\|_{L^p(0,T;X)} = \begin{cases} \left(\displaystyle\int_0^T \|v(t)\|^p_X dt \right)^{1/p} & \text{if } p \in [1,\infty), \\ \operatorname*{ess\,sup}_{t\in[0,T]} \|v(t)\|_X & \text{if } p = \infty. \end{cases}$$

For each $p \in [1,\infty)$ we also equip $L^p(0,T;X)$ with the following equivalent norm:

$$\|v\|^p_{X,p,T} = \begin{cases} \dfrac{1}{T}\|v\|^p_{L^p(0,T;X)} & \text{if } 0 < T \le 1, \\ \displaystyle\sup_{1\le t\le T} \int_{t-1}^t \|v(\tau)\|^p_X d\tau & \text{if } T \ge 1. \end{cases}$$

In what follows, we write $\|v\|_{p,T}$ instead of $\|v\|_{\mathbb{R},p,T}$ for simplicity.

2.2. **Operators.** First recall the well-known orthogonal decomposition of $\boldsymbol{L}^2(\Omega)$ called the *Helmholtz-Weyl decomposition*:

$$\boldsymbol{L}^2(\Omega) = \boldsymbol{L}^2_\sigma(\Omega) \oplus \boldsymbol{G}(\Omega), \quad \boldsymbol{G}(\Omega) = \{\operatorname{grad} q | q \in \boldsymbol{H}^1(\Omega)\}. \tag{7}$$

Let $P : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2_\sigma(\Omega)$ be the orthogonal projection.

We define three operators A_i $(i = 1, 2, 3)$ as follows.

$$D(A_1) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_\sigma(\Omega);$$
$$A_1 u = -(\mu + \chi) P \Delta u \qquad \text{for } u \in D(A_1),$$
$$D(A_2) = \begin{cases} \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega) & \text{if } N = 3, \\ H^2(\Omega) \cap H^1_0(\Omega) & \text{if } N = 2; \end{cases}$$
$$A_2 \omega = \begin{cases} -\alpha \Delta \omega - \beta \operatorname{grad}(\operatorname{div} \omega) & \text{for } \omega \in D(A_2) \quad \text{if } N = 3, \\ -\alpha \Delta \omega & \text{for } \omega \in D(A_2) \quad \text{if } N = 2, \end{cases}$$
$$D(A_3) = \begin{cases} \{b \in \boldsymbol{H}^2(\Omega) | (\operatorname{curl} b) \times n|_{\partial\Omega} = 0 \text{ on } \partial\Omega\} \cap \boldsymbol{H}^1_n(\Omega) & \text{if } N = 3, \\ \{b \in \boldsymbol{H}^2(\Omega) | \operatorname{curl} b|_{\partial\Omega} = 0 \text{ on } \partial\Omega\} \cap \boldsymbol{H}^1_n(\Omega) & \text{if } N = 2; \end{cases}$$
$$A_3 b = \begin{cases} \nu \operatorname{curl}(\operatorname{curl} b) & \text{for } b \in D(A_3) \quad \text{if } N = 3, \\ \nu \widetilde{\operatorname{curl}}\, (\operatorname{curl} b) & \text{for } b \in D(A_3) \quad \text{if } N = 2. \end{cases}$$

It is known that these operators all enjoy the elliptic estimates.

Lemma 2. *Each operator A_i $(i = 1, 2, 3)$ is a linear self-adjoint maximal monotone operator. Moreover, there exist constants C_i $(i = 1, 2, 3)$ depending only on Ω and the physical constants $\mu, \chi, \alpha, \beta, \nu$ such that the following estimates hold.*

$$(i) \quad \|u\|_{\boldsymbol{H}^2(\Omega)} \le C_1 \|A_1 u\|_\sigma \qquad \text{for all } u \in D(A_1),$$
$$(ii) \quad \|\omega\|_{\boldsymbol{H}^2(\Omega)} \le C_2 \|A_2 \omega\| \qquad \text{for all } \omega \in D(A_2),$$
$$(iii) \quad \|b\|_{\boldsymbol{H}^2(\Omega)} \le C_3 \|A_3 b\|_\sigma \qquad \text{for all } b \in D(A_3).$$

Proof. The linearity and monotonicity of A_i $(i = 1, 2, 3)$ is obvious. For the maximality and the elliptic estimates, we refer to [12] for A_1, [9] for A_2 and [11] for A_3. □

2.3. **Abstract formulation.** Here and henceforth $U = (u, \omega, b)$ denotes an element of H with $u, b \in \boldsymbol{L}^2_\sigma(\Omega)$ and $\omega \in \boldsymbol{L}^2(\Omega)$ $(\omega \in L^2(\Omega)$ if $N = 2)$.

We introduce a functional $\Phi : H \to [0, \infty]$ defined by

$$\Phi(U) = \begin{cases} \dfrac{\mu + \chi}{2} \|\nabla u\|^2 + \dfrac{\alpha}{2} \|\nabla \omega\|^2 + \dfrac{\beta}{2} \|\operatorname{div} \omega\|^2_{L^2} + \dfrac{\nu}{2} \|\operatorname{curl} b\|^2 & \text{if } U \in V, \\ \infty & \text{if } U \in H \setminus V, \end{cases}$$

if $N = 3$. When $N = 2$ we put $\|\operatorname{div} \omega\|^2_{L^2} = 0$ in the right-hand side. It is easy to see that Φ is a proper lower semicontinuous convex functional on H and that its subdifferential $\partial\Phi$ is characterized by

$$D(\partial\Phi) = D(A_1) \times D(A_2) \times D(A_3),$$
$$\partial\Phi(U) = (A_1 u, A_2 \omega, A_3 b) \quad \text{for } U = (u, \omega, b) \in D(\partial\Phi).$$

To formulate our problem, we first operate P to equation (1) in order to eliminate the "gradient terms." Then we can reduce the system (1)–(6) to an abstract equation governed by a subdifferential operator:

$$\frac{dU}{dt}(t) + \partial\Phi(U(t)) + L(U(t)) + B(U(t)) = F(t) \quad \text{in } [0,T], \tag{8}$$

$$U(0) = U(T), \tag{9}$$

where

$$L(U) = (-2\chi \operatorname{curl} \omega, -2\chi \operatorname{curl} u + 4\chi\omega, 0),$$

$$B(U) = \begin{cases} (P(u\cdot\operatorname{grad})u - P(b\cdot\operatorname{grad})b, (u\cdot\operatorname{grad})\omega, -\operatorname{curl}(u\times b)) & \text{if } N=3; \\ (P(u\cdot\operatorname{grad})u - P(b\cdot\operatorname{grad})b, (u\cdot\operatorname{grad})\omega, -\widetilde{\operatorname{curl}}\,(u\mathbin{\tilde{\times}} b)) & \text{if } N=2, \end{cases}$$

$$F = (Pf, g, 0).$$

Note that $-\operatorname{curl}(u\times b) = (u\cdot\operatorname{grad})b - (b\cdot\operatorname{grad})u$ (resp. $-\widetilde{\operatorname{curl}}\,(u \mathbin{\tilde{\times}} b) = (u\cdot\operatorname{grad})b - (b\cdot\operatorname{grad})u$) if $\operatorname{div} u = \operatorname{div} b = 0$.

Now our results can be stated as follows.

Theorem 1 (existence). *In the case where $N = 3$, there exists a constant $\rho_1 > 0$ depending only on Ω and the physical constants such that if $F \in L^2(0,T;H)$ satisfies $\|F\|_{H,2,T} \le \rho_1$, then there exists a solution U to (8) and (9) satisfying*

$$(i) \quad U \in C([0,T];V),$$

$$(ii) \quad \frac{dU}{dt},\ \partial\Phi(U(\cdot)),\ L(U(\cdot)),\ B(U(\cdot)) \in L^2(0,T;H).$$

In the case where $N = 2$, for each $F \in L^2(0,T;H)$, there exists a solution U to (8) and (9) satisfying (i) and (ii).

Theorem 2 (stability and uniqueness). *There exist positive constants ρ_2 and ρ_3 depending only on Ω and the physical constants such that if $F \in L^2(0,T;H)$ satisfies $\|F\|_{H,2,T} < \rho_2$, then there exists a unique periodic solution U as in Theorem 1 and if there exists a solution $\widehat{U} \in C([0,T];H) \cap L^2(0,T;V)$ to (8) with the initial condition $\widehat{U}(0) = \widehat{U}_0$ for some $\widehat{U}_0 \in H$, we have*

$$|\widehat{U}(t) - U(t)|_H \le |\widehat{U}_0 - U(0)|_H e^{-\rho_3 t} \quad \text{for all } t \in [0,T].$$

3. Some lemmas

In this section, we collect some lemmas used in sections 4 and 5.

3.1. Some estimates.

Lemma 3. *The following identities hold.*

$$(i) \quad (\operatorname{curl} v, w) = \begin{cases} (v, \operatorname{curl} w) & \text{for all } (v,w) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}_0^1(\Omega), \\ (v, \widetilde{\operatorname{curl}}\, w) & \text{for all } (v,w) \in \boldsymbol{H}^1(\Omega) \times H_0^1(\Omega) \text{ if } N=2. \end{cases}$$

$$(ii) \quad \|\nabla w\|^2 = \begin{cases} \|\operatorname{curl} w\|^2 + \|\operatorname{div} w\|_{L^2}^2 & \text{for all } \omega \in \boldsymbol{H}_0^1(\Omega), \\ \|\widetilde{\operatorname{curl}} w\|^2 & \text{for all } \omega \in H_0^1(\Omega) \text{ if } N=2. \end{cases}$$

Proof. (i) The result immediately follows by integrating by parts.

(ii) In the case where $N = 3$, (i) combined with the well-known formula

$$\operatorname{curl}(\operatorname{curl} w) = -\Delta w + \operatorname{grad}(\operatorname{div} w)$$

gives the result. If $N = 2$, the result immediately follows from the definition of the norm $\|\nabla w\|$ and the operator $\widetilde{\operatorname{curl}}$. □

Lemma 4. *If* $u \in \boldsymbol{H}_n^1(\Omega)$ *and* $v, w \in \boldsymbol{H}^2(\Omega)$ *then*

$$((u \cdot \operatorname{grad})v, w) = -((u \cdot \operatorname{grad})w, v).$$

In particular, if $w = v$, *then* $((u \cdot \operatorname{grad})v, v) = 0$.

Lemma 5. *There exists a constant* C *depending only on* Ω *such that*

$$\|(u \cdot \operatorname{grad})v\| \leq \begin{cases} C\|\nabla u\| \, \|\nabla v\|^{1/2}\|v\|_{\boldsymbol{H}^2}^{1/2} & \text{if } N = 3, \\ C\|u\|^{1/2}\|\nabla u\|^{1/2} \, \|\nabla v\|^{1/2}\|v\|_{\boldsymbol{H}^2}^{1/2} & \text{if } N = 2, \end{cases}$$

for all $(u, v) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^2(\Omega)$.

Lemma 6. *There exists a constant* C *depending only on* Ω *such that*

$$|((u \cdot \operatorname{grad})v, w)| \leq \begin{cases} C\|u\|^{1/2}\|u\|_{\boldsymbol{H}^1}^{1/2}\|\nabla v\| \, \|w\|_{\boldsymbol{H}^1} & \text{if } N = 3, \\ C\|u\|^{1/2}\|u\|_{\boldsymbol{H}^1}^{1/2}\|\nabla v\| \, \|w\|^{1/2}\|w\|_{\boldsymbol{H}^1}^{1/2} & \text{if } N = 2 \end{cases}$$

for all $u, v, w \in \boldsymbol{H}^1(\Omega)$.

For the proofs of Lemmas 4, 5 and 6, see [12]. We here note that Lemmas 5 and 6 are also valid even if v, w are scalar functions.

The following lemma will be used to establish various a priori estimates in sections 4 and 5.

Lemma 7. *Let* y *be a nonnegative absolutely continuous function on* $[0, T]$ *with* $y(0) = y(T)$, $z \in L^1(0, T)$, w *be a nonnegative function belonging to* $L^1(0, T)$, $a_0 > 0$ *and* $a_1 \geq 0$ *satisfying*

$$\frac{dy}{dt}(t) + a_0 y(t) \leq |z(t)| + (a_1 + w(t))y(t) \quad \text{for a.e. } t \in [0, T].$$

If $z \not\equiv 0$ *or* $a_1 \neq 0$, *assume further that* $\|z\|_{1,T} < a_0$ *and that there exists a positive constant* a_2 *such that* $\|y\|_{1,T} \leq a_2\|z\|_{1,T}$. *Then we have*

$$\sup_{t \in [0,T]} y(t) \leq \left(a_2 + 2(1 + a_1 a_2)\left(1 + \frac{1}{a_0 - \|w\|_{1,T}}\right)\right) e^{\|w\|_{1,T}}\|z\|_{1,T}.$$

Proof. For the case where $w \equiv 0$ and $a_1 = 0$, see the proof of Lemma 3.4 in [5]. Here we prove the case that $w \not\equiv 0$ or $a_1 \neq 0$.

The mean value theorem says that there exists a t_0 in $[0, T]$ such that $y(t_0) \leq \|y\|_{1,T}$. For the sake of periodicity, we may assume $t_0 = 0$ without loss of generality. From the given inequality we derive

$$y(t) \leq y(0)\exp\left(-\int_0^t (a_0 - w(s))ds\right) + \int_0^t \exp\left(-\int_s^t (a_0 - w(r))dr\right)(|f(s)| + a_1 y(s))ds.$$

It is easy to see that

$$\int_s^t w(r)dr = \sum_{j=1}^{[t-s]} \int_{s+j-1}^{s+j} w(r)dr + \int_{s+[t-s]}^t w(r)dr \le ([t-s]+1)\|w\|_{1,T} \le (t-s+1)\|w\|_{1,T}$$

for $0 \le s \le t \le T$, where $[r] = \max\{m \mid m \text{ is an integer and } m \le r\}$.

Then we have

$$y(0)\exp\left(-\int_0^t (a_0 - w(s))ds\right) \le e^{\|w\|_{1,T}}\|y\|_{1,T} \le a_2 e^{\|w\|_{1,T}}\|z\|_{1,T}$$

and

$$\begin{aligned}
&\int_0^t \exp\left(-\int_s^t (a_0 - w(r))dr\right)(|f(s)| + a_1 y(s))ds \\
&\le e^{\|w\|_{1,T}}\left(\sum_{j=1}^{[t]} e^{-([t]-j)(a_0-\|w\|_{1,T})}\int_{j-1}^{j}(|z(s)| + a_1 y(s))ds + \int_{[t]}^t (|f(s)| + a_1 y(s))ds\right) \\
&\le e^{\|w\|_{1,T}}\left(\frac{1}{1-e^{-(a_0-\|w\|_{1,T})}} + 1\right)(1 + a_1 a_2)\|z\|_{1,T} \\
&\le (1 + a_1 a_2)\left(2 + \frac{1}{a_0 - \|w\|_{1,T}}\right)e^{\|w\|_{1,T}}\|z\|_{1,T},
\end{aligned}$$

whence the result follows. □

3.2. **Abstract result.** To prove Theorem 1, we make use of the nonmonotone perturbation theory in [10], which is applicable to the equations governed by a subdifferential operator with a nonmonotone perturbation. In the framework of [10], the subdifferential operator could be *time-dependent*, *nonlinear* and *multi-valued* and so could be the perturbation. In our case, however, it is only required that the subdifferential operator is independent of time, linear and single-valued. For the convenience, we here give a simplified version of the theory suitable to our case.

Let $\mathcal{H}$ be a separable real Hilbert space with the norm $|\cdot|_{\mathcal{H}}$, $\psi : \mathcal{H} \to [0,\infty]$ a proper lower semicontinuous convex functional and $\mathcal{A}$ an operator which is linear, self-adjoint and maximal monotone in $\mathcal{H}$. Suppose ψ and $\mathcal{A}$ satisfy the relation:

$$\overline{D(\psi)}^{\mathcal{H}} = \mathcal{H}, \quad D(\psi) = D(\mathcal{A}^{1/2}),$$

$$\psi(u) = \begin{cases} \frac{1}{2}|\mathcal{A}^{1/2}u|^2_{\mathcal{H}} & \text{if } u \in D(\psi), \\ \infty & \text{if } u \in \mathcal{H} \setminus D(\psi), \end{cases}$$

$$D(\partial\psi) = D(\mathcal{A}), \quad \partial\psi = \mathcal{A}.$$

Consider the following abstract periodic problem (AP) in $\mathcal{H}$.

$$\text{(AP)} \begin{cases} \dfrac{dv}{dt}(t) + \partial\psi(v(t)) + \mathcal{B}(v(t)) = \mathcal{F}(t) & \text{in } [0,T], \\ v(0) = v(T), \end{cases}$$

where $\mathcal{B} : D(\mathcal{B}) \to \mathcal{H}$ with $D(\partial\psi) \subset D(\mathcal{B})$ is a (single-valued) nonlinear operator and $\mathcal{F}$ an $\mathcal{H}$-valued function on $[0,T]$. We assume conditions (A.1)–(A.4) for ψ and $\mathcal{B}$ below.

(A.1) There exist constants k_0 and $q \in (1,\infty)$ such that $k_0|v|^q_{\mathcal{H}} \le \psi(v)$ for all $v \in D(\psi)$.

(A.2) For every $\lambda > 0$, the set $\{u \in \mathcal{H}|\ |v|_{\mathcal{H}} + \psi(v) \le \lambda\}$ is compact in $\mathcal{H}$.

(A.3) $\mathcal{B}$ is ψ-demiclosed, i.e., if v_n converges strongly to v in $C([0,T];\mathcal{H})$, $\partial\psi(v_n)$ converges weakly to $\partial\psi(v)$ in $L^2(0,T;\mathcal{H})$, and $\mathcal{B}(v_n)$ converges weakly to ξ in $L^2(0,T;\mathcal{H})$, then $\xi(t) = \mathcal{B}(v(t))$ a.e. $t \in (0,T)$.

(A.4) (i) $\psi(0) = 0$.

(ii) There exist $k \in [0,1)$ and a nondecreasing function $\ell : [0,\infty) \to [0,\infty)$ such that $|\mathcal{B}(v)|^2_{\mathcal{H}} \le k|\partial\psi(v)|^2_{\mathcal{H}} + \ell(|v|_{\mathcal{H}})(\psi(v)+1)^2$ for all $v \in D(\partial\psi)$.

(iii) There exists a positive number δ such that

$$(-\partial\psi(v) - \mathcal{B}(v), v)_{\mathcal{H}} + \delta\psi(v) \le 0 \quad \text{for all } v \in D(\partial\psi).$$

The following Proposition 1 is a direct conclusion of Theorem I in [10]:

Proposition 1. *Assume that conditions (A.1)–(A.4) hold. Then for every function $\mathcal{F}$ belonging to $L^2(0,T;\mathcal{H})$ there exists a solution v to (AP) such that*

$$(i)\quad v \in C([0,T];\mathcal{V}),$$
$$(ii)\quad \frac{dv}{dt},\ \mathcal{A}(v(\cdot)),\ \mathcal{B}(v(\cdot)) \in L^2(0,T;\mathcal{H}).$$

4. Proof of Theorem 1

4.1. **The case $N = 3$.** We begin by considering the following auxiliary problem:

$$\frac{dU}{dt}(t) + \partial\Phi(U(t)) + L(U(t)) = F(t) \quad \text{in } [0,T], \tag{10}$$

$$U(0) = U(T). \tag{11}$$

Lemma 8. *For all $f, g \in L^2(0,T;\boldsymbol{L}^2(\Omega))$ there exists a unique solution U to (10) and (11) such that*

$$(i)\quad U \in C([0,T];V),$$
$$(ii)\quad \frac{dU}{dt},\ \partial\Phi(U(\cdot)),\ L(U(\cdot)) \in L^2(0,T;H).$$

Proof. According to Theorem 1, for the existence we have only to see that the assumptions (A.1)–(A.4) are satisfied.

By the assumption on the physical constants, Lemma 1 and (ii) of Lemma 3, it follows that there exists a constant C_0 depending only on Ω and the physical constants such that $C_0|U|^2_H \le \Phi(U)$ holds for all $U \in V$. Therefore (A.1) is valid with $q = 2$. By virtue of the assumptions on Ω, (A.2) follows from Rellich's embedding theorem. (A.3) and (A.4)(i) is obvious. An easy calculation shows that

$$|L(U)|^2_H \le C_1\Phi(U) \quad \text{for all } U \in D(\partial\Phi),$$

where C_1 depends only on Ω and the physical constants. Hence we can take $k = 0$ and $\ell \equiv C_1$ in (A.4)(ii). We observe that, by Lemma 3, for all $U \in D(\partial\Phi)$

$$(L(U),U)_H = 4\chi\|\omega\|^2 - 4\chi(\operatorname{curl} u, \omega) \geq 4\chi\|\omega\|^2 - 4\chi\left(\|\omega\|^2 + \frac{1}{4}\|\nabla u\|^2\right) = -\chi\|\nabla u\|^2.$$

The above inequality together with the fact that $(\partial\Phi(U), U) = 2\Phi(U)$ yields

$$(\partial\Phi(U) + L(U), U)_H \geq \delta_0 \Phi(U),$$

where $\delta_0 := 2\mu/(\mu+\chi)$. Therefore (A.4)(iii) is valid with $\delta = \delta_0$.

To prove the uniqueness, let U_1 and U_2 be two solutions to (10) and (11). Then $\widetilde{U} = U_1 - U_2$ satisfies

$$\frac{d\widetilde{U}}{dt}(t) + \partial\Phi(\widetilde{U}(t)) + L(\widetilde{U}(t)) = 0 \quad \text{in } [0,T],$$
$$\widetilde{U}(0) = \widetilde{U}(T).$$

Multiplying the above equation by $\widetilde{U}$ and integrating over $[0,T]$, we obtain

$$0 = \int_0^T (\partial\Phi(\widetilde{U}(t)) + L(\widetilde{U}(t)), \widetilde{U}(t))_H dt \geq \delta_0 \int_0^T \Phi(\widetilde{U}(t)) dt \geq \delta_0 C_0 \int_0^T |\widetilde{U}(t)|_H^2 dt,$$

whence follows that $\widetilde{U} \equiv 0$ on $[0,T]$. This completes the proof. □

For any positive number R, define a bounded closed convex subset K_R of $L^2(0,T;H)$ by

$$K_R = \{G \in L^2(0,T;H) \mid \|G\|_{H,2,T}^2 \leq R^2\}.$$

Let an arbitrary $F \in K_R$ be fixed. For each $G \in L^2(0,T;H)$ we denote by U_G the unique solution of (10) with F replaced by $F - G$ and (11). Hence we can define an operator $\mathcal{S}$ of $L^2(0,T;H)$ into itself by

$$\mathcal{S} : L^2(0,T;H) \ni G \mapsto B(U_G) \in L^2(0,T;H).$$

We can show that the operator $\mathcal{S}$ is continuous as a mapping from $\mathfrak{H}_W$ into itself, where $\mathfrak{H}_W$ denotes $L^2(0,T;H)$ endowed with the weak topology. Moreover, if R is sufficiently small, $\mathcal{S}$ maps K_R into itself. Since K_R is a nonempty compact convex subset of $\mathfrak{H}_W$, Tychonoff's fixed point theorem says that there exists a fixed point $\bar{G}$ of $\mathcal{S}$ in K_R such that $\bar{G} = B(U_{\bar{G}})$. Then $U_{\bar{G}}$ turns out to be a solution to (8) and (9).

To show that the assertions on $\mathcal{S}$ are true, we need the following a priori estimates.

Lemma 9 (a priori estimates). *There exist positive constants M_j ($j = 1,2,3,4$) depending only on Ω and the physical constants such that if U is a solution of (10) and (11) then*

(12) $$\sup_{t\in[0,T]} \|U(t)\|_H^2 \leq M_1 \|F\|_{H,2,T}^2,$$

(13) $$\|\Phi(U(\cdot))\|_{1,T} \leq M_2 \|F\|_{H,2,T}^2,$$

(14) $$\sup_{t\in[0,T]} \Phi(U(t)) \leq M_3 \|F\|_{H,2,T}^2,$$

(15) $$\|\partial\Phi(U(\cdot))\|_{H,2,T}^2 \leq M_4 \|F\|_{H,2,T}^2.$$

Proof. Multiplying (10) by $U(t)$ and integrating over $[0,T]$, we have

$$\frac{d}{dt}|U(t)|_H^2 + \delta_0\Phi(U(t)) \le \frac{1}{\delta_0 C_0}|F(t)|_H^2. \tag{16}$$

Hence (12) follows from the fact that $C_0|U|_H^2 \le \Phi(U)$ and Lemma 7. Then integrating (16) over $[t-1,t]$, we obtain (13).

Multiplying (10) by $\partial\Phi(U(t))$ and integrating over $[0,T]$, we have

$$\frac{d}{dt}\Phi(U(t)) + \frac{1}{2}|\partial\Phi(U(t))|_H^2 \le |F(t)|_H^2 + C_1\Phi(U(t)), \tag{17}$$

where we use the well-known formula $d\Phi(U)/dt = (\partial\Phi(U),U)_H$ (see Lemme 3.3 in [2]). Since $2\Phi(U) = (\partial\Phi(U),U)_H$ and $C_0|U|_H^2 \le \Phi(U)$, it easily follows that $4C_0\Phi(U) \le |\partial\Phi(U)|_H^2$. Then we have

$$\frac{d}{dt}\Phi(U(t)) + 2C_0\Phi(U(t)) \le |F(t)|_H^2 + C_1\Phi(U(t)).$$

(14) follows from (13) and Lemma 7. Integration of (17) over $[t-1,t]$ leads to (15). □

By Lemma 5, it follows that there exists a constant C_2 depending only on Ω and the physical constants such that

$$|B(U)|_H^2 \le C_2\Phi(U)^{3/2}|\partial\Phi(U)|_H \quad \text{for all } U \in D(\partial\Phi). \tag{18}$$

Since $F, G \in K_R$, (18) and Lemma 9 imply that

$$\begin{aligned}\|\mathcal{S}(G)\|_{H,2,T}^2 = \|B(U_G)\|_{H,2,T}^2 &\le C_2 \sup_{t\in[0,T]} \Phi(U(t))^{3/2}\|\partial\Phi(U_G(\cdot))\|_{H,1,T} \\ &\le C_2 M_3^{3/2} M_4^{1/2}\|F-G\|_{H,2,T}^4 \\ &\le 16 M_0 M_3^{3/2} M_4^{1/2} R^4.\end{aligned}$$

Let $\rho_0 := (16M_0M_3^{3/2}M_4^{1/2})^{-1/2}$. It is clear that ρ_0 depends only on Ω and the physical constants and $\mathcal{S}$ maps K_{ρ_0} into itself.

Since $L^2(0,T;H)$ is separable, K_{ρ_0} is metrizable in $\mathfrak{H}_W$. Therefore it suffices to show the sequential continuity of $\mathcal{S}$ in $\mathfrak{H}_W$. To this end, let (G_n) be a sequence in K_{ρ_0} converging weakly to some $G \in K_{\rho_0}$. For the sake of brevity, let $U_n = U_{G_n}$ and $U = U_G$. By Lemma 9, (U_n), $(\Phi(U_n))$ and $(\partial\Phi(U_n))$ remain in a bounded subset of $C([0,T];H)$, $C([0,T])$ and $L^2(0,T;H)$ respectively. Hence it follows that $(L(U_n))$, $(B(U_n))$ and (dU_n/dt) are also bounded in $L^2(0,T;H)$. Then it follows that (U_n) forms an equicontinuous family in $C([0,T];H)$. Besides the boundedness of $(\Phi(U_n))$ implies that $(U_n(t))$ lies in a relatively compact subset of H for each fixed $t \in [0,T]$. Therefore, by Ascoli's theorem we can exact a subsequence (U_{n_k}) converging strongly to some $U^* \in C([0,T];H)$. Without loss of generality, we may assume that

$$\begin{aligned}&\frac{dU_{n_k}}{dt} \rightharpoonup \frac{dU^*}{dt} && \text{weakly in } L^2(0,T;H),\\ &\partial\Phi(U_{n_k}) \rightharpoonup \partial\Phi(U^*) && \text{weakly in } L^2(0,T;H),\\ &L(U_{n_k}) \rightharpoonup L(U^*) && \text{weakly in } L^2(0,T;H),\\ &B(U_{n_k}) \rightharpoonup B^* && \text{weakly in } L^2(0,T;H),\end{aligned}$$

where we use the demiclosedness of d/dt, $\partial\Phi$ and L.

By much the same argument in the proof of Theorem II in [5], it follows that B is also Φ-demiclosed. Therefore $B^* = B(U^*)$. In view of (10), U^* must equal the unique solution U. Then we have $B(U_{n_k}) \rightharpoonup B(U)$.

Since the above argument is independent of the choice of subsequences, the original sequence $(B(U_n))$ converges to $B(U)$ weakly in $L^2(0,T;H)$. □

4.2. **The case** $N = 2$. The result follows straightforward from Proposition 1. To see this, let $\widetilde{B}(U) := L(U) + B(U)$. It is easy to see that $\widetilde{B}$ satisfies assumptions (A.1)—(A.4). Here we only show (A.4)(ii) and (iii) are satisfied. By Lemmas 4, 5 and 6 it follows that

$$|\widetilde{B}(U)|_H^2 \leq \frac{1}{2}|\partial\Phi(U)|_H^2 + C(|U|_H^2 + 1)(\Phi(U)+1)^2,$$

where C is a constant depending only on Ω and the physical constants. This assures (A.4)(ii). By virtue of Lemma 4, a simple calculation gives $(B(U),U)_H = 0$. By much the same argument in the case of $N = 3$, it follows that $(\partial\Phi(U) + L(U), U) \geq \delta_0'\Phi(U)$. Therefore (A.4)(iii) holds for $\widetilde{B}$ with $\delta = \delta_0'$. □

5. Proof of Thereom 2

5.1. **The case** $N = 3$. Let $\rho = \|F\|_{H,2,T}$. If $\rho \leq \rho_1$, we can construct a periodic solution U satisfying $\sup_{t\in[0,T]}\Phi(U(t)) \leq 2M_3\rho^2$ as in the proof of Theorem 1. Take $\widehat{U}$ as in the assumption of Theorem 2. Then $\widetilde{U} = \widehat{U} - U$ satisfies

$$\frac{1}{2}\frac{d}{dt}|\widetilde{U}(t)|_H^2 + \delta_0\Phi(\widetilde{U}(t)) = -(B(\widehat{U}(t)) - B(U(t)), \widetilde{U}(t))_H. \tag{19}$$

From Lemma 4, we find that

$$(B(\widehat{U}(t)) - B(U(t)), \widetilde{U}(t))_H$$
$$= ((\tilde{u}\cdot\text{grad})u, \tilde{u}) + ((\tilde{u}\cdot\text{grad})\omega, \tilde{\omega}) + ((\tilde{u}\cdot\text{grad})b, \tilde{b}) - ((\tilde{b}\cdot\text{grad})u, \tilde{b}) - ((\tilde{b}\cdot\text{grad})b, \tilde{u}),$$

where $\widetilde{U} = (\tilde{u}, \tilde{\omega}, \tilde{b})$. By Lemma 6, we get

$$|(B(\widehat{U}(t)) - B(U(t)), \widetilde{U}(t))_H| \leq C_3\Phi(U(t))^{1/2}\Phi(\widetilde{U}(t)),$$

where C_3 is a constant depending only on Ω and the physical constants. Take $\rho_2 > 0$ sufficiently small so that $\rho_2 < \min\{\rho_1, \delta_0 C_3^{-1}(2M_3)^{-1/2}\}$ and $\rho_3 = C_0(\delta_0 - C_3(2M_3)^{1/2}\rho_2) > 0$. Then we obtain by (19)

$$|\widetilde{U}(t)|_H \leq e^{-\rho_3 t}|\widetilde{U}(0)|_H \quad \text{for all } t \in [0,T]. \tag{20}$$

The uniqueness of U follows from (20) at once. □

5.2. **The case** $N = 2$. By much the same argument as in the proof for the case $N = 3$, we find that $\widetilde{U} = \widehat{U} - U$ satisfies the following inequality.

$$\frac{d}{dt}|\widetilde{U}(t)|_H^2 + 2(\delta_0' - C_3'\Phi(U(t))^{1/2})\Phi(\widetilde{U}(t)) \leq 0. \tag{21}$$

We show that if $\|F\|_{H,2,T}$ is sufficiently small, then $\sup_{t\in[0,T]}\Phi(U(t))$ is small. To this end, we need some a priori estimates for solutions to (8) and (9). We can easily derive

$$\sup_{t\in[0,T]}|U(t)|_H^2 \leq M_1'\|F\|_{H,2,T}^2 \text{ and } \|\Phi(U(\cdot))\|_{1,T} \leq M_2'\|F\|_{H,2,T}^2$$

in the analogous way to the proof of (12) and (13).

On the other hand, by multiplying (8) by $\partial\Phi(U(t))$ and Lemma 5, we get

$$\frac{d}{dt}\Phi(U(t)) + C_0'\Phi(U(t)) \le |F(t)|_H^2 + \left(C_1' + \frac{27C_2'^2}{16}|U(t)|_H^2\Phi(U(t))\right)\Phi(U(t)), \tag{22}$$

where we use

$$|B(U)|_H|\partial\Phi(U)|_H \le C_2'^{1/2}|U|_H^{1/2}\Phi(U)^{1/2}|\partial\Phi(U)|_H^{3/2} \le \frac{1}{4}|\partial\Phi(U)|_H^2 + \frac{27C_2'^2}{16}|U|_H^2\Phi(U)^2.$$

Noting that

$$\|\,|U(\cdot)|_H^2\Phi(U(\cdot))\,\|_{1,T} \le M_1'M_2'\|F\|_{H,2,T}^4,$$

we can apply Lemma 7 provided that $\|F\|_{H,2,T}$ is small enough. Thus we find that $\sup_{t\in[0,T]}\Phi(U(t)) \le \ell_*(\|F\|_{H,2,T})$, where ℓ_* is a nonnegative increasing function satisfying $\ell_*(r) \to +0$ as $r \to +0$. Therefore there exists a positive number ρ_2 such that $\rho_3 := \delta_0' - C_3'\ell_*(\rho_2)^{1/2} > 0$. It is now easy to show the uniqueness and stability of U, so we omit the details. □

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