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Unipotent representations of unitary groups in four variables

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1 Introduction

Originally, "unipotent representation" stands for those representations of finite reductive groups whose character is supported on the unipotent set (due to Lusztig). Later this term is used in various fields of representation theory, including Vogan's strategy of describing the unitary dual of real reductive groups and Arthur's conjecture on the discrete automorphic spectrum. Accordingly there seems to be some variant of the definition of unipotent representations. In particular, the characters of unipotent representations of reductive groups over local fields are not supported on the unipotent orbits.

Since we shall be mainly concerned with the reductive groups over p-adic fields, we adopt the definition proposed by Arthur [Art89]. Let G be a connected reductive group defined over a p-adic field F. We fix an algebraic closure $\bar{F}$ of F so that we have the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. Recall that the *Weil group* W_F of F is a variant of

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Gal($\bar{F}/F$) which admits a sufficiently rich complex representations (as opposed to the Galois group). Writing $\widehat{G}$ for the Langlands dual group (the $\mathbb{C}$-group having the root datum dual to that of G), we have the *L-group* ${}^LG = \widehat{G} \rtimes W_F$ of G. In [Art89], Arthur proposed a conjectural parametrization of irreducible representations of $G(F)$ in terms of the homomorphisms

$$\psi : \mathcal{L}_F \times SL(2, \mathbb{C}) \longrightarrow {}^LG,$$

which we call *A-parameters*. Here $\mathcal{L}_F := W_F \times SU(2)$ is the *Langlands group* of F, a variant of the Weil-Deligne group. An irreducible representation of $G(F)$ is *unipotent* if it is parametrized by an A-parameter ψ whose restriction to $\mathcal{L}_F$ is trivial. Although this definition is modulo the conjectural parametrization, the similarity with the Jordan decomposition is obvious. That is, $\psi|_{\mathcal{L}_F}$ is considered as the semisimple part of ψ while $\psi|_{SL(2,\mathbb{C})}$ assigns a unipotent element of $\widehat{G}$ commuting with the semisimple part (by the Jacobson-Morozov theorem).

This also resembles Lusztig's classification of irreducible representations of reductive group $G(\mathbb{F}_q)$ over finite fields via the orbits in the dual group $\widehat{G}(\bar{\mathbb{F}}_q)$ [Lus84, Ch.13]. In that case, the semisimple part of a parameter is really a semisimple element (or orbit) in $\widehat{G}(\bar{\mathbb{F}}_q)$. But in the p-adic case, it is already a tempered Langlands parameter which is of great importance in the arithmetic study of automorphic representations. Thus the A-parameters of mixed type (both semisimple and unipotent parts are non-trivial) will also be worth studying. In this note, we consider the (unique) non-quasisplit unitary group with 4 variables associated to a quadratic extension E of F, and construct some examples of irreducible representations parametrized by such mixed A-parameters. Although they are not literally unipotent representations[1], they are still interesting both representation theoretically and arithmetically. I hope our misuse of the terminology does not confuse anybody.

The plan of this note is as follows. The A-parameters are expected to classify only those irreducible representations which appear as local components of certain automorphic representations. This implies that the representations parametrized will be unitarizable. In § 2, we prepare some notation on the non-quasisplit unitary group G in 4 variables over a p-adic field F, and classify the unitary dual of $G(F)$. In § 3, we construct representations parametrized by an A-parameter for $G(F)$. This relies on a detailed study of local θ-correspondence for unitary dual pairs, especially between unitary groups in two variables.

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2 The unitary dual

Here we classify the unitary dual of the non-quasisplit unitary group G in 4 variables. For this, we have to determine the reducible points of parabolically induced from cuspidal modules. When the inducing representation is generic, the Langlands-Shahidi theory

[1]Of course, the only (elliptic) unipotent representation of unitary groups is the trivial representation. The representations treated in this note are the analogues of the *quadratic unipotent representations* in the sense of Moeglin [Mœgl96].

[Sha90] allows us to reduce the problem to the determination of certain automorphic L-factors (see [Kon01]). But in the present non-quasisplit case, no such theory is available and all we can do is to relate the Plancherel measure of the induced representation to the one on the quasisplit inner form of G by Langlands' functoriality.

2.1 Preliminary

Let E/F be a quadratic extension of non-archimedean local fields of characteristic zero, and σ be the non-trivial element in $\mathrm{Gal}(E/F)$. We have the norm map $\mathrm{N}_{E/F} : E^\times \ni x \mapsto x\sigma(x) \in F^\times$, and write $\omega_{E/F}$ for the non-trivial character of $F^\times/\mathrm{N}_{E/F}(E^\times) \simeq \mathbb{Z}/2\mathbb{Z}$. As usual, $\mathcal{O} \supset \mathfrak{p}$ and $|\ |_F$ denote the ring of integers of F, its unique maximal ideal and the module of F, respectively [Wei]. Fixing an algebraic closure $\bar{F}$ of F containing E, we write $\Gamma := \mathrm{Gal}(\bar{F}/F)$, $W_F := W_{\bar{F}/F}$ for the absolute Galois and Weil groups of F, respectively [Tat79]. Similar notation for E will be used with the subscript E.

For a connected reductive group G defined over F, let us write $\Pi(G(F))$ for the set of isomorphism classes of irreducible admissible representations of $G(F)$. Also $\Pi_{\mathrm{unit}}(G(F)) \supset \Pi_{\mathrm{temp}}(G(F)) \supset \Pi_{\mathrm{disc}}(G(F)) \supset \Pi_0(G(F))$ denote the subsets of unitarizable, tempered, square integrable and (unitarizable) supercuspidal elements in $\Pi(G(F))$, respectively. Recall that an admissible representation of $G(F)$ is *supercuspidal* if its matrix coefficients are compactly supported modulo center.

Recall the classification of hermitian spaces of even dimension over E. Fix $\gamma \in F^\times \smallsetminus \mathrm{N}_{E/F}(E^\times)$ and set

$$V_2^+ := (E^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \quad V_2^- := (E^2, \begin{pmatrix} -1 & \\ & \gamma \end{pmatrix}).$$

V_2^+ is hyperbolic and V_2^- is anisotropic hermitian plane, respectively. Then $V_{2n}^\pm = V_2^\pm \oplus (V_2^+)^{n-1}$ form a complete system of representatives of $2n$-dimensional hermitian space over E. We write G_{2n} and G_{2n}^* for the unitary groups of V_{2n}^- and V_{2n}^+, respectively. To be explicit, these can be realized as the F-algebraic group, which associate to each abelian F-algebra R the groups

$$G_{2n}(R) = \left\{ g \in GL(2n, R \otimes_F E) \,\middle|\, \mathrm{Ad}\left(\begin{array}{c|c|c} & & I_{n-1} \\ \hline & \begin{matrix} -1 & \\ & \gamma \end{matrix} & \\ \hline I_{n-1} & & \end{array}\right) g = {}^t\sigma(g)^{-1} \right\}, \tag{2.1}$$

$$G_{2n}^*(R) = \{ g \in GL(2n, R \otimes_F E) \mid \mathrm{Ad}(I_{2n}) g = {}^t\sigma(g)^{-1} \}.$$

Here

$$I_n := \begin{pmatrix} & & 1 \\ & 1 & \\ \iddots & & \\ 1 & & \end{pmatrix}.$$

Note that G_{2n}^* is the quasisplit inner form of G_{2n}.

Now we turn to the case $G = G_4$. We write $P = MU$ for the unique proper parabolic subgroup, which can be identified with the upper triangular subgroup in the above realization. Thus $M \simeq \mathrm{R}_{E/F}\mathbb{G}_m \times G_2$. Let $P^* = M^*U^*$ be the corresponding standard (with respect to the upper triangular Borel subgroup in the realization (2.1)) parabolic subgroup.

An element in $\Pi_0(M(F))$ is of the form

$$\chi\otimes\tau : M(F)\ni \mathrm{diag}(a,g,\sigma(a)^{-1})\longmapsto \chi(a)\tau(g)\in GL(V),$$

with $\chi\in\Pi_{\mathrm{unit}}(E^\times)$ and (τ,V) is a realization of $\tau\in\Pi_0(G_2(F))$. For $\lambda\in\mathbb{C}$, we write $\chi[\lambda]\otimes\tau := \chi|\,|_E^\lambda\otimes\tau$. We would like to determine the reducible points of the parabolically induced representation $I_P^G(\chi[\lambda]\otimes\tau)$.

2.2 Jacquet-Langlands correspondence for G_2

Local correspondence Recall the local Shimizu-Jacquet-Langlands correspondence. Let D be the unique quaternion division algebra over F. We have the reduced norm $\nu_{D/F}$ of D, so that we can define the *characteristic polynomial* of $\gamma^D\in D^\times$ as $\nu_{D/F}(T-\gamma^D)\in F[T]$. We say γ^D is *regular semisimple* if its characteristic polynomial has two distinct roots. Also regular semisimple $\gamma\in GL(2,F)$ and $\gamma^D\in D^\times$ *correspond* to each other if their characteristic polynomials coincide.

Fact 2.1 (Jacquet-Langlands). *There exists a unique bijection* $\Pi_{\mathrm{disc}}(GL(2,F))\ni\tau\overset{\sim}{\mapsto}\tau^D\in\Pi(D^\times)$*, which is characterized by the character relation*

$$\Theta_\tau(\gamma) = -\Theta_{\tau^D}(\gamma^D)$$

for any regular semisimple $\gamma\in GL(2,F)$, $\gamma^*\in D^\times$ *correspond to each other.*

This can be easily translated into the unitary similitude group case. Let $\widetilde{G}_2 := GU(V_2^-)$ and $\widetilde{G}_2^* := GU(V_2^+)$ be the unitary similitude groups for V_2^- and V_2^+, respectively. We have the isomorphisms [KK, 5.2.1]

$$\widetilde{G}_2 \xrightarrow{\sim} (\mathrm{R}_{E/F}\mathbb{G}_m\times D^\times)/\Delta F^\times,\quad \widetilde{G}_2^* \xrightarrow{\sim} (\mathrm{R}_{E/F}\mathbb{G}_m\times GL(2,F))/\Delta F^\times$$

where Δ stands for the diagonal embedding $z\mapsto(z,z\mathbf{1}_2)$. For a character ω of the center $\widetilde{Z}_2(F)\simeq E^\times$ of $\widetilde{G}_2^*(F)$, $\widetilde{G}_2(F)$, we write $\Pi(\widetilde{G}_2(F))_\omega$ and $\Pi(\widetilde{G}_2^*(F))_\omega$ for the sets of elements in $\Pi(\widetilde{G}_2(F))$ and $\Pi(\widetilde{G}_2^*(F))$ with the central character ω, respectively. We define the Shimizu-Jacquet-Langlands correspondence

$$\Pi_{\mathrm{disc}}(\widetilde{G}_2^*(F))_\omega\ni\omega\otimes\tau\longmapsto\omega\otimes\tau^D\in\Pi(\widetilde{G}_2(F))_\omega \tag{2.2}$$

for $\widetilde{G}_2(F)$, where $\tau\leftrightarrow\tau^D$ is the correspondence in Fact 2.1.

We return to the unitary groups $G_2(F)$, $G_2^*(F)$. $\Pi(G_2^*(F))$ is partitioned into a disjoint union of the finite subsets called *L-packet*. By definition [Rog90, 11.1], each L-packet of $G_2^*(F)$ consists of the irreducible components of $\widetilde{\pi}|_{G_2^*(F)}$ for some $\widetilde{\pi}\in\Pi(\widetilde{G}_2^*(F))$. Let us write $\Phi(G_2^*)$ for the set of L-packets of $G_2^*(F)$, and $\Phi_{\mathrm{disc}}(G_2^*)$ for its subset consisting of L-packets contained in $\Pi_{\mathrm{disc}}(G_2^*(F))$. For $\Pi^*\in\Phi_{\mathrm{disc}}(G_2^*)$, we write

$$\Theta_{\Pi^*}(\gamma) := \sum_{\tau\in\Pi}\Theta_\tau$$

for its *stable character*, a locally constant function on the regular semisimple set. We also have a similar description for $G_2(F)$. Now the correspondence (2.2) gives a correspondence

$$\Phi_{\mathrm{disc}}(G_2^*) \ni \Pi^* \longleftrightarrow \Pi \in \Phi(G_2)$$

which is characterized by the character identity

$$\Theta_{\Pi^*}(\gamma^*) = -\Theta_{\Pi}(\gamma)$$

for any regular semisimple $\gamma^* \in G_2^*(F)$, $\gamma \in G_2(F)$ which are conjugate in $GL(2, E)$. We call this the *Jacquet-Langlands correspondence* for G_2.

A special global correspondence Next we move to a global situation. One can take a quadratic extension K/k of number fields such that $K_{v_i} := (K \otimes_k k_{v_i})/k_{v_i}$ is isomorphic to E/F at two non-archimedean places v_i, $(i = 1, 2)$. Write $\mathbb{A} = \mathbb{A}_k$ for the adele ring of k. Also we have an anisotropic unitary group $\underline{G}_2$ in two variables with respect to K/k such that [Clo91, § 2]

- $\underline{G}_2 \otimes_k k_{v_i} \simeq G_2$ with respect to an isomorphism $k_{v_i} \simeq F$, $(i = 1, 2)$;
- $\underline{G}_2 \otimes_k k_v$ is quasisplit at any place outside of v_1, v_2.

We write $\underline{G}_2^*$ for the quasi-split unitary group in two variables with respect to K/k. For each family $(\Pi_v)_v$ of local L-packets $\Pi_v \in \Phi(G_v)$ such that Π_v contains an unramified representation[2] at all but finite number of non-archimedean places, we can form the associated global L-packet for $\underline{G}_2(\mathbb{A})$:

$$\Pi_{\mathbb{A}} := \left\{ \bigotimes_v{}' \tau_v \;\middle|\; \begin{array}{ll} \text{(i)} & \tau_v \in \Pi_v, \text{ at } \forall v \\ \text{(ii)} & \tau_v \text{ is unramified at } \forall' v \end{array} \right\}.$$

As usual, $\bigotimes_v'$ stands for the *restricted tensor product.*

Using a standard argument on Deligne-Kazhdan's simple version trace formula, one can deduce the following from [Rog90, Ch.13] (or rather [LL79]).

Proposition 2.2. *For each $\tau \in \Pi_0(G_2(F))$, there exists a global L-packet $\Pi_{\mathbb{A}} = \bigotimes_v' \Pi_v$ of $\underline{G}_2(\mathbb{A})$ which have the following properties.*

(i) Π_{v_i} *contains τ for $i = 1, 2$.*

(ii) $\Pi_{\mathbb{A}}$ *is* automorphic, *that is, there exists an irreducible subrepresentation $\tau_{\mathbb{A}} = \bigotimes_v' \tau_v$ of $L^2(\underline{G}_2(k)\backslash\underline{G}_2(\mathbb{A}))$ such that $\tau_v \in \Pi_v$ at all v. Notice that $\tau_{\mathbb{A}}$ is automatically cuspidal since it has supercuspidal local component at v_i. Moreover $\tau_{\mathbb{A}}$ can be chosen in such a way that*

[2] A reductive group G over a non-archimedean local field F is *unramified* if it is quasisplit and split over a non-archimedean extension of F. Then $G(F)$ admits a hyperspecial maximal compact subgroup $\mathbf{K}$, which is unique up to $G(F)$-conjugation. An irreducible admissible representation of $G(F)$ is *unramified* if it contains a non-zero $\mathbf{K}$-fixed vector.

(a) $\tau_{v_i} \simeq \tau$ *for* $i = 1, 2$;

(b) *There exists an irreducible cuspidal automorphic representation* $\tau_{\mathbb{A}}^* = \bigotimes_v' \tau_v^*$ *of* $\underline{G}_2^*(\mathbb{A})$ *contained in the Jacquet-Langlands correspondent* $\Pi_{\mathbb{A}}^*$ *of* $\Pi_{\mathbb{A}}$ *such that*

   i. $\tau_v \simeq \tau_v^*$ *at any* v *other than* v_1, v_2;

   ii. *There exists a non-trivial character* $\psi_{\mathbb{A}} = \bigotimes_v \psi_v : \mathbb{A}/k \longrightarrow \mathbb{C}^\times$ *such that* $\tau_{\mathbb{A}}^*$ *is* $\psi_{\mathbb{A}}$*-generic*[3].

2.3 Lifting Plancherel measures

Going back to the local situation, we put $M(F)^1 := \bigcap_\chi \ker|\chi|_F$, where χ runs over the F-rational character group of M. Since F has a discrete valuation,

$$\widehat{A}_M := \mathrm{Hom}(M(F)/M(F)^1, \mathbb{C}^\times)$$

is a $\mathbb{C}$-torus. Let us write $\Pi_{\mathrm{disc}}(M(F)/A_M(F)) := \{\lambda \otimes \pi \mid \lambda \in \widehat{A}_M,\ \pi \in \Pi_{\mathrm{disc}}(M(F))\}$. Then $\chi \in \widehat{A}_M$ acts on this by $\tau \mapsto \lambda \otimes \tau$. Each $\widehat{A}_M$-orbit $\mathfrak{P}$ in $\Pi_{\mathrm{disc}}(M(F)/A_M(F))$ is a homogeneous space under $\widehat{A}_M$ and hence a $\mathbb{C}$-variety.

Fix such $\widehat{A}_M$-orbit $\mathfrak{P}$. Writing $\bar{P}$ for the parabolic subgroup of G opposite to P with respect to M, consider the intertwining operator

$$J_{\bar{P}|P}(\pi) : I_P^G(\pi) \longrightarrow I_{\bar{P}}^G(\pi), \quad \pi \in \mathfrak{P}.$$

This is defined by an absolutely convergent integral on an open subset of $\mathfrak{P}$, and defines a rational function on $\mathfrak{P}$ [Wal03, IV]. It is known that the induced representation $I_P^G(\pi)$ is irreducible for π in some Zariski dense subset of $\mathfrak{P}$. Hence, by Schur's lemma, there exists a rational function j on $\mathfrak{P}$ such that

$$J_{P|\bar{P}}(\pi) \circ J_{\bar{P}|P}(\pi) = j(\pi) \cdot \mathrm{Id}$$

The Plancherel measure of $\pi \in \mathfrak{P}$ is defined by [Wal03, V.2.]

$$\mu^G(\pi) := j(\pi)^{-1}\gamma(G/M)^2.$$

Here $\gamma(G/M)$ is a certain volume factor defined in [Wal03, I.1.5].

Proposition 2.3. *For* $\chi \otimes \tau \in \Pi_0(M(F))$ *and* $s \in \mathbb{C}$, *we have*

$$\gamma(G_4/M)^{-2}\mu^{G_4}(\chi[s] \otimes \tau) = \gamma(G_4^*/M^*)^{-2}\mu^{G_4^*}(\chi[s] \otimes \tau^*).$$

Here Π *is the unique L-packet of* $G_2(F)$ *containing* τ, *and* τ^* *is any element in the Jacquet-Langlands correspondent* Π^* *of* Π. *Notice that* $\mu^{G_4^*}(\chi[s] \otimes \tau^*)$ *is independent of the choice of* $\tau^* \in \Pi^*$.

[3] That is, τ_v^* admits a ψ_v-Whittaker model at each v.

Proof. (Sketch) The key point is that we have a complete functional equation for global intertwining operators

$$\phi = J_{P|\bar{P}}(\chi_{\mathbb{A}}[s] \otimes \tau_{\mathbb{A}}) \circ J_{\bar{P}|P}(\chi_{\mathbb{A}}[s] \otimes \tau_{\mathbb{A}})\phi,$$

where $\chi_{\mathbb{A}} = \bigotimes'_v \chi_v$ is an idele class character of K such that $\chi_{v_i} = \chi$, $(i = 1, 2)$ and $\tau_{\mathbb{A}}$ is as in Prop. 2.2. It is understood that the vector ϕ is chosen to have an Euler decomposition $\phi = \bigotimes'_v \phi_v$. Using the Gindikin-Karpelevich formula (as for the p-adic case, see [Lan71]) at the places where $\chi_v \otimes \tau_v$ and ϕ_v are unramified, we can write the right hand side as

$$\frac{L^S(s, \tau_{\mathbb{A}}^\vee \times \chi_{\mathbb{A}}) L^S(2s, \chi_{\mathbb{A}}|_{\mathbb{A}_k^\times}) L^S(-s, \tau_{\mathbb{A}} \times \chi_{\mathbb{A}}^{-1}) L^S(-2s, \chi_{\mathbb{A}}^{-1}|_{\mathbb{A}_k^\times})}{L^S(1-s, \tau_{\mathbb{A}} \times \chi_{\mathbb{A}}^{-1}) L^S(1-2s, \chi_{\mathbb{A}}^{-1}|_{\mathbb{A}_k^\times}) L^S(s+1, \tau_{\mathbb{A}}^\vee \times \chi_{\mathbb{A}}) L^S(2s+1, \chi_{\mathbb{A}}|_{\mathbb{A}_k^\times})} \phi^S$$
$$\otimes \bigotimes_{v \in S} j^{G_v}(\chi_v[s] \otimes \tau_v)\phi_v.$$

Here S is the finite set of places where at least one of χ_v, τ_v and ϕ_v is ramified. $L^S(s, \tau_{\mathbb{A}} \times \chi_{\mathbb{A}}^{-1})$ is the partial standard L-function of $\tau_{\mathbb{A}}$ twisted by $\chi_{\mathbb{A}}^{-1}$, and $L^S(s, \chi_{\mathbb{A}}|_{\mathbb{A}_k^\times})$ is just the (partial) Hecke L-function of $\chi_{\mathbb{A}}$. We have a similar description for $I_{P^*}^{G^*}(\chi_{\mathbb{A}}[s] \otimes \tau_{\mathbb{A}}^*)$ of Prop. 2.2:

$$\phi = \frac{L^S(s, \tau_{\mathbb{A}}^\vee \times \chi_{\mathbb{A}}) L^S(2s, \chi_{\mathbb{A}}|_{\mathbb{A}_k^\times}) L^S(-s, \tau_{\mathbb{A}} \times \chi_{\mathbb{A}}^{-1}) L^S(-2s, \chi_{\mathbb{A}}^{-1}|_{\mathbb{A}_k^\times})}{L^S(1-s, \tau_{\mathbb{A}} \times \chi_{\mathbb{A}}^{-1}) L^S(1-2s, \chi_{\mathbb{A}}^{-1}|_{\mathbb{A}_k^\times}) L^S(s+1, \tau_{\mathbb{A}}^\vee \times \chi_{\mathbb{A}}) L^S(2s+1, \chi_{\mathbb{A}}|_{\mathbb{A}_k^\times})} \phi^S$$
$$\otimes \bigotimes_{v \in S} j^{G_v}(\chi_v[s] \otimes \tau_v^*)\phi_v.$$

Notice, for example, $L^S(s, \tau_{\mathbb{A}}^{*,\vee} \times \chi_{\mathbb{A}}) = L^S(s, \tau_{\mathbb{A}}^\vee \times \chi_{\mathbb{A}})$ since $\tau_v^* = \tau_v$ at any $v \notin S$. Comparing these two formulae, we obtain the equality between the v_1, v_2-components:

$$\gamma(G_4/M)^{-4} \mu^{G_4}(\chi[s] \otimes \tau)^2 = \gamma(G_4^*/M^*)^{-4} \mu^{G_4^*}(\chi[s] \otimes \tau^*)^2.$$

This combined with the positivity of the Plancherel measure on the unitary axis yields the assertion on the unitary axis. Then this equality of rational functions on such Zariski dense subset must extends to the whole connected component. □

2.4 Composition series of induced representations

Here we determine the reducible points of the induced representations $I_P^G(\chi[\lambda] \otimes \tau)$, $(\chi \otimes \tau \in \Pi_0(M(F))$, $\lambda \in \mathbb{R})$. To describe the result, we need a classification of $\tau \in \Pi_0(G_2(F))$ in terms of the Jacquet-Langlands correspondence (§ 2.2) and Rogawski's description of $\Phi_{\mathrm{disc}}(G_2^*)$.

Supercuspidal representations of $G_2(F)$ Write G_1 for the unitary group in one variable. We conventionally write η and μ for characters of $E^\times$ such that $\eta|_{F^\times} = \mathbf{1}$ and $\mu|_{F^\times} = \omega_{E/F}$. Thus another such characters are denoted by η', μ', etc. For such η, let η_u be the character $G_1(F) \ni x/\sigma(x) \mapsto \eta(x) \in \mathbb{C}^\times$. We abbreviate the one-dimensional representation $\eta_u \circ \det$ of G_n^* (resp. G_n) as $\eta_{G_n^*}$ (resp. η_{G_n}). The elements of $\Phi_{\mathrm{disc}}(G_2^*)$ are described as follows [Rog90, § 12.1].

(i) The stable cuspidal L-packet $\Pi^*(\pi)$ consists of one irreducible cuspidal representation τ^* whose standard base change lift is a supercuspidal representation π of $GL(2,E)$.

(ii) The endoscopic L-packet $\lambda_{\bar{\mu}}^{G_2^*}(\mathbf{1},\eta)$ with $\eta \neq \mathbf{1}$ consists of two distinct irreducible cuspidal representations. This is the *endoscopic lift* of a character $\mathbf{1}\otimes\eta_u$ of $G_1(F)^2$ to $G_2^*(F)$ with respect to the L-embedding[4] [Rog90, 4.8]:

$$\lambda_{\bar{\mu}}^{G_2^*} : {}^L(G_1^2) \ni (z_1,z_2)\rtimes w \longmapsto \begin{cases} \begin{pmatrix} z_1\mu(w) & \\ & z_2\mu(w)\end{pmatrix} \times w & \text{if } w \in W_E, \\ \begin{pmatrix} & -z_1 \\ z_2 & \end{pmatrix} \rtimes w_\sigma & \text{if } w = w_\sigma \end{cases} \in {}^LG_2^*.$$

(iii) $\{\eta_{G_2^*}\delta^{G_2^*}\}$, where $\delta^{G_2^*}$ is the Steinberg representation of $G_2^*(F)$.

Recall that $\Phi(G_2)$ consists of the Jacquet-Langlands correspondent of these L-packets, we denote each of which as follows.

(i) $\Pi(\pi)$, containing only one irreducible supercuspidal representation.

(ii) $\lambda_{\bar{\mu}}^{G_2}(\mathbf{1},\eta)$ with $\eta \neq \mathbf{1}$. This consists of two distinct supercuspidal representations.

(iii) $\{\eta_{G_2}\}$. Note that η_{G_2} is supercuspidal.

Reducible points and Jordan-Hölder series The reducibility of $I_P^G(\chi[\lambda]\otimes\tau)$, an induced representation from supercuspidal module, is controlled by the behavior of its Plancherel measure as a function in λ:

Fact 2.4 (Harish-Chandra, Silberger). *(i) $I_P^G(\chi[\lambda]\otimes\tau)$ is reducible at $\lambda=0$ if and only if $\chi=\sigma(\chi)^{-1}$ and $\mu^G(\chi\otimes\tau)\neq 0$.*
(ii) [Sil80, Lem. 1.2] *$I_P^G(\chi[\lambda]\otimes\tau)$ is reducible at $\lambda=s>0$ if and only if $\chi=\sigma(\chi)^{-1}$ and $\mu^G(\chi[\lambda]\otimes\tau)$ has a pole at $\lambda=s$.*

Combining Proposition 2.3 with [Sha90, Cor. 3.6], we can calculate the Plancherel measure. The result is

$$\begin{aligned}\mu^G(\chi[s]\otimes\tau) &= \frac{\gamma(G/M)^2}{\gamma(G^*/M^*)^2}\mu^{G^*}(\chi[s]\otimes\tau^*) \\ &= \gamma(G/M)^2\varepsilon(s,\tau^{*\vee}\times\chi,\bar{\psi}_F)\varepsilon(2s,\chi|_{F^\times},\bar{\psi}_F)\varepsilon(-s,\tau^*\times\chi^{-1},\psi_F)\varepsilon(-2s,\chi|_{F^\times}^{-1},\psi_F) \\ &\times\frac{L(1-s,\tau^*\times\chi^\vee)L(1-2s,\chi|_{F^\times}^{-1})L(s+1,\tau^{*\vee}\times\chi)L(2s+1,\chi|_{F^\times})}{L(s,\tau^{*\vee}\times\chi)L(2s,\chi|_{F^\times})L(-s,\tau^*\times\chi^{-1})L(-2s,\chi|_{F^\times}^{-1})}.\end{aligned}$$

Here ψ_F is a fixed non-trivial character of F, $L(s,\chi|_{F^\times})$, $\varepsilon(s,\chi|_{F^\times},\psi_F)$ are the Hecke L and ε-factors of $\chi|_{F^\times}$. We know from [Kon01, Prop. 3.2] that

$$L(s,\tau^*\times\chi) = L_E(s,\pi\times\chi),$$

[4]For the general definition of the functorial lifting of representations, please consult [Bor79].

where $\pi \in \Pi(GL(2,E))$ is the standard base change lift of the L-packet containing τ^*. $L_E(s, \pi \times \chi)$ is the standard L-factor for $GL(2,E)$ defined in [JL70]. Finally note that the ε-factors are all exponential functions so that they do not affect the poles or zeros of the Plancherel measure. These calculations give the following result.

Theorem 2.5. *(1) $I_P^G(\chi[\lambda] \otimes \tau)$ with $\lambda \geq 0$ is irreducible unless $\chi = \sigma(\chi)^{-1}$.*
(2) $I_P^G(\mu[\lambda] \otimes \tau)$ with $\lambda \geq 0$ is irreducible except for the following cases.

(i) τ belongs to either $\Pi(\pi)$ or $\{\eta_{G_2}\}$ and $\lambda = 0$.

(ii) τ belongs to $\lambda^G_{\mu^{-1}}(\mathbf{1}, \eta)$ with $\eta \neq \mathbf{1}$ and $\lambda = 1$.

(3) $I_P^G(\eta[\lambda] \otimes \tau)$ with $\lambda \geq 0$ is irreducible except for the following cases.

(i) $\tau \not\simeq \eta_{G_2}$ and $\lambda = 1/2$.

(ii) $\tau \simeq \eta_{G_2}$ and $\lambda = 3/2$.

Since the length of $I_P^G(\chi[\lambda] \otimes \tau)$ with $\tau \in \Pi_0(G_2(F))$ is at most two [BZ77], its composition series at the reducible points are easy to describe. In fact, Langlands' classification and Casselman's criterion of square integrability [Wal03, Prop.III.1.1] yield the following complete description of composition series at the reducible points in the previous theorem.

Corollary 2.6. *Let χ, τ be as above. For $s > 0$, we write $J_P^{G_4}(\chi[s] \otimes \tau)$ for the Langlands quotient of $I_P^G(\chi[s] \otimes \tau)$.*
(i) If $\tau \in \Pi(\pi)$, we have

$$I_P^G(\mu \otimes \tau) = \tau^G(\mu, \tau)_+ \oplus \tau^G(\mu, \tau)_-.$$
$$0 \longrightarrow \delta^G(\eta, \tau) \longrightarrow I_P^G(\eta[1/2] \otimes \tau) \longrightarrow J_P^G(\eta[1/2] \otimes \tau) \longrightarrow 0,$$

where $\tau^G(\mu, \tau)_\pm \in \Pi_{\mathrm{temp}}(G(F)) \setminus \Pi_{\mathrm{disc}}(G(F))$ and $\delta^G(\eta, \tau) \in \Pi_{\mathrm{disc}}(G(F))$.
(ii) If $\tau \in \lambda_{\mu^{-1}}(\mathbf{1}, \eta)$ with $\eta \neq \mathbf{1}$, we have

$$0 \longrightarrow \delta^G(\mu, \tau) \longrightarrow I_P^G(\mu[1] \otimes \tau) \longrightarrow J_P^G(\mu[1] \otimes \tau) \longrightarrow 0,$$
$$0 \longrightarrow \delta^G(\eta', \tau) \longrightarrow I_P^G(\eta'[1/2] \otimes \tau) \longrightarrow J_P^G(\eta'[1/2] \otimes \tau) \longrightarrow 0.$$

Here $\delta^G(\mu, \tau), \delta^G(\eta', \tau) \in \Pi_{\mathrm{disc}}(G(F))$.
(iii) If $\tau = \eta_{G_2}$, we have

$$I_P^G(\mu \otimes \eta_{G_2}) = \tau^G(\mu, \eta_{G_2})_+ \oplus \tau^G(\mu, \eta_{G_2})_-,$$
$$0 \longrightarrow \delta^G(\eta', \eta_{G_2}) \longrightarrow I_P^G(\eta'[1/2] \otimes \eta_{G_2}) \longrightarrow J_P^G(\eta'[1/2] \otimes \eta_{G_2}) \longrightarrow 0,$$
$$0 \longrightarrow \delta^G(\eta, \eta_{G_2}) \longrightarrow I_P^G(\eta[3/2] \otimes \eta_{G_2}) \longrightarrow J_P^G(\eta[3/2] \otimes \eta_{G_2}) \longrightarrow 0,$$

where $\eta' \neq \eta$. Here $\tau^G(\mu, \eta_{G_2})_\pm \in \Pi_{\mathrm{temp}}(G(F)) \setminus \Pi_{\mathrm{disc}}(G(F))$ and $\delta^G(\eta', \eta_{G_2}), \delta^G(\eta, \eta_{G_2}) \in \Pi_{\mathrm{disc}}(G(F))$.

2.5 The unitary dual

Now the classification of the unitary dual of $G(F)$ is an easy task. We have only to remark that, if $I_P^G(\chi\otimes\tau)$ satisfies $\sigma(\chi)^{-1}=\chi$ but is irreducible, there appears the complementary series representations $I_P^G(\chi[\lambda]\otimes\tau)$ for λ between 0 and the reducible point. Otherwise, no complementary series appears. We summarize the result as a corollary.

Corollary 2.7. *The irreducible unitarizable representations of $G(F)$ are the followings.*
(i) Elements of $\Pi_0(G(F))$.
(ii) The non-supercuspidal square integrable modules $\delta^G(\eta,\tau)$, $(\tau\not\simeq\eta_{G_2})$, $\delta^G(\mu,\tau)$, $(\tau\in\lambda_{\bar\mu}^{G_2}(\mathbf{1},\eta)$ with $\eta\neq\mathbf{1})$, $\delta^G(\eta,\eta_{G_2})$.
(iii) The tempered non square integrable representations $\tau^G(\mu,\tau)_\pm$, (the L-packet of τ is not of the form $\lambda_{\bar\mu}^{G_2}(\mathbf{1},\eta)$), $I_P^G(\mu\otimes\tau)$, $(\tau\in\lambda_{\bar\mu}^{G_2}(\mathbf{1},\eta)$, $\eta\neq\mathbf{1})$, $I_P^G(\chi\otimes\tau)$, $(\chi|_{F^\times}\neq\omega_{E/F}$, $\tau\in\Pi_0(G_2(F)))$.
(iv) The following non-tempered unitarizable representations.

Representation	*Conditions*
$J_P^G(\mu[s]\otimes\tau)$	$\tau\in\lambda_{\bar\mu}^{G_2}(\mathbf{1},\eta)$, $\eta\neq\mathbf{1}$, $1\geq s>0$
$J_P^G(\eta[s]\otimes\eta_{G_2})$	$3/2\geq s>0$
$J_P^G(\eta[s]\otimes\tau)$	$\tau\notin\{\eta_{G_2}\}$, $1/2\geq s>0$

Remark 2.8. *For the purpose of comparison, we recall the corresponding classification for $G^*(F)$. For $\chi\otimes\tau\in\Pi_{\mathrm{disc}}(M^*(F))$ and $\lambda\geq 0$, we have [Kon01, § 4.3, § 5.3]*
(1) $I_{P^*}^{G^*}(\chi[\lambda]\otimes\tau)$ with $\lambda\geq 0$ is irreducible, if $\chi\not\simeq\sigma(\chi)^{-1}$.
(2)$I_{P^*}^{G^*}(\mu[\lambda]\otimes\tau)$ with $\lambda\geq 0$ is irreducible, unless

(i) $\tau\in\Pi^*(\pi)$ or $\{\eta_{G_2^*}\delta^{G_2^*}\}$, and $\lambda=0$.

(ii) $\tau\in\lambda_{\bar\mu}^{G_2^*}(1,\eta)$ with $\eta\neq\mathbf{1}$, and $\lambda=1$.

(3) $I_{P^*}^{G^*}(\eta[\lambda]\otimes\tau)$ with $\lambda\geq 0$ is irreducible, unless

(i) $\lambda=1/2$.

(ii) $\tau=\eta_{G_2^*}\delta^{G_2^*}$ and $\lambda=3/2$.

In this case, $I_{P^*}^{G^*}(\eta[1/2]\otimes\eta_{G_2^*}\delta^{G_2^*})$ is reducible [Kon01, Prop. 5.8]:

$$0\longrightarrow\tau^{G^*}(\eta_{G_2^*}\delta^{G_2^*})\longrightarrow I_{P^*}^{G^*}(\eta[1/2]\otimes\eta_{G_2^*}\delta^{G_2^*})\longrightarrow J_{P^*}^{G^*}(\eta[1/2]\otimes\eta_{G_2^*}\delta^{G_2^*})\longrightarrow 0.$$

Here $\tau^{G^*}(\eta_{G_2^*}\delta^{G_2^*})\in\Pi_{\mathrm{temp}}(G^*(F))\smallsetminus\Pi_{\mathrm{disc}}(G(F))$.

3 Construction of an A-packet

For the moment, let G be a general connected reductive group defined over F. An A-*parameter* is a continuous homomorphism

$$\psi:\mathcal{L}_F\times SL(2,\mathbb{C})\longrightarrow{}^LG,$$

where the Langlands group $\mathcal{L}_F$ is defined in § 1. Two A-parameters are equivalent if they are $\widehat{G}$-conjugate. Let us write $\Psi(G)$ for the set of $\widehat{G}$-conjugacy classes of A-parameters for G. According to Arthur's conjecture, there associates to each $\psi \in \Psi(G)$ a finite subset $\Pi_\psi(G) \subset \Pi_{\text{unit}}(G(F))$ called an *A-packet*. In the cases where these A-packets were determined, namely, $G = GL(n)$, $SL(n)$ and $U_{E/F}(3)$, these played a fundamental role both in local harmonic analysis and in the description of discrete automorphic spectrum. In [KK], we determined a system of candidates for the A-packets of the quasisplit unitary group $G^*(F)$ in 4 variables, by means of the global condition imposed on them. Here we examine a similar construction for its inner form $G(F)$.

3.1 The A-parameter

Since G is an inner form of G^*, the set $\Psi(G)$ coincides with $\Psi(G^*)$ which we described in [KK, Prop. 3.2]. (Precisely speaking, there should be also certain "relevance conditions" on A-parameters, so that $\Psi(G)$ might be a proper subset of $\Psi(G^*)$. But the parameter which we consider below certainly occurs in $\Psi(G)$.) Here let us consider the parameter of type (G.2.b.ii) in [*loc.cit.*, § 3.3]:

$$\psi|_{\mathcal{L}_E \times SL(2)} = \big((\eta \otimes \rho_{2,SL(2)}) \oplus (\eta' \otimes \rho_{2,SU(2)})\big) \times p_{W_E}, \quad \psi(w_\sigma) = \left(\begin{array}{c|c} & 1_2 \\ \hline 1_2 & \end{array}\right) \rtimes w_\sigma$$

with $\eta' \neq \eta$. Here $\rho_{2,SL(2)}$ (resp. $\rho_{2,SU(2)}$) is the two dimensional standard representation of $SL(2)$ (resp. $SU(2)$) and p_{W_E} denotes the projection $\mathcal{L}_E \twoheadrightarrow W_E$. Also we have fixed $w_\sigma \in W_F \setminus W_E$. In the following, we shall construct a candidate of the A-packet $\Pi_\psi(G)$ of G associated to this parameter.

Associated to this, we have a non-tempered Langlands parameter [Art89, § 4]

$$\phi_\psi|_{\mathcal{L}_E} = \big(\eta[1/2] \oplus \eta[-1/2] \oplus (\eta' \otimes \rho_{2,SU(2)})\big) \times p_{W_E}.$$

and the corresponding L-packet $\Pi_{\phi_\psi}(G) = \{J_P^G(\eta[1/2] \otimes \eta'_{G_2})\}$. Since $\Pi_\psi(G)$ contains $\Pi_{\phi_\psi}(G)$, we know that ψ occurs in $\Psi(G)$. On the other hand, the S-group

$$\mathcal{S}_\psi(G) := S_\psi(G)/S_\psi(G)^0 Z(\widehat{G})^\Gamma, \quad S_\psi(G) := \mathrm{Cent}(\psi, \widehat{G})$$

for this ψ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [KK, Prop. 3.7]. According to the conjecture, there should be a pairing

$$\langle\, , \rangle = \langle\, , |_{J_P^G(\eta[1/2]_E \otimes \eta'_{G_2})}\rangle : \mathcal{S}_\psi(G) \times \Pi_\psi(G) \longrightarrow \mathbb{C}^1.$$

We do not know if this is perfect. But if such is the case, $\Pi_\psi(G)$ has another member. Actually, we shall construct a candidate for that representation in what follows.

3.2 Local theta correspondence

Let $(V, (\, ,)) = (V_2^\pm, (\, ,)_2^\pm)$ be the 2-dimensional hermitian space over E § 2.1. Fix a generator δ of E over F such that $\Delta := \delta^2 \in F^\times$. Then any 4-dimensional skew-hermitian space over E is of the form $(W^\pm, \langle\, , \rangle^\pm) := W_2^\pm \oplus \mathbb{H}$, where

$$\mathbb{H} = (W_2^+, \langle\cdot,\cdot\rangle_2^+) := \left(E^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right), \quad (W_2^-, \langle\cdot,\cdot\rangle_2^-) := (E^2, \langle\cdot,\cdot\rangle_2^- = \delta \cdot \sigma((\cdot,\cdot)_2^-)).$$

Let W be any 2 or 4-dimensional skew-hermitian space over E. We have the corresponding unitary groups $G(V)$ and $G(W)$ for V and W, respectively. We introduce the 16-dimensional symplectic space $(\mathbb{W}, \langle\langle\,,\,\rangle\rangle)$:

$$\mathbb{W} := V \otimes_E W, \quad \langle\langle v \otimes w, v' \otimes w' \rangle\rangle := \frac{1}{2}\mathrm{Tr}_{E/F}[(v, v')\sigma(\langle w, w' \rangle)],$$

and write $Sp(\mathbb{W})$ for its symplectic group. Then $(G(V), G(W))$ is a *dual reductive pair* in $Sp(\mathbb{W})$:

$$\iota_{V,W} = \iota_W \times \iota_V : G(V) \times G(W) \ni (g, g') \longmapsto g \otimes g' \in Sp(\mathbb{W}).$$

Fixing a non-trivial character ψ_F of F, we have the *metaplectic group* of $Sp(\mathbb{W})$ which is a central extension

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp_{\psi_F}(\mathbb{W}) \longrightarrow Sp(\mathbb{W}) \longrightarrow 1.$$

This admits a unique Weil representation ω_{ψ_F} on which the subgroup $\mathbb{C}^1$ acts by the multiplication [RR93]. For a character η of $E^\times$ with $\eta|_{F^\times} = \mathbf{1}_{F^\times}$, there corresponds a lifting $\widetilde{\iota}_{V,W,\eta} : G(V) \times G(W) \to Mp_{\psi_F}(\mathbb{W})$ of $\iota_{V,W}$:

$$\begin{array}{ccc} G(V) \times G(W) & \xrightarrow{\widetilde{\iota}_{V,W,\eta}} & Mp_{\psi_F}(\mathbb{W}) \\ \| & & \downarrow \\ G(V) \times G(W) & \xrightarrow{\iota_{V,W}} & Sp(\mathbb{W}) \end{array}$$

The composite $\omega_{V,W,\eta} = \omega_{\psi_F} \circ \widetilde{\iota}_{V,W,\eta}$ is the *Weil representation* of $G(V) \times G(F)$ associated to η, a smooth representation. We write $\omega_{W,\mathbf{1}}$, $\omega_{V,\eta}$ for its restriction to $G(V)$, $G(W)$, respectively.

We write $\mathscr{R}(G(V), \omega_{W,\mathbf{1}})$ for the set of isomorphism classes of irreducible admissible representations of $G(V)$ which appear as quotients of $\omega_{W,\mathbf{1}}$. We denote $\mathcal{S}_{V,W,\eta}$ a realization of $\omega_{V,W,\eta}$. For $\tau_V \in \mathscr{R}(G(V), \omega_{W,\mathbf{1}})$, we write $\mathcal{S}_{V,W,\eta}(\tau_V)$ for the maximal quotients of $\mathcal{S}_{V,W,\eta}$ on which $G(V)$ acts by some copies of τ_V. Thus we have smooth representations $\Theta_\eta(\tau_V, W)$ of $G(W)$ such that

$$\mathcal{S}_{V,W,\eta}(\tau_V) \simeq \tau_V \otimes \Theta_\eta(\tau_V, W).$$

Similarly, we have $\mathscr{R}(G(W), \omega_{V,\eta})$ and $\Theta_\eta(\tau_W, V)$ for $\tau_W \in \mathscr{R}(G(W), \omega_{V,\eta})$. Now the local Howe duality conjecture, proved by Waldspurger if the residual characteristic of F is odd, asserts the following.

- $\Theta_\eta(\tau_V, W)$ (resp. $\Theta_\eta(\tau_W, V)$) is an admissible representation of finite length.
- $\Theta_\eta(\tau_V, W)$ (resp. $\Theta_\eta(\tau_W, V)$) admits a unique maximal submodule and hence a unique irreducible quotient $\theta_\eta(\tau_V, W)$ (resp. $\theta_\eta(\tau_W, V)$).
- $\tau_V \mapsto \theta_\eta(\tau_V, W)$ and $\tau_W \mapsto \theta_\eta(\tau_W, V)$ are bijections converse to each other between $\mathscr{R}(G(V), \omega_{W,\mathbf{1}})$ and $\mathscr{R}(G(W), \omega_{V,\eta})$.

Moreover, when $\dim_E W = 2$ we have the following. Notice that $G(V_2^+) \simeq G(W_2^+) \simeq G_2^*(F)$ and $G(V_2^-) \simeq G(W_2^-) \simeq G_2(F)$.

Theorem 3.1 (ε-dichotomy [KK], Th. 5.4). *We write $\epsilon(V_\pm) := \pm 1$.*
(i) Let Π^* be an L-packet of $G_2^*(F)$ and $\tau^* \in \Pi^*$. For a 2-dimensional hermitian space V over E, $\Theta_\eta(\tau^*, V) \neq 0$ if and only if

$$\varepsilon(1/2, \Pi^* \times \eta^{-1}, \psi_F)\omega_{\Pi^*}(-1)\lambda(E/F, \psi_F)^{-2} = \epsilon(V). \tag{3.1}$$

Here $\varepsilon(s, \Pi^* \times \omega, \psi_F)$ denotes the standard ε-factor for Π^* twisted by $\omega \in \Pi(E^\times)$ defined by the Langlands-Shahidi theory [Sha90], ω_{Π^*} is the central character of τ^* and $\lambda(E/F, \psi_F)$ is Langlands' λ-factor [Lan70].
(ii) If this is the case, we have a bijection

$$\Pi^* \ni \tau^* \longmapsto \Theta_\eta(\tau^*, V) \in \begin{cases} \eta_G \Pi^{*,\vee} & \text{if } \epsilon(V) = 1, \\ \eta_G \Pi^\vee & \text{otherwise,} \end{cases}$$

where Π is the Jacquet-Langlands correspondent of Π^*.

3.3 A candidate for the A-packet

Now we are in position to determine the partner of $J_P^G(\eta[1/2] \otimes \eta'_{G_2})$. First we apply Th. 3.1 to $\tau = \eta'_{G_2^*}\delta^{G_2^*}$. We know from [KK, § 6.1.2 (2.b.ii)] that

$$\varepsilon(1/2, \eta'_{G_2^*}\delta^{G_2^*} \times \eta^{-1}, \psi_F)\lambda(E/F, \psi_F)^{-2} = 1.$$

and hence

$$\Theta_\eta(\eta'_{G_2^*}\delta^{G_2^*}, V_2^+) = (\eta\bar{\eta}')_{G_2^*}\delta^{G_2^*}, \quad \Theta_\eta(\eta'_{G_2^*}\delta^{G_2^*}, V_2^-) = \{0\}. \tag{3.2}$$

Then this determines the local θ-correspondent of $(\eta\bar{\eta}')_{G_2} \in \Pi_0(G_2(F))$. In fact, the dichotomy property of local Howe duality for supercuspidal representations of general unitary groups over p-adic fields is established in [HKS96]. This means that exactly one of $\Theta_\eta((\eta\bar{\eta}')_{G_2}, W_2^\pm)$ is non zero. If $\Theta_\eta((\eta\bar{\eta}')_{G_2}, W_2^+) \neq \{0\}$, it must be $\eta'_{G_2^*}\delta^{G_2^*}$ since the Howe duality is a correspondence. This contradicts (3.2) so that we have

$$\Theta_\eta((\eta\bar{\eta}')_{G_2}, W_2^+) = \{0\}, \quad \Theta_\eta((\eta\bar{\eta}')_{G_2}, W_2^-) = \eta'_{G_2}.$$

Let us calculate $\theta_\eta((\eta\bar{\eta}')_{G_2}, W^-)$. For this, we use the *induction principle* for local θ-correspondence [MVW87, Ch.3 IV.4]. This tells us:

(i) Since $\Theta_\eta((\eta\bar{\eta}')_{G_2}, W_2^-)$ is non-zero, so is $\Theta_\eta((\eta\bar{\eta}')_{G_2}, W^-)$;

(ii) Its Jacquet module along P is $\eta[-1/2] \otimes \eta'_{G_2}$.

Since $J_P^G(\eta[1/2] \otimes \eta'_{G_2})$ is the unique irreducible representation of $G(F)$ with the Jacquet module $\eta[-1/2] \otimes \eta'_{G_2}$ along P, we see

$$\Theta((\eta\bar{\eta}')_{G_2}, W^-) \simeq J_P^G(\eta[1/2] \otimes \eta'_{G_2}).$$

To construct another element in $\Pi_\psi(G)$, we consider the Jacquet-Langlands correspondent $(\eta\bar{\eta}')_{G_2^*}\delta^{G_2^*}$ of $(\eta\bar{\eta}')_{G_2}$. Although $\Theta_\eta((\eta\bar{\eta}')_{G_2^*}\delta^{G_2^*}, W_2^-) = \{0\}$ (3.2), one can show that $\Theta_\eta((\eta\bar{\eta}')_{G_2^*}\delta^{G_2^*}, W^-)$ is non-trivial by a global argument. This must be the desired representation. On the other hand, the induction principle cited above asserts the following.

(a) $\Theta_\eta((\eta\bar{\eta}')_{G_2^*}\delta^{G_2^*}, W^-)$ is an admissible representation of finite length, so that this admits an irreducible quotient (but not necessarily unique).

(b-ii) Let $\bar{B}_2$ be the lower triangular Borel subgroup of G_2^* in the realization (2.1). The Jacquet module of $(\eta\bar{\eta}')_{G_2^*}\delta^{G_2^*}$ along $\bar{B}_2$ is $\eta\bar{\eta}'[-1/2]$. Then any irreducible quotient of $\Theta_\eta((\eta\bar{\eta}')_{G_2^*}\delta^{G_2^*}, W^-)$ is a quotient of $I_P^G(\eta'[-1/2]\otimes\eta_{G_2})$.

We know from the classification Cor. 2.6 that $\delta^G(\eta', \eta_{G_2})$ is the unique irreducible quotient of $I_P^G(\eta'[-1/2]\otimes\eta_{G_2})$. We conclude

$$\theta_\eta((\eta\bar{\eta}')_{G_2^*}\delta^{G_2^*}, W^-) \simeq \delta^G(\eta', \eta_{G_2})$$

and set

$$\Pi_\psi(G) := \{J_P^G(\eta[1/2]\otimes\eta'_{G_2}), \delta^G(\eta', \eta_{G_2})\}.$$

Our construction can be summarized as the following figure.

$$\begin{tikzcd}
 & & & \delta^G(\eta',\eta_{G_2}) & G(W^-)=G_4(F) \arrow[ddd, dash, "\text{Witt tower}"] \\
 & & & J_P^G(\eta[1/2]\otimes\eta'_{G_2}) & \\
G(V_2^+) & (\eta\bar{\eta}')_{G_2^*}\delta^{G_2^*} \arrow[uurr, "\theta_\eta(\bullet,W^-)"] & & & \\
G(V_2^-) & (\eta\bar{\eta}')_{G_2} \arrow[u, "\text{J-L. corr.}"] \arrow[uurr, "\theta_\eta(\bullet,W^-)"'] & & \eta'_{G_2} \arrow[ll, "\theta_\eta(\bullet,V_2^-)"'] & G(W_2)=G_2(F)
\end{tikzcd}$$

Figure 1: Construction of the A-packet

Remark 3.2. *By transposing* (η, η'), *we have another A-packet* $\Pi_{\psi'}(G) = \{J_P^G(\eta'[1/2]\otimes\eta_{G_2}), \delta^G(\eta, \eta'_{G_2})\}$ *for*

$$\psi'|_{\mathcal{A}_E} = (\eta'\otimes\rho_{2,SL(2)})\oplus(\eta\otimes\rho_{2,SU(2)}) \quad \psi'(w_\sigma) = \left(\begin{array}{c|c} & \mathbf{1}_2 \\ \hline \mathbf{1}_2 & \end{array}\right)\rtimes w_\sigma.$$

Also there is a discrete L-packet $\Pi_\varphi(G) = \{\delta^G(\eta, \eta'_{G_2}), \delta^G(\eta', \eta_{G_2})\}$ *for*

$$\varphi|_{\mathcal{L}_E} = (\eta\oplus\eta')\otimes\rho_{2,SU(2)}, \quad \varphi(w_\sigma) = \left(\begin{array}{c|c} & \mathbf{1}_2 \\ \hline \mathbf{1}_2 & \end{array}\right)\rtimes w_\sigma.$$

Let us compare these with the corresponding packets for G^*. *They were given in [KK, § 6.1.2] (2.b.ii):*

$$\Pi_\psi(G^*) = \{J_{P^*}^{G^*}(\eta[1/2]\otimes\eta'_{G_2^*}\delta^{G_2^*}), \pi_c\},$$
$$\Pi_{\psi'}(G^*) = \{J_{P^*}^{G^*}(\eta'[1/2]\otimes\eta_{G_2^*}\delta^{G_2^*}), \pi_c\},$$
$$\Pi_\varphi(G^*) = \{\delta^G(\eta, \eta'), \pi_c\}$$

Here $\pi_c := \Theta_\eta((\eta\bar{\eta}')_{G(V_-)}, W^+)$ *is an irreducible supercuspidal representation. Notice that* $\delta^G(\eta, \eta'_{G_2}) \not\simeq \delta^{G^*}(\eta', \eta_{G_2})$ *in the* G*-case but* $\delta^{G^*}(\eta, \eta') \simeq \delta^{G^*}(\eta', \eta)$ *in the* G^**-case. Thus a supercuspidal representation* π_c *appears for* $G^*(F)$*. It will be interesting to investigate the character relations between these packets. We would like to return this question in some near future.*

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