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Multiple Solutions of the Dirichlet Problem for H-systems

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1 Introduction.

Let $\mathsf{D} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ be the unit disk in $\mathbb{R}^2$ and $\gamma \in H^{1/2}(\partial\mathsf{D}, \mathbb{R}^3) \cap L^\infty(\partial\mathsf{D})$ a non-constant mapping. For $H > 0$ and $u \in H^1(\mathsf{D}, \mathbb{R}^3)$, we consider the following equations known as H-systems:

$$\begin{cases} \Delta u = 2H u_{x_1} \wedge u_{x_2} & \text{in } \mathsf{D} \\ u = \gamma & \text{on } \partial\mathsf{D}, \end{cases} \tag{1}$$

where $\wedge$ is the exterior product in $\mathbb{R}^3$ and subscripts denote partial derivatives. It arises when we seek surfaces in $\mathbb{R}^3$ with mean curvature H bounded by $\gamma(\partial\mathsf{D})$: If a solution u of (1) is conformal, i.e., $|u_{x_1}|^2 - |u_{x_2}|^2 = u_{x_1} \cdot u_{x_2} = 0$, $u(\mathsf{D})$ represents a surface with mean curvature H at all points $x \in \mathsf{D}$ where the rank of $du(x)$ is 2.

(1) is the Euler-Lagrange equation of the functional $\mathcal{E}_H$ in $H^1_\gamma(\mathsf{D}, \mathbb{R}^3) = \{u \in H^1(\mathsf{D}, \mathbb{R}^3) : u = \gamma \text{ on } \partial\mathsf{D}\}$:

$$\mathcal{E}_H(u) = \int_{\mathsf{D}} |\nabla u|^2 \, dx + \frac{4}{3} H \int_{\mathsf{D}} u \cdot u_{x_1} \wedge u_{x_2} \, dx.$$

Let us assume that $\gamma(\partial\mathsf{D})$ is contained in the closed ball of radius R with center at the origin. Under the condition $HR \leq 1$, Hildebrandt [6] proved the existence of a solution to (1). His solution, denoted by $\underline{u}_H$, is characterized as a solution of the minimization problem: $\mathcal{E}_H(\underline{u}_H) = \inf_{u \in S_R} \mathcal{E}_H(u)$, where $S_R = \{u \in H^1_\gamma(\mathsf{D}, \mathbb{R}^3) : \|u\|_\infty \leq R\}$. Thus $\underline{u}_H$ is a relative minimum of $\mathcal{E}_H$ with respect to $H^1 \cap L^\infty$-topology. In fact, it is proved in [3] (see also [13]) that relative minimum of $\mathcal{E}_H$ with respect to $H^1 \cap L^\infty$-topology is unique. The second solution to (1) is obtained independently by Brezis-Coron [3] and Struwe [11], [12] under the assumption $HR < 1$. Their large solution (here, we generally call non-minimal critical points of $\mathcal{E}_H$ as large solutions) is obtained as a mountain pass type critical point of $\mathcal{E}_H$ and it is written as the form $\overline{u}_H = \underline{u}_H + \frac{J_H(v_H)}{2H} v_H$, where $J_H(v) = \int_{\mathsf{D}} |\nabla v|^2\, dx + 4H \int_{\mathsf{D}} \underline{u}_H \cdot v_{x_1} \wedge v_{x_2}\, dx$ and v_H is a solution to the minimization problem:

$$\inf\{J_H(v) : v \in H^1_0(\mathsf{D}, \mathbb{R}^3), \quad Q(v) = -1\},$$

where $Q(v) = \int_{\mathsf{D}} v \cdot v_{x_1} \wedge v_{x_2}\, dx$.

Contrary to the small solution, large solution is not necessary unique. The following example is due to H. Wente (see the book of Struwe [13]). Let $0 < H < 1$ and consider $\gamma(x) = (x_1, 0, 0)$. It is shown in [13] that in this case, there are infinitely many large solutions to (1).

Let us consider other example. Let us assume $0 < H < 1$ and $\gamma(x) = (x_1, x_2, 0)$. In this case, by a geometric meaning of the equations (1), it is generally believed that there is exactly one large solutions. However, at least to the author's knowledge, there is no proof of it, c.f., [2, p169] . The lesson to be learned from these examples is the following one: There are more than one large solutions in general and non-uniqueness may depend on boundary data. Thus we are naturally led to the following problem posed by some authors (see [2, Remark 11] and [13, p126, Example 3.7]).

Problem. Determine under what conditions on γ, there exist more than two solutions to (1).

Note that for a critical point v of J_H in $\mathcal{M} = \{v \in H^1_0(\mathsf{D}, \mathbb{R}^3) : Q(v) = -1\}$, $\underline{u}_H + \frac{J_H(v)}{2H} v$ is a solution to (1). Thus the above problem is equivalent to finding a suitable condition of γ such that J_H admits at least two distinct critical points in $\mathcal{M}$.

In this note, we report results obtained in [10] concerning the above problem. Before stating the results, we introduce some notations. For $\gamma \in H^{1/2}(\partial\mathsf{D}, \mathbb{R}^3)$, h_γ denotes the harmonic extension of γ in D: $\Delta h_\gamma = 0$ in D and $h_\gamma = \gamma$ on $\partial\mathsf{D}$. $e_1 = {}^t(1,0,0)$, $e_2 = {}^t(0,1,0)$ and $e_3 = {}^t(0,0,1)$ denote the standard orthonormal basis of $\mathbb{R}^3$. $SO(3)$ is the special orthogonal group of $\mathbb{R}^3$: $SO(3) = \{R \in GL(\mathbb{R}^3) : R^tR = 1, \ \det R = 1\}$.

Our first result is the following:

Theorem 1.1 *Let $\gamma \in H^{1/2}(\mathsf{D}, \mathbb{R}^3) \cap L^\infty(\partial\mathsf{D})$. Assume that γ satisfies the following condition:*

(C-1) *h_γ is regular at some point in D (that is, the rank of dh_γ is 2 at some $a \in \mathsf{D}$, or equivalently, $(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2} \not\equiv 0$ in D) and $|\nabla h_\gamma|^2 - 2|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}|$ is not identically equal to 0 in D.*

Then there exists $H_0 > 0$ such that one of the following (A-1) and (A-2) holds for $0 < H \le H_0$:

(A-1) *There exists a non-minimal critical point v_H of J_H in $\mathcal{M}$, that is, v_H is a critical point of J_H in $\mathcal{M}$ satisfying $J_H(v_H) > \inf_{v \in \mathcal{M}} J_H(v)$.*

(A-2) *There exists infinitely many minimizers of J_H in $\mathcal{M}$.*

In particular, under the assumption (C-1), for $0 < H \le H_0$, (1) admits at least three distinct solutions.

The next theorem gives another criterion for γ such that (1) admits at least three distinct solutions.

Theorem 1.2 *Let $\gamma \in H^{1/2}(\partial\mathsf{D}, \mathbb{R}^3) \cap L^\infty(\partial\mathsf{D})$. Assume that γ satisfies the following:*

(C-2) *There exist $a \in \mathsf{D}$ and $\delta > 0$ such that the set $\{R \in SO(3) : (h_\gamma)_{x_1}(a) \cdot Re_1 + (h_\gamma)_{x_2}(a) \cdot Re_2 > \delta\} \subset SO(3)$ is not contractible in $SO(3)$.*

Then there exists $H_0 > 0$ such that for $0 < H \leq H_0$, there exist at least three distinct solutions to (1).

One can show that (C-1) implies (C-2), see Example 1 in §3. Thus we can deduce the existence of the third solution under (C-1) from Theorem 1.2. However, the conclusion of Theorem 1.2 is weaker than that of Theorem 1.1.

Let us return to the boundary condition γ. For non-constant γ, there are only three possibilities:

(P-1) $(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2} \not\equiv 0$ and $|\nabla h_\gamma|^2 - 2|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}| \not\equiv 0$.

(P-2) $(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2} \not\equiv 0$ and $|\nabla h_\gamma|^2 - 2|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}| \equiv 0$.

(P-3) $(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2} \equiv 0$ and $|\nabla h_\gamma|^2 - 2|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}| \not\equiv 0$.

The case (P-1) is considered in Theorem 1.1 and in such a case, there are at least three distinct solutions for (1). The case $\gamma(x_1, x_2) = (x_1, x_2, 0)$ satisfies (P-2) and we think that in such a case one can not expect more than two solutions in general. We will see in Example 2 in §3.2 that (P-3) implies (C-2), and by Theorem 1.2, there are at least three solutions for such a case. From these observations, we guess that the condition (C-2) is the best one for γ such that (1) admits three solutions.

As for the case (P-3), we have in fact:

Theorem 1.3 *Assume $\gamma \in H^{1/2}(\partial \mathrm{D}, \mathbb{R}^3) \cap L^\infty(\partial \mathrm{D})$ satisfies (P-3). There exists $H_0 > 0$ such that for $0 < H \leq H_0$, (1) admits (uncountably)infinitely many distinct solutions.*

The following corollaries are easy consequences of Theorem 1.1, Theorem 1.2 and Theorem 1.3.

Corollary 1.1 *Let $\gamma \in H^{1/2}(\partial \mathrm{D}, \mathbb{R}^3) \cap L^\infty(\partial \mathrm{D})$. Assume that the function $|\nabla h_\gamma|^2 - 2|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}|$ is not identically equal to 0 in D. Then there exists $H_0 > 0$ such that for $0 < H \leq H_0$, (1) admits at least three distinct solutions.*

Corollary 1.2 *There exists an open dense subset $\mathfrak{U} \subset H^{1/2}(\partial \mathrm{D}, \mathbb{R}^3) \cap L^\infty(\partial \mathrm{D})$ such that for $\gamma \in \mathfrak{U}$, there exists $H_0 > 0$ such that for $0 < H \leq H_0$, (1) admits at least three distinct solutions.*

Thus for almost all $\gamma \in H^{1/2}(\partial \mathrm{D}, \mathbb{R}^3) \cap L^\infty(\partial \mathrm{D})$, (1) admits at least three distinct solutions for small $H > 0$.

Here we give some remarks about the function $|\nabla h_\gamma|^2 - 2|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}|$.

Remark 1.1 • *The function $|\nabla h_\gamma|^2 - 2|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}|$ is always non-negative (by the Cauchy-Schwartz inequality) and it is 0 at $x \in \mathrm{D}$ if and only if h_γ is conformal at x, that is, $|(h_\gamma)_{x_1}(x)|^2 - |(h_\gamma)_{x_2}(x)|^2 = (h_\gamma)_{x_1}(x) \cdot (h_\gamma)_{x_2}(x) = 0$. The last claim follows from an easily checked fact: For $a, b \in \mathbb{R}^3$, $|a|^2 + |b|^2 - 2|a \wedge b| \geq 0$ and equality holds if and only if $|a| = |b|$ and $a \cdot b = 0$.*

• *We have either $|\nabla h_\gamma|^2 - 2|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}| \equiv 0$ in D or the zeros of the function $|\nabla h_\gamma|^2 - 2|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}|$ are isolated. The proof is given in the next section.*

Our idea for the proofs of the above results are based on the invariance of the first equation of (1) under the natural action of $SO(3)$ (acting as: $SO(3) \times \mathbb{R}^3 \ni (R, u) \mapsto Ru \in \mathbb{R}^3$). In general, a Lie group action of a variational problem leads to conservation laws (Noether's theorem). For example, an action of $\mathbb{R}^2$ as translation $H^1(\mathrm{D}, \mathbb{R}^3) \times \mathbb{R}^2 \ni (u, a) \mapsto u(\cdot + a) \in H^1(\mathrm{D}, \mathbb{R}^3)$ leads to a conservation of "momentum" (more precisely, the conservation of "stress-energy momentum"), and from it a well-known Pohozaev identity follows (see the book of Hélein [5] for a derivation of Pohozaev identity from the conservation of stress-energy momentum tensor). From it, one can show that for a constant boundary date γ, (1) has a unique solution, the constant solution which is a result of Wente [16].

On the other hand, $SO(3)$-action leads to a conservation of "angular momentum", and our question is: What can we say about solutions of (1) from this conservation law. In other words, we study the role of $SO(3)$ for our equations (1). In fact, it turns out that the topological properties of $SO(3)$ play an important role to our problem. It is also important for the asymptotic analysis presented in [7], [8] and [9].

In the next section, we give outlines of the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3. For complete arguments, see [10].

2 Outlines of the Proofs of Main Theorems.

We first give functional analytic properties of Q defined in $H^1(\mathsf{D},\mathbb{R}^3)$. It is obvious that $Q(v) = \int_{\mathsf{D}} v \cdot v_{x_1} \wedge v_{x_2}\, dx$ is well-defined for $v \in H^1(\mathsf{D},\mathbb{R}^3) \cap L^\infty(\mathsf{D})$. However, the space $H^1(\mathsf{D},\mathbb{R}^3) \cap L^\infty(\mathsf{D})$ is not useful in order to develop a variational theory. We want to work in $H^1(\mathsf{D},\mathbb{R}^3)$ (or affine spaces modeled on $H^1_0(\mathsf{D},\mathbb{R}^3)$) directly. The following result, essentially due to H. Wente [15] (see also [3], [13], [1], [4] and [5] for recent developments), asserts that it has also a well-defined meaning for $v \in u + H^1_0(\mathsf{D},\mathbb{R}^3)$, where $u \in H^1(\mathsf{D},\mathbb{R}^3) \cap L^\infty(\mathsf{D})$ is arbitrary.

Lemma 2.1 *Let $u \in H^1(\mathsf{D},\mathbb{R}^3) \cap L^\infty(\mathsf{D})$ be given. The functional Q defined in $H^1(\mathsf{D},\mathbb{R}^3) \cap L^\infty(\mathsf{D})$ extends to an analytic functional on $u + H^1_0(\mathsf{D},\mathbb{R}^3)$. Q has the following expansion for $\varphi \in H^1_0(\mathsf{D},\mathbb{R}^3)$:*

$$Q(u+\varphi) = Q(u) + \langle dQ(u), \varphi\rangle + \frac{1}{2} d^2Q(u)\langle \varphi, \varphi\rangle + Q(\varphi).$$

Here

1. *$\langle dQ(v), \varphi\rangle = 3\int_{\mathsf{D}} \varphi \cdot v_{x_1} \wedge v_{x_2}\, dx$ for $\forall v \in u + H^1_0(\mathsf{D},\mathbb{R}^3)$ and $\forall \varphi \in H^1_0(\mathsf{D},\mathbb{R}^3) \cap L^\infty(\mathsf{D})$ and dQ extends continuously to a map $dQ : H^1(\mathsf{D},\mathbb{R}^3) \to H^{-1}(\mathsf{D},\mathbb{R}^3)$ which satisfies the estimate*

   $$|\langle dQ(v), \varphi\rangle| \leq C\|\nabla v\|^2_{L^2(\mathsf{D})} \|\nabla \varphi\|_{L^2(\mathsf{D})}$$

   for any $v \in H^1(\mathsf{D},\mathbb{R}^3)$ and any $\varphi \in H^1_0(\mathsf{D},\mathbb{R}^3)$,

2. *$d^2Q(u)(\varphi,\psi) = 3\int_{\mathsf{D}} u \cdot (\varphi_{x_1} \wedge \psi_{x_2} + \psi_{x_1} \wedge \varphi_{x_2})\, dx$ for $\varphi, \psi \in H^1_0(\mathsf{D},\mathbb{R}^3)$ and it extends continuously to a map $d^2Q : H^1(\mathsf{D},\mathbb{R}^3) \to S^2H^{-1}(\mathsf{D},\mathbb{R}^3)$ which satisfies the estimate*

   $$|d^2Q(u)(\varphi,\psi)| \leq C\|\nabla u\|_{L^2(\mathsf{D})} \|\nabla \varphi\|_{L^2(\mathsf{D})} \|\nabla \psi\|_{L^2(\mathsf{D})}$$

   for any $u \in H^1(\mathsf{D},\mathbb{R}^3)$ and $\varphi, \psi \in H^1_0(\mathsf{D},\mathbb{R}^3)$, where $S^2H^{-1}(\mathsf{D},\mathbb{R}^3)$ denotes the 2-fold symmetric product of $H^{-1}(\mathsf{D},\mathbb{R}^3)$.

Proof. 1. It is obvious that $v_{x_1} \wedge v_{x_2} \in L^1(\mathsf{D})$, but $\varphi \in H^1(\mathsf{D}, \mathbb{R}^3)$ is not included in $L^\infty(\mathsf{D})$ in general. However, one can show that (using a determinant structure of the nonlinearity) $v_{x_1} \wedge v_{x_2} \in \mathcal{H}^1$ ($\mathcal{H}^1$ is the Hardy space) and $\varphi \in H^1(\mathsf{D}, \mathbb{R}^3) \subset \mathrm{BMO}$ (by the Poincaré inequality. Here BMO is the space of functions with Bounded Mean Oscillation). From these and Fefferman-Stein's $\mathcal{H}^1$-BMO duality theorem, one can consider the integral $\int_{\mathsf{D}} \varphi \cdot v_{x_1} \wedge v_{x_2}\, dx$ as the duality pairing between $\mathcal{H}^1$ and BMO.

2. is proved similarly. □

Set $\mathcal{M} = \{v \in H_0^1(\mathsf{D}, \mathbb{R}^3) : Q(v) = -1\}$. Then by the above lemma, for $v \in \mathcal{M}$, we have $\langle dQ(v), v\rangle = 3Q(v) = -3 \neq 0$ and -1 is a regular value of $Q : H_0^1(\mathsf{D}, \mathbb{R}^3) \to \mathbb{R}$. Thus by the inverse function theorem, $\mathcal{M} \subset H_0^1(\mathsf{D}, \mathbb{R}^3)$ is a codimension 1 submanifold. Let $u \in \mathcal{M}$. From the inclusion $T_u\mathcal{M} \subset H_0^1(\mathsf{D}, \mathbb{R}^3)$, $T_u\mathcal{M}$ is equipped with a metric. Since $\mathcal{M} \subset H_0^1(\mathsf{D}, \mathbb{R}^3)$ is closed it is complete and is a Hilbert manifold.

Under these preparations, we first give an outline of the proof of Theorem 1.1.

2.1 Outline of the Proof of Theorem 1.1.

By the observation given in Remark 1.1, the function $|\nabla h_\gamma|^2 - 2|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}|$ vanishes at $x \in \mathsf{D}$ if and only if the holomorphic quadratic differential Φ_{h_γ} associated to h_γ is 0 at x. Here $\Psi_{h_\gamma} = |(h_\gamma)_{x_1}|^2 - |(h_\gamma)_{x_2}|^2 - 2i(h_\gamma)_{x_1} \cdot (h_\gamma)_{x_2}$. Since $\Delta h_\gamma = 0$ and $\Psi_{h_\gamma} = 4\frac{\partial h_\gamma}{\partial z} \cdot \frac{\partial h_\gamma}{\partial z}$, we have $\frac{\partial}{\partial \bar{z}}\Psi_{h_\gamma} = 0$ and Ψ_{h_γ} is holomorphic. The condition (C-1), the observation given in Remark 1.1 and the holomorphy of Ψ_{h_γ} imply that there exists $a \in \mathsf{D}$ such that $(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a) \neq 0$ and $|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)| \neq 0$.

The crucial step for the proof of Theorem 1.1 is the following result:

Lemma 2.2 *Assume (C-1) holds. There exist $H_0 > 0$ and $\delta_0 > 0$ such that for $0 < H \leq H_0$ and $0 < \delta \leq \delta_0$, we have $\pi_1(J_H^{S-\delta}) \neq 0$. Here $J_H^{S-\delta} = \{v \in \mathcal{M} : J_H(v) < S - \delta\}$ and $\pi_1(J_H^{S-\delta})$ is the fundamental group of $J_H^{S-\delta}$.*

Proof. We first construct $\Theta : J_H^{S-\delta} \to SO(3)$. For this purpose, we observe that if $v \in J_H^{S-\delta}$, there exists $C > 0$ (independent of v) such that

$$(1 - CH) \int_{\mathsf{D}} |\nabla v|^2 \, dx \le J_H(v) < S - \delta < S.$$

From this, we have

$$\int_{\mathsf{D}} |\nabla v|^2 \, dx < S(1 - CH)^{-1}. \tag{2}$$

We need the following lemma:

Lemma 2.3 *For any $\epsilon > 0$, there exists $\eta > 0$ such that the following holds:*
For any $v \in \mathcal{M}$ with $\int_{\mathsf{D}} |\nabla v|^2 \, dx < S + \eta$, there exist $R \in SO(3)$, $a \in \mathsf{D}$ and $\lambda > 0$ satisfying $\lambda / d(a, \partial \mathsf{D}) < \epsilon$ such that

$$\left\| \nabla \left(\frac{S}{2} v - RPU_{\lambda,a} \right) \right\|_{L^2(\mathsf{D})} < \epsilon.$$

Here $U_{\lambda,a}(x) = \frac{2\lambda}{\lambda^2 + |x-a|^2} \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ -\lambda \end{pmatrix}$, $PU_{\lambda,a} = U_{\lambda,a} - h_{\lambda,a}$, $\Delta h_{\lambda,a} = 0$ *and* $h_{\lambda,a}|_{\partial \mathsf{D}} = U_{\lambda,a}|_{\partial \mathsf{D}}$.

For the proof of this lemma, see [10].

By the above lemma and (2), for any $\epsilon > 0$, there exist $H_1 > 0$, $R \in SO(3)$, $a \in \mathsf{D}$ and $\lambda > 0$ with $\lambda / d(a, \partial \mathsf{D}) < \epsilon$ such that

$$\left\| \nabla \left(\frac{S}{2} v - RPU_{\lambda,a} \right) \right\| < \epsilon. \tag{3}$$

For $\epsilon > 0$, define

$$\begin{aligned} M(\epsilon) = \{ v \in H_0^1(\mathsf{D}, \mathbb{R}^3) : \exists R \in SO(3),\ \exists \lambda > 0 \text{ with} \\ \lambda / d(a, \partial \mathsf{D}) < \epsilon \text{ such that } \| \nabla (v - RPU_{\lambda,a}) \|_{L^2(\mathsf{D})} < \epsilon \}. \end{aligned}$$

It is proved in [7], [8] that there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \le \epsilon_0$ and $v \in M(\epsilon)$, the problem

$$\begin{aligned} \inf \{ \| \nabla (v - \mu RPU_{\lambda,a}) \|_{L^2(\mathsf{D})} : 1/2 < \mu < 2, \\ R \in SO(3),\ a \in \mathsf{D},\ \lambda > 0 \text{ with } \lambda / d(a, \partial \mathsf{D}) < 2\epsilon \} \end{aligned} \tag{4}$$

has a unique solution.

In (3), we take $\epsilon = \epsilon_0$. For $v \in J_H^{S-\delta}$ $(0 < H \leq H_1)$, consider the unique solution μ, R, a and λ to the problem (4) and define $\Theta(v) = R$. By the uniqueness of the solution, Θ is a continuous function.

In the following, we give a construction of an essential loop in $J_H^{S-\delta}$. Consider $F : SO(3) \to \mathbb{R}$ defined by

$$F(R) = -(h_\gamma)_{x_1}(a) \cdot Re_1 - (h_\gamma)_{x_2}(a) \cdot Re_2.$$

Here $a \in \mathsf{D}$ satisfies $(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a) \neq 0$ and $|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)| \neq 0$ (see the beginning of this section). For this choice of a, it can be shown that F is a Morse function in $SO(3)$, see [10] for details. (In fact, $(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a) \neq 0$ and $|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)| \neq 0$ are necessary and sufficient conditions for F to be a Morse function). The critical values of F are $-(|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}$ (Morse index 0), $-(|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}$ (Morse index 1), $(|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}$ (Morse index 2) and $(|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}$ (Morse index 3). From this and Morse theory, one can show that there exists a loop $R : \mathsf{S}^1 \to SO(3)$ which is not homotopically trivial such that

$$\begin{aligned}\sup_{R \in R(\mathsf{S}^1)} F(R) &= \inf\{\sup_{R \in \ell(\mathsf{S}^1)} F(R) : \ell : \mathsf{S}^1 \to SO(3),\ \ell \sim R(\cdot)\} \\ &= -(|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}. \qquad (5)\end{aligned}$$

R is obtained as a parametrization of the unstable manifold of the negative gradient flow of F associated with the critical point of F whose Morse index is 1.

Under these preparations, we define $\alpha_0 : \mathsf{S}^1 \to H_0^1(\mathsf{D}, \mathbb{R}^3)$ by

$$\alpha_0(\theta) = \frac{R(\theta) PU_{\lambda_a, a}}{|Q(R(\theta) PU_{\lambda_a, a})|^{1/3}},$$

where

$$\lambda_a = \frac{(|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)} H,$$

$$\begin{cases} \Delta h_a^i = 0 \quad \text{in D} \\ h_a^i(x) = \frac{2(x_i - a_i)}{|x-a|^2} \quad \text{on } \partial\text{D}. \end{cases}$$

We then have

Lemma 2.4 *There exists $\delta > 0$ such that $\alpha_0 : \mathrm{S}^1 \to J_H^{S-\delta}$ is not homotopically trivial.*

Proof. We have the following expansion (see [7] for the proof):

$$\begin{aligned} J_H(\alpha_0(\theta)) &= S + \frac{S}{2}\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda_a^2 \\ &- S((h_\gamma)_{x_1}(a) \cdot R(\theta)e_1 + (h_\gamma)_{x_2}(a) \cdot R(\theta)e_2)\lambda_a H + o(H^2). \end{aligned}$$

From this, (5) and the definition of λ_a, we have

$$\begin{aligned} &J_H(\alpha_0(\theta)) \\ &\leq S - \frac{S}{2}\frac{|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)} H^2 + o(H^2). \end{aligned}$$

By this, there exists $\delta > 0$ such that for small $H > 0$, $\alpha_0(\mathrm{S}^1) \subset J_H^{S-\delta}$.

We claim that α_0 is not homotopically trivial in $J_H^{S-\delta}$. Assume by contradiction that $\alpha_0 \sim 0$ in $J_H^{S-\delta}$. Then there exists a homotopy $H : \mathrm{S}^1 \times [0,1] \to J_H^{S-\delta}$ between α_0 and a constant loop: $H(\cdot, 0) = \alpha_0$, $H(\cdot, 1) = v_1 \in J_H^{S-\delta}$. Consider $\tilde{H} = \Theta \circ H : \mathrm{S}^1 \times [0,1] \to SO(3)$. Since $\Theta(\alpha_0) = R(\cdot)$, $\tilde{H}$ gives a homotopy between $R(\cdot)$ and a constant loop $\Theta(v_1)$ in $SO(3)$. This is a contradiction since $R(\cdot)$ is not homotopically trivial in $SO(3)$. □

By Lemma 2.4, we have completed the proof of Lemma 2.2. □

To proceed, we recall the following notion:

Definition 2.1 *Let M be a complete Finsler manifold, $J \in C^1(M)$, $\beta \in \mathbb{R}$. J satisfies $(PS)_\beta$-condition if any $\{v_n\} \subset M$ satisfying $J(v_n) \to \beta$ and $dJ(v_n) \to 0$ is relatively compact in M.*

We then have

Lemma 2.5 *For any $\beta < S$, J_H satisfies $(PS)_\beta$ in $\mathcal{M}$.*

For the proof, see [3], [10], [11], [13].

We now complete the proof of Theorem 1.1.

Completion of the proof of Theorem 1.1. We define

$$\beta = \inf\left\{ \sup_{v\in\alpha(\mathsf{S}^1)} J_H(v) : \alpha : \mathsf{S}^1 \to J_H^{S-\delta} \text{ is homotopic to } \alpha_0 \right\}.$$

By Lemma 2.4, we have $\beta < S - \delta$ and by Lemma 2.5, β is a critical value of J_H. There are two possibilities:

- $\beta > \beta_{min}$.
- $\beta = \beta_{min}$.

Here $\beta_{min} = \inf_{v\in\mathcal{M}} J_H(v)$.

It is obvious that the first case implies (A-1). We claim that the second case implies (A-2). The idea of the proof is as follows:

We assume that there are only finitely many minimizers $v_1, \ldots, v_p$ $(p \geq 1)$ in $\mathcal{M}$.

•1. Fix $\theta_0 \in \mathsf{S}^1$. By a compactness argument, it can be shown that there exists a sequence of loops $\{\alpha_n\}_{n\geq 1}$, $\alpha_n : \mathsf{S}^1 \to J_H^{S-\delta}$ such that $\alpha_n \sim \alpha_0$ and $\mathrm{dist}(\alpha_n(\theta_0), \{v_1, \ldots, v_p\}) \to 0$. Without loss of generality, we may assume that $\alpha_n(\theta_0) \to v_1$ as $n \to \infty$.

•2. It can be shown that for any $\kappa > 0$, there exists $N \in \mathbb{N}$ such that $\alpha_n(\mathsf{S}^1) \subset \mathsf{B}_\kappa(v_1) := \{v \in \mathcal{M} : \|\nabla(v - v_1)\|_{L^2(\mathsf{D})} < \kappa\}$ for $n \geq N$.

•3. For all small $\kappa > 0$, it can be shown that $\mathsf{B}_\kappa(v_1) \subset J_H^{S-\delta}$ and $\mathsf{B}_\kappa(v_1)$ is contractible in $J_H^{S-\delta}$.

•4. By 3, we have $\alpha_0 \sim \alpha_n \sim 0$. This is a contradiction. Thus the second case implies (A-2). □

2.2 Outline of the proof of Theorem 1.2.

Proof of Theorem 1.2. We argue by contradiction. So assume that there is exactly one critical point v_0 of J_H in $\mathcal{M}$ (v_0 is necessary a minimizer of

J_H in $\mathcal{M}$ and it is obtained by Brezis-Coron [3] and Struwe [11], [12]). We derive a contradiction from this.

We first prepare

Lemma 2.6 *There exists $H_0 > 0$ such that the following holds: For any $0 < H \leq H_0$ and $\epsilon > 0$ with $\beta_{min} < S - \epsilon$, $J_H^{S-\epsilon}$ is contractible in itself.*

Proof. We only give the outline of the proof.

•1. First, it can be shown that there exists $\kappa > 0$ such that $\mathsf{B}_\kappa(v_0) \subset J_H^{S-\epsilon}$ and $\mathsf{B}_\kappa(v_0)$ is contractible in $J_H^{S-\epsilon}$.

•2. By a compactness argument, one can show that there exists $\epsilon' > 0$ such that $J_H^{\beta_{min}+\epsilon'} \subset \mathsf{B}_\kappa(v_0)$.

•3. Then, since J_H satisfies $(PS)_\beta$ for any $\beta < S$ (Lemma 2.5), by Morse theory, $J_H^{\beta_{min}+\epsilon'}$ is a strong deformation retract of $J_H^{S-\epsilon}$.

From $1, 2, 3$, the conclusion follows. □

Completion of the proof of Theorem 1.2. Take $a \in \mathsf{D}$ and $\delta > 0$ satisfying the assumption of the theorem. Define $E = \{R \in SO(3) : (h_\gamma)_{x_1}(a) \cdot Re_1 + (h_\gamma)_{x_2}(a) \cdot Re_2 > \delta\}$.

Define $\Psi : E \to \mathcal{M}$ by the formula ($\epsilon > 0$ is determined later)

$$\Psi(R) = \frac{RPU_{\lambda(R),a}}{|Q(RPU_{\lambda(R),a})|^{1/3}},$$

here

$$\lambda(R) = \frac{(h_\gamma)_{x_1}(a) \cdot Re_1 + (h_\gamma)_{x_2}(a) \cdot Re_2}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)} H.$$

We then compute

$$
\begin{aligned}
&J_H(\Psi(R)) \\
&\leq S + \frac{S}{2}\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda(R)^2 \\
&\quad - S((h_\gamma)_{x_1}(a)\cdot Re_1 + (h_\gamma)_{x_2}(a)\cdot Re_2)\lambda(R)H + o(H^2) \\
&\leq S - \frac{S}{2}\frac{((h_\gamma)_{x_1}(a)\cdot Re_1 + (h_\gamma)_{x_2}(a)\cdot Re_2)^2}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)}H^2 + o(H^2) \\
&\leq S - \frac{S}{2}\frac{\delta^2}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)}H^2 + o(H^2).
\end{aligned}
$$

From this, for small $H > 0$ and $\epsilon > 0$, we have $\Psi(E) \subset J_H^{S-\epsilon}$.

In the next step, we consider the following composition of maps:

$$\Theta \circ \Psi : E \xrightarrow{\Psi} J_H^{S-\epsilon} \xrightarrow{\Theta} SO(3).$$

By our definition of Θ, $\Theta \circ \Psi(R) = R$ for any $R \in E$. On the other hand, Lemma 2.6 implies that $\Theta \circ \Psi \sim 0$. Thus E is contractible in $SO(3)$. This is a contradiction. Thus we complete the proof of Theorem 1.2. □

2.3 Outline of the Proof of Theorem 1.3.

We first show that, under the assumption (P-3), $h_\gamma(\mathrm{D})$ is contained in a one dimensional affine space in $\mathbb{R}^3$.

More precisely, we have

Lemma 2.7 *Assume γ satisfies (P-3). Then there exists a harmonic function $h : \mathrm{D} \to \mathbb{R}$ and $\boldsymbol{e}$, $\boldsymbol{f} \in \mathbb{R}^3$ such that $h_\gamma = h\boldsymbol{e}+\boldsymbol{f}$.*

This follows from the following lemma.

Lemma 2.8 *Let $G : \mathrm{D} \to \mathbb{C}^3$ be a holomorphic map with $G \wedge \bar{G} \equiv 0$. Here D is equipped with the standard complex structure and $\wedge : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$ is defined as the extension of $\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ by complex bilinearity. Then there exists a holomorphic function $g : \mathrm{D} \to \mathbb{C}$ and $\boldsymbol{e} \in \mathbb{R}^3$ such that $G = g\boldsymbol{e}$.*

For the proof of these lemmas, see [10].

Under these preparations, we now complete the proof of Theorem 1.3.

Completion of the proof of Theorem 1.3. Since the equation $\Delta u = 2Hu_{x_1} \wedge u_{x_2}$ is invariant under the natural action of the Euclidean motion $SO(3) \ltimes \mathbb{R}^3$, by Lemma 2.7, we may assume without loss of generality that $h_\gamma = {}^t(h, 0, 0)$ for some harmonic function h.

The proof consists in three steps:

•1. Since $\Delta h_\gamma = 0$ and $(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2} = 0$, h_γ is a solution to (1). By the maximum principle, $H\|h_\gamma\|_{L^\infty(\mathrm{D})} \leq H\|\gamma\|_{L^\infty(\partial\mathrm{D})} < 1$ (if $H > 0$ is small). From this and the characterization of the small solution by L^∞-norm, h_γ is equal to the small solution of Hildebrandt.

•2. By the result of Brezis-Coron [3] and Struwe [?], [?], there exists a large solution $\overline{u}_H$ of (1). We claim $\overline{u}_H(\mathrm{D}) \not\subset \{{}^t(x_1, 0, 0) : x_1 \in \mathbb{R}\}$. In fact, if $\overline{u}_H$ is contained in the x_1-axis, then $(\overline{u}_H)_{x_1} \wedge (\overline{u}_H)_{x_2} \equiv 0$ and $\Delta\overline{u}_H = 2(\overline{u}_H)_{x_1} \wedge (\overline{u}_H)_{x_2} = 0$. So we have $\overline{u}_H = h_\gamma$. This is a contradiction.

•3. By •2, there exists $a \in \mathrm{D}$ such that $\overline{u}_H(a)$ has a nonzero x_2 or x_3 component. For $\theta \in \mathrm{S}^1$, define $R_\theta \in SO(3)$ by $R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$.

Then $\{R_\theta \overline{u}_H\}_{\theta\in\mathrm{S}^1}$ are S^1-parametrized distinct solutions to (1). □

3 Examples.

3.1 Example 1.

Here we show (C-1) implies (C-2).

Assume $a \in \mathrm{D}$ satisfies $(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a) \neq 0$ and $|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)| \neq 0$. We take $\delta = \frac{1}{2}(|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}$. Since $(|\nabla h_\gamma(a)|^2 - 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}$ is a critical point of the function $SO(3) \ni R \mapsto (h_\gamma)_{x_1}(a) \cdot Re_1 + (h_\gamma)_{x_2}(a) \cdot Re_2 \in \mathbb{R}$ with Morse index 2, it can be shown by Morse theory that the set $E = \{R \in SO(3) :$

$(h_\gamma)_{x_1}(a) \cdot Re_1 + (h_\gamma)_{x_2}(a) \cdot Re_2 > \delta\}$ is not contractible. In fact, it is homotopy equivalent to a 1-cell of $SO(3)$ which generates the first homology group $H_1(SO(3);\mathbb{Z}) = \mathbb{Z}_2$.

3.2 Example 2.

Here we show (P-3) implies (C-2).

By the result of the previous section, we may assume without loss of generality that $h_\gamma = {}^t(h, 0, 0)$, where h is a harmonic function. Then $E = \{R \in SO(3) : h_{x_1}(a)R_{11} + h_{x_2}(a)R_{12} > \delta\}$. Here $R = (R_{ij})$. Let $a \in \mathrm{D}$ be such that $dh(a) \neq 0$. We claim that E is not contractible for small $\delta > 0$. The proof of the claim consists of three steps:

•1. Let $P_{SO}(\mathrm{S}^2) \to \mathrm{S}^2$ be the oriented orthonormal frame bundle of S^2. There is a natural identification $P_{SO}(\mathrm{S}^2) \cong SO(3)$: A point of $P_{SO}(\mathrm{S}^2)$ is specified by three mutually orthogonal unit vectors in $\mathbb{R}^3$. One corresponds to a base point of the fibration $P_{SO}(\mathrm{S}^2) \to \mathrm{S}^2$ and other two correspond to an oriented orthonormal basis at that point. Moreover, these vectors form an oriented orthonormal basis of $\mathbb{R}^3$. Since $SO(3)$ is naturally identified with the set of all oriented orthonormal bases of $\mathbb{R}^3$, it is identified with $P_{SO}(\mathrm{S}^2)$.

•2. For small $\delta > 0$, the set $U = \{{}^t(x_1, x_2, x_3) \in \mathrm{S}^2 : h_{x_1}(a)x_1 + h_{x_2}(a)x_2 > \delta\}$ is topologically a disk in S^2. Therefore it is contractible and $P_{SO}(\mathrm{S}^2)|_U \to U$ is isomorphic to the trivial bundle $U \times SO(2) \to U$. Thus E has the same homotopy type of $SO(2)$, the fiber over a point $p \in U$.

•3. From •1 and •2, E is homotopically equivalent to a subset of $SO(3)$ consisting of rotations about the axis $\langle p \rangle = \{tp : t \in \mathbb{R}\}$. The latter set is not contractible in $SO(3)$. In fact, it generates the first homology group of $SO(3)$, see [10] for details.

Thus we complete the proof of the implication (P-3) $\Rightarrow$ (C-2).

3.3 Example 3.

We consider the case $\gamma = {}^t(x_1, x_2, 0)$. We show in this case that $E = \{R \in SO(3) : (h_\gamma)_{x_1}(a) \cdot Re_1 + (h_\gamma)_{x_2}(a) \cdot Re_2 > \delta\}$ is empty or contractible for any $a \in \mathsf{D}$ and $\delta > 0$.

In this case, we observe that $F(R) = R_{11} + R_{22}$ and the critical values are -2 (with Morse index 0), 0 (corresponding critical points are degenerate) and 2 (with Morse index 3). From this, by Morse theory, E is empty (if $\delta > 2$) or contractible (if $0 < \delta \leq 2$).

This example also support our conjecture: For $\gamma = {}^t(x_1, x_2, 0)$, there are exactly two solutions to (1).

3.4 Example 4.

Here we give a condition of γ such that if γ satisfies it, then the conclusion of (A-1) in Theorem 1.1 holds.

Let γ be sufficiently smooth (for example, $\gamma \in C^{2,\alpha}(\partial\mathsf{D})$ for some $\alpha > 0$ is sufficient). We assume the set

$$\{a \in \mathsf{D} : K^+(a) = \max_{x \in \mathsf{D}} K^+(x)\}$$

consists of isolated points in D and for any $a \in \mathsf{D}$ with $K^+(a) = \max_{x \in \mathsf{D}} K^+(x)$, $(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a) \neq 0$. Here

$$K^+(x) = \frac{|\nabla h_\gamma(x)|^2 + 2|(h_\gamma)_{x_1}(x) \wedge (h_\gamma)_{x_2}(a)|}{\frac{\partial h_a^1}{\partial x_1}(x) + \frac{\partial h_a^2}{\partial x_2}(x)}.$$

Then we showed in [10] that (A-1) in Theorem 1.1 holds.

Since the above condition of γ is satisfied for generic γ, for generic boundary data γ, (1) admits at least three distinct solutions $\underline{u}_H$, $\overline{u}_H$ and u_H whose energies satisfy $\mathcal{E}_H(\underline{u}_H) < \mathcal{E}_H(\overline{u}_H) < \mathcal{E}_H(u_H)$.

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