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Oscillation Problem for Elliptic Equations with Nonlinear Perturbed Terms

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1. INTRODUCTION

We consider the semilinear elliptic equation

$$\Delta u+p(x)u+\phi(x,u)=0 \tag{1}$$

in an unbounded domain Ω containing $G_a=\{x\in\mathbb{R}^N\colon |x|>a\}$ for some $a>0$ and $N\geq 3$. Throughout this paper, we call such a domain an *exterior domain* of $\mathbb{R}^N$. We assume that $p:\Omega\to[0,\infty)$ and $\phi:\Omega\times\mathbb{R}\to\mathbb{R}$ are locally Hölder continuous with exponent $\alpha\in(0,1)$.

For convenience, let $C^{2+\alpha}(\overline{M})$ denote the space of all continuous functions on the closure $\overline{M}$ of a bounded domain $M\subset\Omega$ such that the usual Hölder norm $||\cdot||_{2+\alpha,\overline{M}}$ is finite. A *solution* of (1) in Ω is defined to be a function $u\in C^{2+\alpha}(\overline{M})$ for every bounded subdomain $M\subset\Omega$ such that u satisfies equation (1) at every point $x\in\Omega$. A solution of (1) is called *oscillatory* if it keeps neither positive nor negative in any exterior domain. On the other hand, it is called *nonoscillatory* if it never changes the sign throughout some exterior domain.

Equation (1) naturally includes the linear equation

$$\Delta u+p(x)u=0 \tag{2}$$

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which has been widely studied by many authors. For example, see [2–5, 9]. When

$$p(x) = \frac{\mu_1}{|x|^2}, \qquad |x| > 1 \tag{3}$$

with μ_1 positive, it is well known that equation (2) has the radial solution

$$u(x) = \begin{cases} \sqrt{1/t(x)}\{K_1 + K_2 \log t(x)\} & \text{if } \mu_1 = \lambda_N, \\ \sqrt{1/t(x)}\{K_3 t(x)^\zeta + K_4 t(x)^{-\zeta}\} & \text{if } \mu_1 \neq \lambda_N, \end{cases}$$

where $t(x) = (N-2)|x|^{N-2}$ and $\lambda_N = (N-2)^2/4$, and where K_i $(i = 1, 2, 3, 4)$ are arbitrary constants and ζ is a number satisfying

$$(N-2)^2\zeta^2 = \lambda_N - \mu_1.$$

For this reason, equation (2) with (3) has nonoscillatory solutions if $0 < \mu_1 \leq \lambda_N$; otherwise, all radial solutions are oscillatory. From Sturm's separation theorem for linear elliptic equations it follows that all non-radial solutions are also oscillatory. As for this point, see [1, 4] and [9, p. 187]. Hence, for equation (2) with (3) the critical value of μ_1 is λ_N.

Next, consider the case that

$$p(x) = \frac{\lambda_N}{|x|^2} + \frac{\mu_2}{|x|^2\{\log t(x)\}^2}, \qquad |x| > e \tag{4}$$

with μ_2 positive. Then radial solutions of (2) are represented as the form of

$$u(x) = \begin{cases} \sqrt{\log t(x)/t(x)}\{K_1 + K_2 \log(\log t(x))\} & \text{if } \mu_2 = \lambda_N, \\ \sqrt{\log t(x)/t(x)}\{K_3 (\log t(x))^\zeta + K_4 (\log t(x))^{-\zeta}\} & \text{if } \mu_2 \neq \lambda_N, \end{cases}$$

where K_i $(i = 1, 2, 3, 4)$ are arbitrary constants and ζ is a number satisfying

$$(N-2)^2\zeta^2 = \lambda_N - \mu_2.$$

Hence, the situation is the same as in the case (3), in other words, the critical value of μ_2 is also λ_N for equation (2) with (4). From this point of view, we may regard cases (3) and (4) as the first and the second stages for equation (2), respectively.

To go on to the nth stage for equation (2), we introduce three sequences of functions as follows:

$$\log_1 t = |\log t| \quad \text{and} \quad \log_{k+1} t = \log(\log_k t);$$

$$l_1(t) = 1 \quad \text{and} \quad l_{k+1}(t) = l_k(t)\log_k t;$$

$$S_0(t) = 0 \quad \text{and} \quad S_k(t) = \sum_{i=1}^{k} \frac{1}{\{l_i(t)\}^2}$$

for $k \in \mathbb{N}$. The sequences are well-defined for $t > 0$ sufficiently small or sufficiently large. To make sure, we enumerate the sequences $\{l_k(t)\}$ and $\{S_k(t)\}$:

$$l_2(t) = |\log t|, \qquad l_3(t) = |\log t|(\log|\log t|),$$

$$l_4(t) = |\log t|(\log|\log t|)(\log(\log|\log t|)), \cdots\cdots\cdots\cdots\cdots ;$$

$$S_1(t) = 1, \qquad S_2(t) = 1 + \frac{1}{(\log t)^2},$$

$$S_3(t) = 1 + \frac{1}{(\log t)^2} + \frac{1}{(\log t)^2(\log|\log t|)^2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

We may consider the case

$$p(x) = \frac{\lambda_N}{|x|^2} S_{n-1}(t(x)) + \frac{\mu_n}{|x|^2\{l_n(t(x))\}^2}, \qquad |x| > e_{n-1} \tag{5}$$

to be the nth stage for equation (2), where μ_n is a positive parameter and $\{e_k\}$ is a sequence satisfying

$$e_0 = 1 \quad \text{and} \quad e_k = \exp(e_{k-1}) \quad \text{for } k \in \mathbb{N}.$$

The reason for this is that equation (2) with (5) has the radial solution

$$u(x) = \begin{cases} \sqrt{l_n(t(x))/t(x)}\{K_1 + K_2 \log_n t(x)\} & \text{if } \mu_n = \lambda_N, \\ \sqrt{l_n(t(x))/t(x)}\{K_3(\log_{n-1} t(x))^{\zeta} + K_4(\log_{n-1} t(x))^{-\zeta}\} & \text{if } \mu_n \neq \lambda_N, \end{cases}$$

where K_i $(i = 1, 2, 3, 4)$ are arbitrary constants and ζ is a number satisfying

$$(N-2)^2\zeta^2 = \lambda_N - \mu_n.$$

The critical value of μ_n is also λ_N for equation (2) with (5).

For simplicity, let

$$p_n(x) = \frac{\lambda_N}{|x|^2} S_n(t(x)).$$

Then, as shown above, the linear equation

$$\Delta u + p_n(x)u = 0 \tag{6}$$

has nonoscillatory solutions. Let us add a linear perturbation of the form $q(x)u$ to equation (6). If

$$|x|^2\{l_{n+1}(t(x))\}^2 q(x) \leq \lambda_N$$

for $|x|$ sufficiently large, then nonoscillatory solutions remain in the equation

$$\Delta u + p_n(x)u + q(x)u = 0. \tag{7}$$

On the other hand, if there exists a $\nu > \lambda_N$ such that

$$|x|^2\{l_{n+1}(t(x))\}^2 q(x) \geq \nu$$

for $|x|$ sufficiently large, then all nonoscillatory solutions disappear from equation (7). It is safe to say that the linear perturbation problem is solved. However, there remains an unsettled question: what is the lower limit of the nonlinear perturbed term $\phi(x,u)$ for all solutions of the elliptic equation

$$\Delta u + p_n(x)u + \phi(x,u) = 0$$

to be oscillatory?

The purpose of this paper is to answer the above question and to discuss whether or not equation (1) has nonoscillatory solutions under the assumption that equation (2) has nonoscillatory solutions.

2. PRESERVATION OF NONOSCILLATORY SOLUTIONS

To begin with, we define a *supersolution* (resp., *subsolution*) of (1) in Ω as a function $u \in C^{2+\alpha}(\overline{M})$ for every bounded domain $M \subset \Omega$ such that u satisfies the inequality $\Delta u + p(x)u + \phi(x,u) \leq 0$ (resp., ≥ 0) at every point $x \in \Omega$. Using the so-called "supersolution-subsolution" method due to Noussair and Swanson [7], we have the following result.

Lemma 1. *If there exists a positive supersolution $\overline{u}$ of* (1) *and a positive subsolution $\underline{u}$ of* (1) *in G_b such that $\underline{u}(x) \leq \overline{u}(x)$ for all $x \in G_b \cup C_b$, where $b \geq a$ and $C_b = \{x \in \mathbb{R}^N$: $|x| = b\}$, then equation* (1) *has at least one solution u satisfying $u(x) = \overline{u}(x)$ on C_b and $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ through G_b.*

To find a suitable pair of positive supersolution and positive subsolution of (1) in Ω, we consider the nonlinear differential equation

$$w'' + \frac{2}{t}w' + \frac{1}{4t^2}S_n(t)w + \frac{1}{t^2}g(w) = 0, \qquad t > a, \tag{8}$$

where $' = d/dt$, and $g(w)$ is locally Lipschitz continuous on $\mathbb{R}$ and satisfies the signum condition

$$wg(w) > 0 \quad \text{if } w \neq 0. \tag{9}$$

We say that a solution of (8) (or its equivalent equation) is *oscillatory* if the set of its zeros is unbounded; otherwise it is *nonoscillatory*. Recently, the second author [8] has presented a sufficient condition which guarantees the existence of a nonoscillatory solution of (8) as follows.

Proposition 1. *Assume* (9) *and suppose that*

$$\frac{g(w)}{w} \leq \frac{1}{4\{l_{n+1}(w^2)\}^2} \tag{10}$$

for $w > 0$ or $w < 0$, $|w|$ sufficiently small. Then equation (8) *has a nonoscillatory solution.*

Letting $s = \log t$, we can transform equation (8) into the system of Liénard type

$$\begin{aligned} \dot{\xi} &= \eta - \xi, \\ \dot{\eta} &= -\frac{1}{4}S_n(e^s)\xi - g(\xi), \end{aligned} \tag{11}$$

where $\dot{} = d/ds$ and $\xi(s) = w(e^s) = w(t)$. Since the global asymptotic stability of the zero solution of (11) is shown in [8], all solutions $w(t)$ of (8) tend to zero as $t \to \infty$, that is,

$$\lim_{t \to \infty} w(t) = 0.$$

Also, we can estimate the decaying speed of nonoscillatory solutions of (8).

Lemma 2. *Assume* (9). *If* $g(w)$ *satisfies* (10) *for* $w > 0$ *sufficiently small, then there exist a* $b > 0$ *and a positive solution of* (8) *such that*

$$w(t) \geq \frac{bw(b)}{t} \quad \textit{for } t \geq b.$$

Proof. As in the proof of Proposition 1, we can find a solution $w(t)$ of (8) which is eventually positive. Hence, there exists a $b > 0$ such that

$$w(t) > 0 \quad \text{for } t \geq b.$$

Let $(\xi(s), \eta(s))$ be the solution of (11) corresponding to $w(t)$. Then we have the relation

$$(\xi(s), \eta(s)) = (w(t), w'(t)t + w(t)).$$

Since $\xi(s) > 0$ for $s \geq \log b$,

$$\dot{\eta}(s) < 0 \quad \text{for } s \geq \log b \tag{12}$$

by (9). Hence, we see that $\eta(s) \geq 0$ for $s \geq \log b$. In fact, if $\eta(s_0) < 0$ for some $s_0 \geq \log b$, then by (12) we obtain

$$\dot{\xi}(s) = \eta(s) - \xi(s) < \eta(s_0)$$

for $s \geq s_0$. Integrating this inequality from s_0 to s, we get

$$\xi(s) < \xi(s_0) + \eta(s_0)(s - s_0) \to -\infty \quad \text{as } s \to \infty.$$

This contradicts the fact that $\xi(s)$ is eventually positive. Because $\eta(s) > 0$ for $s \geq \log b$,

$$\dot{\xi}(s) = \eta(s) - \xi(s) \geq -\xi(s)$$

for $s \geq \log b$. Integrate the both sides to obtain

$$\xi(s) \geq b\xi(\log b)e^{-s} \quad \text{for } s \geq \log b,$$

namely, $w(t) \geq bw(b)/t$ for $t \geq b$. Thus, the lemma is proved. □

We shall construct a positive supersolution of (1) and a positive subsolution of (1) which is not larger than the supersolution by using the functions $w(t)$ and $bw(b)/t$ in Lemma 2, respectively. Hence, by virtue of Lemma 1, we can supply the following answer to our question in Section 1.

Theorem 1. *Suppose that there exists an $n \in \mathbb{N}$ such that*

$$0 \le p(x) \le p_n(x), \qquad x \in \Omega. \tag{13}$$

Also suppose that there exists a locally Lipschitz continuous function $h(u)$ with $h(0) = 0$ and $h(u) > 0$ if $u > 0$ such that

$$0 \le \phi(x,u) \le \frac{h(u)}{|x|^2}, \qquad x \in \Omega, \quad u \ge 0. \tag{14}$$

If $h(u)$ satisfies

$$\frac{h(u)}{u} \le \frac{\lambda_N}{\{l_{n+1}(u^2)\}^2} \tag{15}$$

for $u > 0$ sufficiently small, then equation (1) has a positive solution $u(x)$ in an exterior domain with $\lim_{|x|\to\infty} u(x) = 0$.

Remark 1. *Theorem 1 is true even for $n = 0$. In this case, however, $p(x)$ is identically equal to zero, in other words, equation (1) has no linear term. Hence, this case deviates from the main subject.*

Proof of Theorem 1. Define

$$g(w) = \begin{cases} h(w)/4\lambda_N & \text{if } w \ge 0, \\ -h(-w)/4\lambda_N & \text{if } w < 0. \end{cases}$$

Then $g(w)$ is locally Lipschitz continuous on $\mathbb{R}$ and satisfies the signum condition (9). Also, it follows from (15) that

$$\frac{g(w)}{w} \le \frac{1}{4\{l_{n+1}(w^2)\}^2}$$

for $w > 0$ sufficiently small. Hence, by Lemma 2 equation (8) has a solution $w(t)$ which is positive for $t \ge b$ with some $b \ge a$ and tends to zero as $t \to \infty$.

Let $\overline{u}(x)$ be the function defined in G_b by

$$\overline{u}(x) = v(r) = w(t), \qquad r = |x|, \qquad t = (N-2)r^{N-2}.$$

Then, by (13) and (14) we have

$$\begin{aligned} &\Delta\overline{u}(x) + p(x)\overline{u}(x) + \phi(x,\overline{u}(x)) \le \Delta\overline{u}(x) + p_n(x)\overline{u}(x) + \frac{1}{|x|^2}h(\overline{u}(x)) \\ &= \frac{d^2}{dr^2}v(r) + \frac{N-1}{r}\frac{d}{dr}v(r) + \frac{(N-2)^2}{4r^2}S_n((N-2)r^{N-2})v(r) + \frac{1}{r^2}h(v(r)) \\ &= \frac{(N-2)^2}{r^2}\left\{t^2w''(t) + 2tw'(t) + \frac{1}{4}S_n(t)w(t) + g(w(t))\right\} = 0, \end{aligned}$$

and therefore, $\overline{u}(x)$ is a positive supersolution of (1) in G_b. We next define $\underline{u}(x) = bw(b)/t$ for $t \geq b$. Then, by (13) and (14) again, we obtain

$$\begin{aligned}\Delta\underline{u}(x) + p(x)\underline{u}(x) + \phi(x,\underline{u}(x)) &\geq \Delta\underline{u}(x)\\ &= \frac{(N-2)^2}{r^2}\left\{t^2\left(\frac{bw(b)}{t}\right)'' + 2t\left(\frac{bw(b)}{t}\right)'\right\}\\ &= \frac{(N-2)^2}{r^2}\left\{t^2\frac{2bw(b)}{t^3} - 2t\frac{bw(b)}{t^2}\right\} = 0,\end{aligned}$$

so that $\underline{u}(x)$ is a positive subsolution of (1) in G_b.

From Lemma 2 we see that

$$\underline{u}(x) = \frac{bw(b)}{t} \leq w(t) = \overline{u}(x)$$

for $|x| \geq b$. Hence, by means of Lemma 1, we conclude that there exists a positive solution $u(x)$ of (1) satisfying $\underline{u}(x) = u(x) = \overline{u}(x)$ on C_b and $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ through $x \in G_b$. Since $w(t)$ approaches zero as $t \to \infty$, the positive solution $u(x)$ also tends to zero as $|x| \to \infty$. This completes the proof. □

3. DISAPPEARANCE OF NONOSCILLATORY SOLUTIONS

We next give a converse theorem to Theorem 1 in some sense. To this end, we add a nonlinear perturbation of the form $h(u)/|x|^2$ to equation (6), that is, we consider the equation

$$\Delta u + p_n(x)u + \frac{h(u)}{|x|^2} = 0. \tag{16}$$

In the case that $h(u)$ satisfies

$$\frac{h(u)}{u} = \frac{\lambda_N}{\{l_{n+1}(u^2)\}^2}$$

for $u > 0$ sufficiently small, from Theorem 1 we see that there exists a nonoscillatory solution $u(x)$ of (16) satisfying $\lim_{|x|\to\infty} u(x) = 0$. However, Theorem 1 is inapplicable to the case that

$$\frac{h(u)}{u} = \frac{\mu}{\{l_{n+1}(u^2)\}^2}, \qquad \mu > \lambda_N$$

for $u > 0$ sufficiently small. As a matter of fact, all nontrivial solutions of (16) are oscillatory in this case.

Theorem 2. *Suppose that there exists an $n \in \mathbb{N}$ such that*

$$p(x) = p_n(x), \qquad x \in \Omega. \tag{17}$$

Also suppose that there exists a locally Lipschitz continuous function $h(u)$ with $uh(u) > 0$ if $u \neq 0$ such that

$$\phi(x,u) \geq \frac{h(u)}{|x|^2}, \qquad x \in \Omega, \quad u \geq 0 \tag{18}$$

and

$$\phi(x,u) \leq \frac{h(u)}{|x|^2}, \qquad x \in \Omega, \quad u < 0. \tag{19}$$

If $h(u)$ satisfies

$$\frac{h(u)}{u} \geq \frac{\mu}{\{l_{n+1}(u^2)\}^2}, \qquad \mu > \lambda_N \tag{20}$$

for $|u| > 0$ sufficiently small, then all nontrivial solutions of (1) are oscillatory.

Remark 2. *It is unnecessary to assume that $p(x)$ and $\phi(x,u)$ are locally Hölder continuous with exponent $\alpha \in (0,1)$ in Theorem 2.*

For the proof of Theorem 2, we need to prepare the following lemmas on the nonlinear differential equation associated with (1):

$$\frac{d}{dr}\left(r^{N-1}\frac{d}{dr}v\right) + r^{N-1}\left\{\frac{\lambda_N}{r^2}S_n((N-2)r^{N-2})v + \frac{1}{r^2}h(v)\right\} = 0. \tag{21}$$

Lemma 3. *If $h(u)$ satisfies (20) with $uh(u) > 0$ if $u \neq 0$, then all nontrivial solutions of (21) are oscillatory.*

Lemma 4. *Assume (17) and (18). Suppose that equation (1) has a positive solution $u(x)$ existing on $|x| \geq b$ with some $b \geq a$. Then the associated equation (21) has a positive solution $v(r)$ on $[b,\infty)$ such that*

$$0 < v(r) \leq \min_{|x|=r} u(x).$$

To prove Lemma 3, we use the oscillation theorem mentioned below. For the proof, see [8].

Proposition 2. *Assume (9) and suppose that there exists a $\nu > 1/4$ such that*

$$\frac{g(w)}{w} \geq \frac{\nu}{\{l_{n+1}(w^2)\}^2}$$

for $|w| > 0$ sufficiently small. Then all nontrivial solutions of (8) are oscillatory.

By putting $w(t) = v(r)$ and $t = (N-2)r^{N-2}$, equation (21) is transformed into equation (8) with $g(w) = h(w)/4\lambda_N$. In fact, we have

$$\begin{aligned}
&t^2w''(t) + 2tw'(t) + \frac{1}{4}S_n(t)w(t) + g(w(t)) \\
&= (t^2w'(t))' + \frac{1}{4}S_n(t)w(t) + \frac{1}{4\lambda_N}h(w(t)) \\
&= \frac{1}{(N-2)^2r^{N-3}}\frac{d}{dr}\left(r^{N-1}\frac{d}{dr}v(r)\right) + \frac{1}{4}S_n((N-2)r^{N-2})v(r) + \frac{1}{4\lambda_N}h(v(r)) \\
&= \frac{1}{4\lambda_N r^{N-3}}\left[\frac{d}{dr}\left(r^{N-1}\frac{d}{dr}v(r)\right) + r^{N-1}\left\{\frac{\lambda_N}{r^2}S_n((N-2)r^{N-2})v(r) + \frac{1}{r^2}h(v(r))\right\}\right] \\
&= 0.
\end{aligned}$$

From $wh(w) > 0$ if $w \neq 0$, we see that $g(w)$ satisfies assumption (9). Let $\nu = \mu/4\lambda_N$. Then, by (20) we obtain

$$\frac{g(w)}{w} = \frac{h(w)}{4\lambda_N w} \geq \frac{\mu}{4\lambda_N\{l_{n+1}(w^2)\}^2} = \frac{\nu}{\{l_{n+1}(w^2)\}^2}$$

with $\nu > 1/4$. Hence, from Proposition 2 we conclude that Lemma 3 is true.

Naito *et al.* [6] have shown that the existence of a positive solution for the elliptic equation $\Delta u + \psi(x, u) = 0$ implies the existence of a positive solution for its associated ordinary differential equation. In the same way, we can prove Lemma 4 which guarantees the simultaneity of positive solutions for equations (1) and (21). As space is limited, we omit the proof.

Remark 3. *Similarly, under the assumptions* (17) *and* (19), *we can show that if equation* (1) *has a negative solution on G_b with some $b \geq a$, then equation* (21) *also has a negative solution on* $[b, \infty)$.

We are now ready to prove Theorem 2.

Proof of Theorem 2. By way of contradiction, we suppose that equation (1) has a nonoscillatory solution $u(x)$ in some exterior domain. Then there exists a $b \geq a$ such that $u(x)$ is positive for $|x| \geq b$ or negative for $|x| \geq b$.

In the former case, by (17), (18) and Lemma 4, equation (21) has a positive solution $v(r)$ for $r \geq b$. On the other hand, since $h(u)$ satisfies (20) with $uh(u) > 0$ if $u \neq 0$, all nontrivial solutions of (21) are oscillatory by Lemma 3. This is a contradiction.

Noticing Remark 3, we can carry out the proof of the latter case in the same manner. We have thus proved the theorem. □

4. AN EXAMPLE

To show the value of Theorems 1 and 2, we consider the equation

$$\Delta u + \frac{\mu_1}{|x|^2} u + \phi(x, u) = 0, \qquad |x| > 1, \tag{22}$$

where μ_1 is positive, and $\phi(x, u)$ is locally Hölder continuous and satisfies

$$\phi(x, -u) = -\phi(x, u) \quad \text{for } u \in \mathbb{R}$$

and

$$\phi(x, u) = \begin{cases} \dfrac{\mu_2}{|x|^2}\left(\dfrac{3}{4}u - \dfrac{1}{2e}\right) & \text{if } u \geq \dfrac{1}{e}, \\ \dfrac{\mu_2}{|x|^2}\dfrac{u}{(\log u^2)^2} & \text{if } 0 < u < \dfrac{1}{e}, \end{cases}$$

where μ_2 is positive. Let us examine an effect of positive parameters μ_1 and μ_2 on the oscillation of solutions of (22).

Case 1. $\mu_1 > \lambda_N$. Suppose that equation (22) has a nonoscillatory solution $u(x)$ in G_b with some $b \geq 1$. We may assume that $u(x)$ is positive on G_b, because the argument of the case that $u(x)$ is negative is carry out in the same way. The positive solution $u(x)$ also satisfies the linear equation

$$\Delta u + \left(\frac{\mu_1}{|x|^2} + \frac{\phi(x, u(x))}{u(x)}\right) u = 0, \qquad |x| > b. \tag{23}$$

On the other hand, as mentioned in Section 1, all nontrivial solutions of the equation

$$\Delta u + \frac{\mu_1}{|x|^2} u = 0, \qquad |x| > 1$$

are oscillatory. Hence, from Sturm's comparison theorem, we see that all nontrivial solutions of (23) are oscillatory. This contradicts the fact that equation (23) has the positive solution $u(x)$. We therefore conclude that all nontrivial solutions of (22) are oscillatory.

Case 2. $\mu_1 \leq \lambda_N$. We can not judge whether or not all nontrivial solutions of (22) are oscillatory by means of Sturm's comparison theorem. Using Theorems 1 and 2, we give judgment on the matter. For this purpose, we classify Case 2 into three subcases as follows.

(i) $\mu_i \le \lambda_N$ for $i = 1, 2$. Since

$$\frac{\mu_1}{|x|^2} \le \frac{\lambda_N}{|x|^2} = p_1(x)$$

for $|x| > 1$, condition (13) is satisfied with $n = 1$. Let

$$h(u) = \begin{cases} \mu_2(3u/4 - 1/2e) & \text{if } u \ge 1/e, \\ \mu_2 u/(\log u^2)^2 & \text{if } 0 < u < 1/e, \\ 0 & \text{if } u = 0. \end{cases} \tag{24}$$

Then condition (14) holds and condition (15) is satisfied with $n = 1$. Hence, by Theorem 1 equation (22) has a nonoscillatory solution which decays at infinity.

(ii) $\mu_1 = \lambda_N < \mu_2$. It is clear that condition (17) is satisfied with $n = 1$. Let $h(u)$ be the odd function satisfying (24). Then conditions (18) and (19) hold. Since

$$\frac{h(u)}{u} = \frac{\mu_2}{(\log u^2)^2} = \frac{\mu_2}{\{l_2(u^2)\}^2}$$

for $|u| > 0$ sufficiently small, condition (20) is also satisfied with $n = 1$. Hence, from Theorem 2 it turns out that all nontrivial solutions of (22) are oscillatory.

(iii) $\mu_1 < \lambda_N < \mu_2$. Let $\tilde{p}(x) \equiv 0$ and

$$\tilde{\phi}(x, u) = \frac{\mu_1}{|x|^2} u + \phi(x, u).$$

We show that $\tilde{p}(x)$ and $\tilde{\phi}(x, u)$ satisfy conditions (13)–(15). Since $p_0(x) \equiv 0$, condition (13) is satisfied with $n = 0$. Define

$$h(u) = \begin{cases} \mu_1 u + \mu_2(3u/4 - 1/2e) & \text{if } u \ge 1/e, \\ \mu_1 u + \mu_2 u/(\log u^2)^2 & \text{if } 0 < u < 1/e, \\ 0 & \text{if } u = 0. \end{cases}$$

Then we have

$$0 \le \tilde{\phi}(x, u) \le \frac{h(u)}{|x|^2}$$

for $|x| > 1$ and $u \ge 0$, namely, condition (14). We also see that

$$\frac{h(u)}{u} \le \mu_1 + \frac{\mu_2}{(\log u^2)^2} < \lambda_N = \frac{\lambda_N}{\{l_1(u^2)\}^2}$$

for $u > 0$ sufficiently small. Hence, condition (15) is satisfied with $n = 0$. Thus, from Theorem 1 we see that equation (22) has a decaying nonoscillatory solution.

	$\mu_1 < \lambda_N$	$\mu_1 = \lambda_N$	$\mu_1 > \lambda_N$
$\mu_2 < \lambda_N$	Case 2(i) by Theorem 1 ($n = 1$) $^{\exists}$sol. of (22): nonosci.	Case 2(i) by Theorem 1 ($n = 1$) $^{\exists}$sol. of (22): nonosci.	Case 1 by Sturm's theorem $^{\forall}$sol. of (22): osci.
$\mu_2 = \lambda_N$	Case 2(i) by Theorem 1 ($n = 1$) $^{\exists}$sol. of (22): nonosci.	Case 2(i) by Theorem 1 ($n = 1$) $^{\exists}$sol. of (22): nonosci.	Case 1 by Sturm's theorem $^{\forall}$sol. of (22): osci.
$\mu_2 > \lambda_N$	Case 2(iii) by Theorem 1 ($n = 0$) $^{\exists}$sol. of (22): nonosci.	Case 2(ii) by Theorem 2 ($n = 1$) $^{\forall}$sol. of (22): osci.	Case 1 by Sturm's theorem $^{\forall}$sol. of (22): osci.

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