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A blowup mechanism with high emergence - theory of self-organization

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1 Introduction

We have discovered quantized blowup mechanism in the mean field of many self-interacting particles, which is subject to the total mass conservation, decrease of free energy, and compensated compactness via the symmetrization of potential kernel. A typical model is the elliptic-parabolic system of cross-diffusion,

$$\left.\begin{array}{l} u_t = \nabla\cdot(\nabla u - u\nabla v) \\ 0 = \Delta v - av + u \end{array}\right\} \quad \text{in} \quad \Omega\times(0,T)$$
$$\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega\times(0,T)$$
$$u|_{t=0} = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{1}$$

where $\Omega\subset\mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $a>0$ is a constant, and ν is the unit outer normal vector on $\partial\Omega$. It is the simplified system of chemotaxis in mathematical biology, describing chemotctic feature of cellular slime molds, but is also the description of non-equilibrium mean field of self-attractive particles subject to the second law of thermodynamics in the theory of statistical mechanics.

Unique existence of the classical solution locally in time is proven, and the solution becomes positive and regular for $t>0$ if $u_0\not\equiv 0$. Supremum of the existence time is denoted by $T_{\max}\in(0,+\infty]$, and $T_{\max}<+\infty$ is referred to as the blowup in finite time. In this case, it holds that

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \tag{2}$$

as $t \uparrow T_{\max}$ in $\mathcal{M}(\overline{\Omega})$ with

$$m(x_0) = m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega), \end{cases} \tag{3}$$

where $\mathcal{M}(\overline{\Omega})$ denotes the set of measures on $\overline{\Omega}$, $\rightharpoonup$ the $*$-weak convergence there (Suzuki [17]). Actually, $\mathcal{S}$ in (2) is the blowup set of $u(\cdot,t)$, and $x_0 \in \mathcal{S}$ if and only if there are $x_k \to x_0$ and $t_k \uparrow T_{\max}$ in $x_k \in \overline{\Omega}$ satisfying $u(x_k,t_k) \to +\infty$. We have

$$\lim_{t\uparrow T_{\max}} \|u(t)\|_\infty = +\infty$$

and hence $\mathcal{S} \neq \emptyset$ holds in the case of $T_{\max} < +\infty$. Therefore, (2) with (3) implies the sharp estimate of the number of blowup points,

$$\sharp(\partial\Omega \cap \mathcal{S}) + 2 \cdot \sharp(\Omega \cap \mathcal{S}) \leq \|u_0\|_1 / (4\pi).$$

Equality $m(x_0) = m_*(x_0)$ in (2) is referred to as mass quantization of collapses. It has been suspected from the hierarchy of systems in statistical mechanics, that is, global existence of weak solution to the Fokker-Planck equation and mass and location quantization of the blowup family to the Liouville-Gel'fand equation, which describe kinetic and equilibrium states of the mean field, respectively. Actual proof is associated with the backward self-similar transformation, and the blowup point $x_0 \in \mathcal{S}$ is classified into two types. Namely, it is of type (I) if

$$\limsup_{t\uparrow T_{\max}} \sup_{x\in\Omega,\ |x-x_0|\leq CR(t)} R(t)^2 u(x,t) < +\infty$$

for any $C > 0$ and of type (II) for the other case that

$$\limsup_{t\uparrow T_{\max}} \sup_{x\in\Omega,\ |x-x_0|\leq CR(t)} R(t)^2 u(x,t) = +\infty$$

for some $C > 0$, where $R(t) = (T_{\max} - t)^{1/2}$. Important notion introduced there is the *parabolic envelope*, infinitely wide parabolic region as $b \uparrow +\infty$ of $\left\{(x,t) \in \overline{\Omega} \times [0, T_{\max} \mid |x - x_0| \leq bR(t)\right\}$. That is, the whole blowup mechanism is enveloped there and it holds that

$$\lim_{R\to+\infty} \lim_{t\uparrow T_{\max}} \int_\Omega \psi_{x_0,bR(t)}(x) u(x,t) dx = m(x_0),$$

where $\psi = \psi_{x_0,R}(x)$ is the cut-off function around x_0 with the support radius $2R > 0$ and $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$.

If $x_0 \in \mathcal{S}$ is of type (II) and $t_k \uparrow T_{\max}$ satisfies

$$\lim_{k\to\infty} \sup_{x\in\Omega,\ |x-x_0|\leq CR(t_k)} R(t_k)^2 u(x,t_k) = +\infty$$

for some $C > 0$, then it holds that

$$z(y,s_k)dy \rightharpoonup m_*(x_0)\delta_0(dy)$$

in $\mathcal{M}(\mathbf{R}^2)$ as $k \to \infty$. Here, $z(y,s) = R(t)^2 u(x,t)$ with zero extension taken where it is not defined, $y = (x - x_0)/R(t)$, and $s = s_k$ is defined from $t = t_k$ by $s = -\log(T_{\max} - t)$. Thus, type (II) blowup point is fixed at first. Then, it attracts the distribution mass of particles asymptotically radially symmetric and create a quantized collapse. Concetration, compared with aggregation, is strong, and the rescaled solution $z = z(y,s)$ develops delta singularity $m_*(x_0)\delta_0(dz)$, called the sub-collapse, at the origin. While actual existence of type (I) blowup point is open, if it exists then it takes a profile of emergence in the sense of Kaufmann as

$$\lim_{t\uparrow T_{\max}} \mathcal{F}_{x_0,bR(t)}(u(t)) = +\infty$$

holds for any $b > 0$, where $\mathcal{F}_{x_0,R}(u)$ denotes the local free energy defined by

$$\begin{aligned}\mathcal{F}_{x_0,R}(u) = &\int_\Omega \psi_{x_0,R}(x) u (\log u - 1)(x)dx \\ &-\frac{1}{2}\int\int_{\Omega\times\Omega} \psi_{x_0,R}(x)\psi_{x_0,R}(x')G(x,x')u\otimes u(x,x')dxdx'.\end{aligned}$$

Here, $G = G(x,x')$ denotes the Green's function for $-\Delta + a$ under Neumann boundary condition. In this connection, it should be noted that the global free energy $\mathcal{F}(u(t))$ always decreases as requirement of the second law of thermodynamics, where

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1)(x)dx - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u\otimes u(x,x')dxdx'.$$

On the other hand, formation of collapses around this type of blowup point may be non-radially symmetric or decay slowly at infinite in the rescaled variable, as the rescaled system is formally provided with the Lyapunov function. Actually, it is formal because the convergence of that Lyapunov function needs fast decay at infinity of the rescaled solution. If this condition is assured, then the classical solution must be stationary in the rescaled system, while radially symmetric sationary solution to this system does not exist. In this way, we can conclude that around type (I) blowup point, if it exists, mass and free energy are exchanged at the wedge of the parabolic envelope, with a "self" of quantized mass creating.

2 Equilibria

The quantized blowup mechanism of non-stationary state described above comes from that of the stationary state, and this story is called the nonlinear quantum mechanics. In fact, the equilibrium state of (1) is realized as the nonlinear eigenvalue problem

$$-\Delta v + av = \frac{\lambda e^v}{\int_\Omega e^v dx} \quad \text{in} \quad \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \tag{4}$$

with $\lambda = \|u_0\|_1$, and the quantized blowup mechanism at this level arises in the blowup family of solutions. Namely, if $\{(\lambda_k, v_k)\}_{k=1}^{\infty}$ is a family of solutions to (4) for $\lambda = \lambda_k$ and $v = v_k$, satisfying $\lambda_k \to \lambda_0 \in [0, \infty)$ and $\|v_k\|_\infty \to +\infty$, then the blowup set of $\{v_k\}$, denoted by $\mathcal{S} \subset \overline{\Omega}$ is finite, and passing through a subsequence, it holds that

$$u_k(x)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m_*(x_0)\delta_{x_0}(dx)$$

in $\mathcal{M}(\overline{\Omega})$ as $k \to \infty$, where

$$u_k = \frac{\lambda_k e^{v_k}}{\int_\Omega e^{v_k} dx}.$$

In particular, $\lambda_0 \in 4\pi\mathcal{N}$, and furthermore, we have

$$\nabla_x \left(m_*(x_0)K(x, x_0) + \sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m_*(x_0')G(x, x_0') \right)\Bigg|_{x=x_0} = 0 \tag{5}$$

for each $x_0 \in \mathcal{S}$, where only tangential derivative is taken in (5) if $x_0 \in \partial\Omega$, and

$$K(x, x') = G(x, x') + \begin{cases} \frac{1}{2\pi}\log|x - x'| & (x \in \Omega) \\ \frac{1}{\pi}\log|x - x'| & (x \in \partial\Omega) \end{cases}$$

represents the regular part of the Green's function $G(x,x')$.

This kind of quantization was first observed by Nagasaki and Suzuki [11], [12] for the Gel'fand problem

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v dx} \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega, \tag{6}$$

and Baraket and Pacard [1] proved the converse, that is, singular perturbation. Calculation of topological degree was initiated by Y.Y. Li [8] based on those facts, and it was completed by C.C. Chen and C.S. Lin. On the other hand, Suzuki [10], [15] established uniqueness of the solution to (6) for simply connected Ω and $\lambda \in (0, 8\pi)$. There, Morse index of the stationary solution v is shown to be equal to the number of eigenvalues in $\mu < 1$ minus one of the eigenvalue problem

$$\begin{aligned} &-\Delta\phi = \mu u \phi \qquad \text{in} \qquad \Omega \\ &\phi = \text{constant} \qquad \text{on} \qquad \partial\Omega \\ &\int_{\partial\Omega} \frac{\partial\phi}{\partial\nu} ds = 0, \end{aligned} \tag{7}$$

where

$$u = \frac{\lambda e^v}{\int_\Omega e^v dx}. \tag{8}$$

This Morse index is induced from the variational structure of (6), associated with the functional

$$\mathcal{J}(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v dx\right) + c \tag{9}$$

defined for $v \in H_0^1(\Omega)$, where c is a constant.

Independently, Wolansky [21] showed the same fact for $u = u(x) > 0$ satisfying

$$(-\Delta_D)^{-1} u = \log u + \text{constant} \quad \text{in} \quad \Omega, \qquad \|u\|_1 = 1. \tag{10}$$

Actually, he introduced this problem as the stationary state for the similar system to (1),

$$\left.\begin{aligned} &u_t = \nabla\cdot(\nabla u - u\nabla v) \\ &0 = \Delta v + u \end{aligned}\right\} \quad \text{in} \quad \Omega\times(0,T)$$

$$\left.\begin{aligned} &\tfrac{\partial u}{\partial\nu} - u\tfrac{\partial v}{\partial\nu} = 0 \\ &v = 0 \end{aligned}\right\} \quad \text{on} \quad \partial\Omega\times(0,T). \tag{11}$$

It is subject to the decrease of free energy defined by

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1)(x)dx - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u(x)u(x')dxdx',$$

where $G = G(x,x')$ denotes the Green's function for $-\Delta$ under Dirichlet boundary condition. Problem (10) is nothing but the Euler equation for the variational problem $\delta\mathcal{F}(u) = 0$ under the constraint that $\|u\|_1 = \lambda$, and the Morse index of its solution is defined by the maximal dimension of linear sub-spaces where the associated quadratic form is negative. Thus, what he showed is that this index is equal to the number of eigenvalues in $\mu < 1$ minus 1 for (7). On the other hand, those problems (7) are equivalent through (8) and

$$v = (-\Delta_D)^{-1} u, \tag{12}$$

and in this way, those two variational structures concerning v and u are equivalent up to Morse indices. This is very important for our point of view, because the structure of elliptic problem (6) is known in details and Morse index is easier to calculate, while the dynamics of (11) is subject to the decrease of free energy and local dynamics around the stationary solution is controled by its Morse index. Actually, this observation, combined with the global bifurcation diagram of the equilibrium state led us to the conjecture of mass quantization of collapses of the non-stationary solution described above, and particularly, the relation between dynamical and linearized stabilities is important ([16]).

However, dynamical equivalence of those two variations is a consequence of the general theory, *dual variation.*

3 Variation

Above dynamical equivalence of those variational structures can be proven directly. For this purpose, Lyapunov function for the full system takes a role, and we describe that situation for

$$\left.\begin{array}{l} u_t = \nabla\cdot(\nabla u - u\nabla v) \\ \tau v_t = \Delta v + u \end{array}\right\} \quad \text{in} \quad \Omega\times(0,T)$$
$$\left.\begin{array}{l} \frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu} = 0 \\ v = 0 \end{array}\right\} \quad \text{on} \quad \partial\Omega\times(0,T) \tag{13}$$

to make the description to be simple. In fact, if $\mathcal{W}$ is defined by

$$\mathcal{W}(u,v) = \int_\Omega u(\log u - 1)dx + \frac{1}{2}\|\nabla v\|_2^2 - \langle v, u\rangle,$$

then we have

$$\frac{d}{dt}\mathcal{W}(u(t),v(t)) + \tau\|v_t(t)\|_2^2 + \int_\Omega u\left|\nabla(\log u - v)\right|^2(\cdot,t)dx = 0, \tag{14}$$

where $u = u(\cdot,t)$, $v = v(\cdot,t)$ is the classical solution to (13) and $\langle\ ,\ \rangle$ denotes the duality:

$$\langle v, u\rangle = \int_\Omega uvdx.$$

Actually, in the simplified system we have (12), and this $\mathcal{W}$ is reduced to the free energy as

$$\mathcal{W}|_{v=(-\Delta_D)^{-1}u} = \mathcal{F}. \tag{15}$$

On the other hand, we have

$$u = \frac{\lambda e^v}{\int_\Omega e^v dx}$$

in the stationary state, because

$$\log u - v = \text{constant} \qquad \text{and} \qquad \|u\|_1 = \lambda$$

follows from (14) in this case. If we take $c = \lambda\log\lambda - \lambda$ in (9), that is,

$$J(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda\log\left(\int_\Omega e^v dx\right) + \lambda\log\lambda - \lambda$$

for $v \in H_0^1(\Omega)$, then it holds that

$$\mathcal{W}|_{u=\frac{\lambda e^v}{\int_\Omega e^v dx}} = \mathcal{J}. \tag{16}$$

We call those relations, (15) and (16) the *unfolding Legendre transformation.* Let us confirm that stationary states given by u and v, (10) and (6), respectively, are realized by $\delta\mathcal{F}(u) = 0$ on $\|u\|_1 = \lambda$ and $\delta J(v) = 0$, respectively. On the other hand, we have the *minimality* indicated as

$$\mathcal{W}(u,v) \geq \max\{\mathcal{F}(u), \mathcal{J}(v)\}, \tag{17}$$

where $\|u\|_1 = \lambda$. In fact, the first inequality is a direct consequence of Schwarz' inequality, while the second inequality is proven by Jensen's inequality. It is applied to show the global existence of the solution to (11) or (13) in the case of $\lambda = \|u_0\|_1 < 8\pi$, but is regarded as the dual form of the Trudinger-Moser inequality as is described later.

Theory of *dual variation* guarantees the splitting og the equilibrium to each component in the general stataioniary system, provided with the variational and dynamical equivalence. However, those unfolding and minimality are sufficient for its stability. Some systems describing mean field, such as the Penrose-Fife theory, take only semi-unfolding and semi-minimality, from which we can derive the stability of one component.

4 Duality

We now develop the general theory of dual variation. Let X be a Banach space over $\mathbf{R}$. Its dual space and the paring are denoted by X^* and $\langle\ ,\ \rangle = \rangle\ ,\ \rangle_{X,X^*}$, respectively. Given $F : X \to [-\infty, +\infty]$, we define its Legendre transformation by

$$F^*(p) = \sup_{x\in X} \{\langle x, p\rangle - F(x)\} \qquad (p \in X^*).$$

Then, Fenchel-Moreau's theorem says that if $F : X \to (-\infty, +\infty]$ is proper, convex, lower semi-continuous, then so is $F^* : X^* \to (-\infty, +\infty]$, and the second Legendre transformation defined by

$$F^{**}(x) = \sup_{p\in X^*} \{\langle x, p\rangle - F^*(p)\} \qquad (x \in X)$$

is equal to $F(x)$ ([5]).

Let $F, G : X \to (-\infty, +\infty]$ be proper, convex, lower semi-continuous, and put that

$$\Phi(x, y) = F(x + y) - G(x).$$

Let $D(F)$, $D(G)$ be the effective domains of F, G, respectively. Each $x \in D(G)$ induces proper, convex, lower semi-continuous mapping

$$y \in X \mapsto \Phi(x, y) \in (-\infty, +\infty]$$

and its Legendre transformation is given by

$$W(x, p) = \sup_{y\in X} \{\langle y, p\rangle - \Phi(x, y)\} \qquad (p \in X^*)$$

and thus,

$$W(x,\cdot) : X^* \to (-\infty, +\infty]$$

is proper, convex, lower semi-continuous. Sometimes

$$L(x,p) = -W(x,p)$$

is referred to as the *Lagrange function*. Then, we have for $(x,p) \in D(G) \times X^*$ that

$$\begin{aligned} W(x,p) &= \sup_{y \in X} \{\langle y + x, p\rangle - F(x+y) + G(x) - \langle x, p\rangle\} \\ &= F^*(p) + G(x) - \langle x, p\rangle . \end{aligned} \tag{18}$$

Putting $W(x,p) = +\infty$ for $x \notin D(G)$, we get (18) for any $(x,p) \in X \times X^*$.

Next, given $p \in X^*$, we put that

$$J^*(p) = \begin{cases} F^*(p) - G^*(p) & (p \in D(F^*)) \\ +\infty & (\text{otherwise}) . \end{cases} \tag{19}$$

Then, we obtain

$$\begin{aligned} \inf_{x \in X} W(x,p) &= F^*(p) - \sup_{x \in X} \{\langle x, p\rangle - G(x)\} \\ &= F^*(p) - G^*(p) = J^*(p) \end{aligned}$$

for $p \in D(F^*)$. It is valid even for $p \notin D(F^*)$ by (18) and (19). Similarly, we set for $x \in X$ that

$$J(x) = \begin{cases} G(x) - F(x) & (x \in D(G)) \\ +\infty & (\text{otherwise}) \end{cases} \tag{20}$$

and obtain

$$\begin{aligned} \inf_{p \in X^*} W(x,p) &= G(x) - \sup_{p \in X^*} \{\langle x, p\rangle - F^*(p)\} \\ &= G(x) - F^{**}(x) = J(x) \end{aligned}$$

for $x \in D(G)$, which is valid even for $x \notin D(G)$ by (18) and (20). Thus, we have

$$\begin{aligned} D(J) &= \{x \in X \mid J(x) \neq \pm\infty\} = D(G) \cap D(F) \\ D(J^*) &= \{p \in X^* \mid J^*(p) \neq \pm\infty\} = D(G^*) \cap D(F^*) \end{aligned}$$

and

$$\begin{array}{ll} \inf_{x\in X} W(x,p) = J^*(p) & (p \in X^*) \\ \inf_{p\in X^*} W(x,p) = J(x) & (x \in X)\,. \end{array} \tag{21}$$

Relation (21) implies

$$\inf_{(x,p)\in X\times X^*} W(x,p) = \inf_{p\in X^*} J^*(p) = \inf_{x\in X} J(x), \tag{22}$$

called the *Toland duality* ([19], [20]).

5 Sub-differential

Above global theory can be localized in use of the sub-differential. In fact, given $F : X \to [-\infty, +\infty]$, $x \in X$, and $p \in X^*$, we say $p \in \partial F(x)$, $x \in \partial F^*(p)$ if

$$\begin{array}{ll} F(y) \geq F(x) + \langle y - x, p\rangle & (\text{for any } \; y \in X), \\ F^*(q) \geq F^*(p) + \langle x, q - p\rangle & (\text{for any } \; q \in X^*), \end{array}$$

respectively. It is obvious that $\partial F(x) \neq \emptyset$ implies $x \in D(F)$, but if $F : X \to (-\infty, +\infty]$ is proper, convex, lower semi-continuous, then

$$x \in \partial F^*(p) \qquad \Leftrightarrow \qquad p \in \partial F(x), \tag{23}$$

and Fenchel-Moreau's identity

$$F(x) + F^*(p) = \langle x, p\rangle \tag{24}$$

holds in this case ([5]).

Under those prepartions, we can show the first part of the theory of dual variation, the variational equivalence.

Theorem 1 *Let $F, G : X \to (-\infty, +\infty]$ be proper, convex, lower semi-continuous, and $W = W(x,p)$ be defined by (18). Given $\hat{x} \in X$, $\hat{p} \in X^*$, we take the set of minimizers of $p \in X^*$, $x \in X$ in*

$$J(\hat{x}) = \inf_{p\in X^*} W(\hat{x}, p), \qquad J^*(\hat{p}) = \inf_{x\in X} W(x, \hat{p}),$$

denoted by $A^*(\hat{x})$, $A(\hat{p})$, respectively. We say that $\hat{x} \in X$, $\hat{p} \in X^*$ are critical points of J, J^* if $\partial G(\hat{x}) \cap \partial F(\hat{x}) \neq \emptyset$, $\partial G^*(\hat{p}) \cap \partial F^*(\hat{p}) \neq \emptyset$, respectively, *and that* $(\hat{x}, \hat{p})$ *is a critical point of* W *if* $0 \in \partial_x W(\hat{x}, \hat{p})$, $0 \in \partial_p W(\hat{x}, \hat{p})$ *holds true. Then, first, we have for any* $(x, p) \in X \times X^*$ *that*

$$A^*(x) = \partial F(x), \qquad A(p) = \partial G^*(p). \tag{25}$$

Furthermore, the following items are equivalent:

1. $(\hat{x}, \hat{p}) \in X \times X^*$ *is a critical point of* W.
2. $\hat{x} \in X$ *is a critical point of* J *and it holds that* $\hat{p} \in \partial G(\hat{x}) \cap \partial F(\hat{x})$.
3. $\hat{p} \in X^*$ *is a critical point of* J^* *and it holds that* $\hat{x} \in \partial F^*(\hat{p}) \cap \partial G^*(\hat{p})$.

Finally, we have

$$W(\hat{x}, \hat{p}) = J(\hat{x}) = J^*(\hat{p}) \tag{26}$$

in this case.

Proof: In fact, we have from (18) and (23) that

$$\begin{array}{lllll} 0 \in \partial_x W(x,p) = 0 & \Leftrightarrow & p \in \partial G(x) & \Leftrightarrow & x \in \partial G^*(p) \\ 0 \in \partial_p W(x,p) = 0 & \Leftrightarrow & x \in \partial F^*(p) & \Leftrightarrow & p \in \partial F(x) \end{array} \tag{27}$$

for any $(x, p) \in X \times X^*$. Given $x \in X$, we take $p \in A^*(x)$. This means that it attains

$$J(x) = \inf_{p \in X^*} W(x, p),$$

which is equivalent to $0 \in \partial_p W(x, p)$. Thus, $A^*(x) = \partial F(x)$ holds by (27). Relation $A(p) = \partial G^*(p)$ follows similarly, and the first part, (25), is proven. The second part, the equivalence of those three items are obtained also by (27), because $(\hat{x}, \hat{p}) \in X \times X^*$ is a critical point of

$$W = W(x, p) = F^*(p) + G(x) - \langle x, p \rangle$$

if and only if $\hat{p} \in \partial G(\hat{x})$ and $\hat{x} \in \partial F^*(\hat{p})$. Finally, (26) follows from (25) and $\hat{p} \in \partial G(\hat{x})$, $\hat{x} \in \partial F^*(\hat{p})$, as

$$\begin{aligned} W(\hat{x}, \hat{p}) &= F^*(\hat{p}) + G(\hat{x}) - \langle \hat{x}, \hat{p} \rangle \\ &= F^*(\hat{p}) - G^*(\hat{p}) \\ &= G(\hat{x}) - F(\hat{x}). \end{aligned}$$

The proof is complete. $\square$

We have the equivalence of

$$\hat{p} \in \partial G(\hat{x}) \cap \partial F(\hat{x}) \qquad \Leftrightarrow \qquad \hat{x} \in \partial F^*(\hat{p}) \cap \partial G^*(\hat{p}),$$

and therefore, each critical point of J, J^* produces that of J^*, J, respectively. This correspondence, we call, the Legendre transformation of critical points, or their duality. *Principle of dual variation* indicates the production of those critical points of J, J^* from that duality. This means equivalently that the critical point $(\hat{x}, \hat{p}) \in X \times X^*$ of $W = W(x,p)$ is characterized as for each element $\hat{x}$, $\hat{p}$ to be a critical point of the separted J, J^*, respectively. We can prove the equivalence of those critical points up to their Morse indices under natural assumptions, as in the special case of (1) described in [17], that is, the second part of the theory of dual variation, indicated as the dynamical equivalence. If a (local) dynamical system

$$t \in [0,T) \mapsto (x(t), p(t))$$

is given and $W = W(x,p)$ acts as a Lyapunov function, then we call critical points of W the *equilibrium*.

6 Stability

Remarkable structures (15), (16), and (17) of unfolding and minimality are natural consequences of the abstract setting of dual variation. We can show the third part of the theory of dual variation, the unfolding - minimality.

Theorem 2 *Given proper, convex, lower semi-continous functionals*

$$F, G : X \to (-\infty, +\infty],$$

we take $W = W(x,p)$, $J = J(x)$, and $J^* = J^*(p)$ by (18), (20), and (19), respectively. Then, it holds that

$$W|_{p \in \partial F(x)} = J, \qquad W|_{x \in \partial G^*(p)} = J^*, \tag{28}$$

and

$$W(x,p) \geq \max\{J(x), J^*(p)\}, \tag{29}$$

where $(x,p) \in X \times X^*$.

Proof: For the unfolding (28) to prove, we note that $p \in \partial F(x)$ implies

$$F^*(p) - \langle x, p \rangle = -F(x)$$

from Fenchel-Moreau's identity (24). This implies the first equality of (28), and the second equality is proven similarly. On the other hand, the minimality (29) is a direct consequence of (21). □

Unfolding and minimality imply the stability of equilibrium in the following way.

Theorem 3 *Let a proper, convex, lower semi-continuous functional*

$$F : X \to (-\infty, +\infty]$$

be given with $J : X \to [-\infty, +\infty]$ *and* $W : X \times X^* \to [-\infty, +\infty]$ *satisfying*

$$W|_{p\in\partial F(x)} = J \qquad \text{and} \qquad W(x,p) \geq J(x)$$

for any $(x,p) \in X \times X^*$. *Let* $(\hat{x}, \hat{p}) \in D(W) \subset X \times X^*$ *be in*

$$\hat{p} \in \partial F(\hat{x}) \cap Y_* \qquad \text{and} \qquad \hat{x} \in Y_0,$$

where Y_0 *is a closed subset of a Banach space* Y *continuously imbedded in* X, *and* Y_* *is a Banach space continuously embedded in* X^*. *Spppose that* $\hat{x}$ *is a linearized stable local minimizer of* $J|_{Y_0}$ *in the sense that for some* $\varepsilon_0 > 0$, *any* $\varepsilon \in (0, \varepsilon_0/4]$ *admits* $\delta > 0$ *such that*

$$x \in Y_0, \quad \|x - \hat{x}\|_Y < \varepsilon_0, \quad J(x) - J(\hat{x}) < \delta \qquad \Rightarrow \qquad \|x - \hat{x}\|_Y < \varepsilon. \tag{30}$$

Suppose, finally, that $W|_{Y_0 \times Y_*}$ *is continuous at* $(\hat{x}, \hat{p})$. *Then, if*

$$\{(x(t), p(t))\}_{0 \leq t < T} \subset Y_0 \times Y_*$$

is given with $t \in [0, T) \mapsto x(t) \in Y_0$ *continuous and*

$$t \in [0, T) \quad \mapsto \quad W(x(t), p(t)) \tag{31}$$

non-increasing, then any $\varepsilon \in (0, \varepsilon_0/4]$ *admits* $\delta > 0$ *such that*

$$\|x(0) - \hat{x}\|_Y < \delta \qquad \text{and} \qquad \|p(0) - \hat{p}\|_{Y_*} < \delta \tag{32}$$

imply

$$\|x(t) - \hat{x}\|_Y < \varepsilon \qquad (0 \le t < T). \tag{33}$$

Similarly, if $G : X \to (-\infty, +\infty]$ *is proper, convex, lower semi-continuous,* $J^* : X^* \to [-\infty, +\infty]$ *satsifies that*

$$W|_{x \in \partial G^*(p)} = J^* \qquad \textit{and} \qquad W(x,p) \ge J^*(x)$$

for any $(x,p) \in X \times \hat{X}$, $(\hat{x}, \hat{p}) \in D(W)$ *is in*

$$\hat{x} \in \partial G^*(\hat{p}) \qquad \textit{and} \qquad \hat{p} \in Y_{0*},$$

respectively, where Y_{0*} *is a closed set in* Y_*, $\hat{p}$ *is a linearized stable local minimizer of* $J^*|_{Y_{0*}}$ *in the sense that any* $\varepsilon \in (0, \varepsilon_0]$ *admits* $\delta > 0$ *such that*

$$p \in Y_{0*}, \quad \|p - \hat{p}\|_{Y_*} < \varepsilon_0, \quad J^*(p) - J^*(\hat{p}) < \delta \qquad \Rightarrow \qquad \|p - \hat{p}\|_{Y_*} < \varepsilon,$$

$t \in [0, T) \mapsto p(t) \in Y_{0^*}$ *is continuous with (31) decreasing,* $\hat{x} \in \partial G^*(\hat{p})$, *and* $W|_{Y \times Y_{0*}}$ *is continuous at* $(\hat{x}, \hat{p})$, *then any* $\varepsilon \in (0, \varepsilon_0/4]$ *admits* $\delta > 0$ *such that (32) implies*

$$\|p(t) - \hat{p}\|_{Y_*} < \varepsilon \qquad (0 \le t < T).$$

Proof: We show the former part. In fact, given $\varepsilon \in (0, \varepsilon_0/4]$, we take $\delta = \delta_1 > 0$ in (30). Because $W|_{Y_0 \times Y_*}$ is continuous at $(\hat{x}, \hat{p})$, there exists $\delta \in (0, \varepsilon_0/2]$ such that

$$\|x(0) - \hat{x}\|_Y < \delta \qquad \text{and} \qquad \|p(0) - \hat{p}\|_{Y_*} < \delta \tag{34}$$

imply

$$W(x(0), p(0)) - W(\hat{x}, \hat{p}) < \delta_1. \tag{35}$$

On the other hand, we have

$$W(x,p) \ge J(x) \ge J(\hat{x}) = W(\hat{x}, \hat{p})$$

for any $(x,p) \in Y_0 \times X^*$ with $\|x - \hat{x}\|_Y < \varepsilon_0$ from the assumption. Therefore, as far as

$$\|x(t) - \hat{x}\|_Y < \varepsilon_0 \tag{36}$$

we have

$$\begin{aligned} 0 &\leq J(x(t)) - J(\hat{x}) \leq W(x(t), p(t)) - J(\hat{x}) \\ &\leq W(x(0), p(0)) - W(\hat{x}, \hat{p}) < \delta_1. \end{aligned} \tag{37}$$

Now, we have

$$\|x(0) - \hat{x}\|_Y < \delta \leq \varepsilon_0/2.$$

Then, if there is $t_0 \in (0, T)$ such that $\|x(t_0) - \hat{x}\|_Y = \varepsilon_0/2$, then we have (36) and hence (37) for $t = t_0$. This implies from (30) (with $\delta = \delta_1$) that

$$\|x(t) - \hat{x}\|_Y < \varepsilon \leq \varepsilon_0/4, \tag{38}$$

a contradiction. Therefore, because $t \in [0, T) \mapsto x(t) \in Y_0 \subset Y$ is continuous, the relation

$$\|x(t) - \hat{x}\|_Y < \varepsilon_0/2$$

keeps to hold for $t \in [0, T)$, and hence (36) im particular. Again this implies (37) and (38) for any $t \in [0, T)$, and the proof is complete. □

Continuity of W at $(x, p = (\hat{x}, \hat{p})$ can be replaced by the first case of (34) and (35) for the initial value $(x(0), p(0))$, to imply (32).

7 Applications

By Damlamian [4], Toland duality in applied science was first observed in free boundary problem for plasma confinement, between the formulations of Berestycki and Brezis [2] and Temam [18]. That duality can be realized in a slightly different way, with the Nehari principle involving, and then we can localize their equivalence up to Morse indices. In the Penrose-Fife system [13], on the other hand, exact duality cannot be observed, while semi-unfolding and semi-minimality are valid, which provide stability of the field component. We have several examples of those dual variation or semi-dual variation in mean field theories. Here, we show how the abstract theory is realized in the system of chemotaxis, particularly in (11), where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$.

For this problem, we take $X = H_0^1(\Omega)$ with the Gel'fand triple $X \hookrightarrow L^2(\Omega) \hookrightarrow X^*$. Then, *dual entropy functional* $F : X \to (-\infty, +\infty]$ is defined by

$$F(v) = \lambda \log \left(\int_\Omega e^v dx \right) - \lambda \log \lambda + \lambda,$$

which is proper, convex, lower semi-continuous. We have

$$D(F) = \left\{ v \in X \mid \int_\Omega e^v dx < +\infty \right\},$$

$\partial F(v) \neq \emptyset$ for any $v \in D(F)$, and

$$u \in \partial F(v) \qquad \Leftrightarrow \qquad u = \frac{\lambda e^v}{\int_\Omega e^v dx}.$$

Entropy functional is defined by its Legendre transformation,

$$F^*(u) = \begin{cases} \int_\Omega u(\log u - 1)dx & (u \in X^* \cap L^1(\Omega),\ u \geq 0,\ \|u\|_1 = \lambda) \\ +\infty & (\text{otherwise}). \end{cases}$$

It holds that

$$D(F^*) = \{u \in X^* \mid u \geq 0,\ u \in L \log L(\Omega),\ \|u\|_1 = \lambda\},$$

$v \in \partial F^*(u)$ if and only if $u \in D(F^*)$ and

$$v = \log u + \text{constant} \in X,$$

where $L \log L(\Omega)$ denotes the Zygmund space. On the other hand, taking

$$G(v) = \frac{1}{2} \|\nabla v\|_2^2,$$

we get the proper, convex, lower semi-continuous mapping

$$G : X \to (-\infty, +\infty).$$

Operator $-\Delta_D$ induces the isomorphism $\hat{A} : X \to X^*$, and we have

$$G^*(u) = \frac{1}{2} \left\langle \hat{A}^{-1} u, u \right\rangle$$

for $u \in X^*$. Then, Lyapunov function of this system is realized as

$$W(v, u) = F^*(u) + G(v) - \langle v, u \rangle$$

and the equilibrium is described by

$$0 \in \partial_v W(\overline{v}, \overline{u}), \qquad 0 \in \partial_u W(\overline{v}, \overline{u})$$

or equivalently,

$$\overline{u} = \hat{A}^{-1}\overline{v}, \qquad \overline{v} \in \partial F^*(\overline{u}).$$

From Theorem 1, this relation is transformed into the conditions on $\overline{u}$ and $\overline{v}$, separately, that is, to be critical points of

$$\begin{aligned} J(v) &= G(v) - F(v) \\ &= \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v dx\right) + \lambda \log \lambda - \lambda \end{aligned}$$

defined for $v \in X$ and

$$\begin{aligned} J^*(u) &= F^*(u) - G^*)u) \\ &= \int_\Omega u(\log u - 1)dx - \frac{1}{2}\left\langle \hat{A}^{-1}u, u\right\rangle \end{aligned}$$

defined for $u \in X^* \cap L^1(\Omega)$, $u \geq 0$, $\|u\|_1 = \lambda$, respectively. Those conditions are equivalent to

$$\begin{aligned} &\overline{v} \in X, \quad \int_\Omega e^{\overline{v}} dx < +\infty \\ &\hat{A}\overline{v} = \frac{\lambda e^{\overline{v}}}{\int_\Omega e^{\overline{v}} dx} \in X^* \end{aligned} \tag{39}$$

and

$$\begin{aligned} &\overline{u} \in X \cap L \log L(\Omega), \quad \overline{u} \geq 0, \quad \|\overline{u}\|_1 = \lambda \\ &\hat{A}^{-1}\overline{u} = \log \overline{u} + \text{constant} \in X, \end{aligned} \tag{40}$$

respectively. The exact correspondence of Morse indices can be derived from the general theory, but here we only admit the equivalence of linearized stability of those $\overline{u}$ and $\overline{v}$. Then we can apply Theorem 3 for

$$Y = D(J) = D(G) \cap D(F) = \left\{ v \in X \mid \int_\Omega e^v dx < +\infty \right\}$$

and

$$\begin{aligned} Y_{0*} &= D(J^*) = \{u \in X^* \mid u \in L \log L(\Omega),\ u \geq 0,\ \|u\|_1 = \lambda\} \\ &\subset Y_* = X^* \cap L \log L(\Omega), \end{aligned}$$

and in this case there is $\varepsilon_0 > 0$ such that if $u = u(\cdot, t)$, $v = v(\cdot, t)$ is a solution to (11) for $t \in [0, T)$, then any $\varepsilon \in (0, \varepsilon_0/4]$ admits $\delta > 0$ such that

$$\|v(\cdot, 0) - \overline{v}\|_X < \delta, \quad \|u(\cdot, 0) - \overline{u}\|_{X^* \cap L \log L} < \delta, \quad \|u(\cdot, 0)\|_1 = \lambda = \|\overline{u}\|_1$$

implies

$$\|v(t) - \overline{v}\|_X < \varepsilon, \qquad \|u(\cdot,t) - \overline{u}\|_{X^* \cap L\log L} < \varepsilon$$

for any $t \in [0,T)$. This result is valid to any space dimension and also to the full system

$$\left.\begin{array}{l} u_t = \nabla \cdot (\nabla u - u\nabla v) \\ \tau v_t = \Delta v + u \end{array}\right\} \quad \text{in} \quad \Omega \times (0,T)$$

$$\left.\begin{array}{l} \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = 0 \\ v = 0 \end{array}\right\} \quad \text{on} \quad \partial\Omega \times (0,T)$$

with $\tau > 0$.

Next, we examine the role of unfolding - minimality in this system of (1). In fact, first, from the Trudinger-Moser inequality, we have

$$\inf_{v \in X} J(v) > -\infty \tag{41}$$

in the case of $n = 2$ and $\lambda = 8\pi$. Next, Theorem 1 guarantees the equuivalence of boundedness from below of J on X and J^* on X^*, and hence it follows that

$$\inf_{u \in X^* \cap L\log L,\ u \geq 0,\ \|u\|_1 = \lambda} J^*(u) > -\infty$$

in this case. Furthermore, the Trudinger-Moser inequality again guarantees $L\log L(\Omega) \hookrightarrow X^*$ for $n = 2$, and hence it holds that

$$\begin{aligned} \inf\{ \int_\Omega u(\log u - 1)dx - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u \otimes u dx dx' \\ \mid u \geq 0,\ \|u\|_1 = 8\pi\} > -\infty, \end{aligned} \tag{42}$$

where $G = G(x,x')$ denotes the Green's function. Inequality (42), valid to $n = 2$ is regarded as the dual form of the Trudinger-Moser-Onofri inequality.

We have from (41) that each $\lambda < 8\pi$ admits a constant C_1 such that

$$\begin{aligned} J(v) &= \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v dx\right) + \lambda\log\lambda - \lambda \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda}{8\pi}\right)\|\nabla v\|_2^2 - C_1 \end{aligned}$$

for any $v \in X$. Therefore, if $\|u_0\|_1 = \lambda < 8\pi$ in (11), then we have

$$\sup_{t \in [0,T)} \|\nabla v(\cdot,t)\|_2 \leq C_2$$

with a constant $C_2 > 0$ determine by λ, because of (29) and

$$\sup_{t\in[0,T)} W\left(v(t), u(t)\right) \leq W\left(u(0), v(0)\right).$$

Similarly, from (42) we have

$$J^*(u) \geq \left(1 - \frac{\lambda}{8\pi}\right) \int_\Omega u \log u dx - C_3$$

for any $u \in L\log L(\Omega)$ in $u \geq 0$ and $\|u\| = \lambda < 8\pi$, and this implies that

$$\sup_{t\in[0,T)} \int_\Omega (u\log u)(x,t)dx \leq C_4. \tag{43}$$

Then, from Moser's iteration scheme or the maximal reguality we can derive $T_{\max} = +\infty$ and the uniform boundedness of $u(\cdot,t)$:

$$\sup_{t\in[0,T)} \|u(\cdot,t)\|_\infty \leq C_5.$$

In the original work of Biler [3], Gajewski and Zacharias [6], Nagai, Senba, and Yosida [9], inequality (43) is derived from the Trudinger-Moser inequality and

$$\int_\Omega \left(u\log u - uv\right) dx + \lambda \log\left(\int_\Omega e^v dx\right) - \lambda\log\lambda \geq 0 \tag{44}$$

valid to $u \in L\log L(\Omega)$, $u \geq 0$, $\|u\|_1 = \lambda$. Inequality (44) follows from Jensen's inequality, but it is also a consequence of the minimality

$$W(v,u) \geq J(v).$$

In the simplified system, we have

$$W(v,u) = \mathcal{F}(u)$$

and hence from the minimality it follows that

$$\mathcal{F}\left(u(t)\right) \geq J\left(v(t)\right) \qquad \left(0 \leq t < T_{\max}\right).$$

On the other hand, from the quantization of blowup mechanism of the stationary state we have

$$j_\lambda = \inf\left\{J(v) \mid v \in E_\lambda\right\} > -\infty$$

for $\lambda \in [0, \infty) \setminus \mathcal{N}$, where E_λ denotes the set of critical points of J on X. Furthermore, we have

$$\frac{d}{dt}\int_\Omega u \log u dx \leq 2K^2\lambda$$
$$+4|\Omega| \exp\left(4K^2 \int_\Omega u \log u dx + 4K^2 e^{-1} |\Omega|\right)$$

with a constant $K > 0$ determined by Ω, and if $T_{\max} = +\infty$ and

$$\lim_{t \to +\infty} \int_\Omega u \log u(x,t) dx < +\infty,$$

then we have $t_k \to +\infty$, $\delta > 0$, and $C > 0$ such that

$$\int_\Omega (u \log u)(x,t) dx \leq C \qquad (t \in [t_k, t_k + \delta]) .$$

This implies the non-empty of the omega limit set of $(u(t), v(t))$ so that

$$W(v_0, u_0) \geq \lim_{t \to +\infty} W(v(t), u(t)) \geq j_\lambda$$

holds true. Because

$$\lim_{t \uparrow T_{\max}} \int_\Omega (u \log u)(x,t) dx = +\infty$$

follows in the case of $T_{\max} < +\infty$, we obtain the criterion of Horstmann and Wang [7] that

$$W(v_0, u_0) < j_\lambda \qquad \Rightarrow \qquad \lim_{t \to T_{\max}} \int_\Omega (u \log u)(x,t) dx = +\infty. \tag{45}$$

Relation (45) is valid even in the full system, in use of the argument developed in [14].

8 Conclusion

Nonlinear quantum mechanics is just an episode of the mathematical theory of statistical mechanics, where each hierarchy of equations to the mean field of many particles has its own physical and mathematical principles. It asserts the control of the total set of equilibrium states over the global dynamics of non-equilibrium states. This story, we are convinced, is efficient to the most theory on mean fields, where self-interaction is caused in term of the field created by particles. The principle of dual variation aries in this context, study of the equilibrium states in the nonlinear system, with the interaction desribed in terms of the field and particles. It assures that the equilibrium state in those hierarchies splits into the problems on the field and on particles, each of which is provided with the variational structure, dynamically equivalent each other. We have unified such a structure in the formulation of Toland duality for the system of chemotaxis and also in the free boundary problem in plasma confinement, where concentration of the particle distribution is widely observed. On the other hand, the Penrose-Fife system succeeds a part of this duality, unfolding - minimality, only in the field component and the same is true for the Euler-Possion equation describing the evolution of gaseous stars. Consequently, we can discuss the stability of equilibrium field in those systems, by introducing variational structure for the field component. Skew gradient system, on the other hand, is also under the control of dual variation, but the leading structure is different and the stable equilibrium is realized as the saddle point of the Lagrange function. Actally, it is associated with the Kuhn-Tucker duality and especially, the dynamics around degenerate stable equilibrium is quite strange. All of those materials are still in progress and will be published someday in future.

In the context of the theory of self-organization, type (I) blowup point is quite interesting. It assures the emergence coming from the wedge of the parabolic envelope, where entropy and mass are exchanged to create a clean self with the mass quntized, which reminds us of the principle asserted in system biology that the expanding cosmos is the origin of life. This remarkable fact is derived mathematically, motivated by the theory of nonlinear quantum mechanics, where the theory of dual variation takes a role to control the set of equilibirum states as well as the local dynamics around them. See [17].

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