数理解析研究所講究録 1348 巻 2003 年 61-74

「Lie Theory のひろがりと新たな進展」研究集会

Fundamental solutions, Cauchy problems and Huygens principle for invariant differential operators on prehomogeneous vector spaces of commutative parabolic type

(July 23, 2003)

Masakazu Muro[1] (Gifu University)

Abstract

Huygens principle and propagation of singularity of Cauchy problems for linear invariant differential operators on symmetric or Hermitian matrix spaces are discussed in this paper. Let $P(\partial)$ be an invariant linear differential operator on a prehomogeneous vector space of commutative parabolic type. We consider a Cauchy problem of $P(\partial)$ with the initial plane that $P(\partial)$ is hyperbolic with respect to. We construct an explicit fundamental solution of $P(\partial)$ by using a Laurent expansion coefficient of the Laurent expansion of the complex power of the determinant function. As a consequence we obtain the exact support of the fundamental solution and hence we can give a necessary and sufficient condition that Huygens principle for $P(\partial)$ holds. Next we construct the fundamental solution for the Cauchy problem and give the singularity spectrum of it explicitly. Then we can obtain an accurate result on the propagation of singularity of the hyperfunction solution to the Cauchy problem.

Introduction.

The purpose of this paper is to construct explicit fundamental solutions to invariant differential operators and determine their support and singularity spectrum of them on a kind of vector space with group action. These differential operators are hyperbolic with respect to some initial planes. We prove that Huygens principle holds for these differential operators by the precise investigation of the fundamental solutions. In addition we can clarify how the singularity of the solutions to Cauchy problems with respect to the initial plane propagates by determining the singularity spectrum of the fundamental solutions.

Let us begin with an explanation of a typical examaple of hyperbolic differential operator. The most primitive hyperbolic differential operator may be the wave operator, $\Box = \partial^2/\partial t^2 + \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_n^2$, which is called "d'Alembertian". A distinguished phenomenon we observe in d'Alembertian is the Huygens principle. Namely when the dimension of the space-time is even and ≥ 4, the support of the fundamental solution of d'Alembertian concentrates on the boundary of the convex cone in the time-positive direction. We prove in this paper that similar phenomena are observed for the differential operators we are concerned. Another important problem is the description of propagation of singularity in the solutions of Cauchy problems. Since d'Alembertian is a *strongly* hyperbolic differential operator, the singularity of the solutions of Cauchy problems propagates along bicharacteristic strips of d'Alembertian (see Kashiwara, Kawai and Kimura [9, Chapter 6, §6], Duistermaat [2, §5.1]). However, the differential operators in this paper is *not* strongly hyperbolic and the singularity propagates along not only bicharacteristic strips but also other varieties. In order to see the propagation of singularity, we have to determine the singularity spectrum of the fundamental solution. Though Hörmander's theorem[6, Theorem 12.6.2 in Page 125] gives an upper estimate of the singularity spectrum, it does not give the exact singularity spectrum. We give in this paper the exact singularity spectrum of the fundamental solutions of Cauchy problems for the hyperbolic differential operators.

D'Alembertian is an invariant differential operator under the action of Lorentz group. It is natural to ask whether the same properties are valid for similar invariant differential operators. Indeed, Gårding[3]

[1]The author is supported in part by the grant-in-aid of The Mitsubishi Foundation and Grant-in-Aid for Scientific Research (B)(2)15340042 of Japan Society for the Promotion of Science.

constructed solutions for the Cauchy problem of matrix-type differential operator on the symmetric matrix space $\mathrm{Sym}_n(\mathbb{R})$ and the complex Hermitian matrix space $\mathrm{Her}_n(\mathbb{C})$ by using the approach of Riesz[18]. Gindikin[5] enlarged their calculus to more general type of cones on which Lie groups operate homogeneously and proved the Huygens principle for invariant differential operators on them systematically. On the other hand, they never mentioned about the propagation of singuarity of the Cauchy problem.

In this paper, we present more precise results on these problems by utilizing the author's results in the preceding papers in [13], [14]. The results of this paper are the followings.

1. To construct the explicit fundamental solutions of invariant differential operator $P(\partial)$ on the real symmetric matrix space $\mathrm{Sym}_n(\mathbb{R})$, the complex Hermitian matrix space $\mathrm{Her}_n(\mathbb{C})$ and the quaternion Hermitian matrix space $\mathrm{Her}_n(\mathbb{H})$ in terms of Laurent expansion coeffcients of the complex powers of the determinant functions (Theorem 7.1).
2. To determine the exact support and the singularity spectrum of the fundamental solutions of $P(\partial)$ (Theorem 7.1 and Theorem 8.1).
3. To give a necessary and sufficient condition in order that the Huygens principle holds (Corollary 7.2).
4. To give a law of the propagation of the singularity for the Cauchy problems with an initial plane which $P(\partial)$ is hyperbolic with respect to (Theorem 9.2).

The results on the exact support of the fundamental solutions of $P(\partial)$ have been partly obtained in some preceding papers. For example, Gindikin[5, p. 112, Example 2] and Atiyah, Bott, and Gårding[1, p. 181, Example 8.8] mentioned about the exact support of the fundamental solutions of invariant differential operators on $\mathrm{Her}_n(\mathbb{C})$. However the complete computations of the exact support seems to be carried out for the first time, especially on $\mathrm{Sym}_n(\mathbb{R})$ and on $\mathrm{Her}_n(\mathbb{H})$, in this paper. Our method is based on the author's results on invariant hyperfunctions ([13], [14]). We give the complete answer to the Huygens principle of the differential operators. The results on the exact singularity spectrum of the fundamental solutions of $P(\partial)$ and the propagation of singularity are derived for the first time in this paper. It is well known that the singularity spectrum propagates along the bicharacteristic strip for a *strongly* hyperbolic differential operators. However, since these operators are hyperbolic but not strongly hyperbolic, the singularity specrum of the hyperfunction solutions propagates not only along the bicharacteristic strips. In fact, we can observe that the singularity spectrum propagates along the varieties which does not consists of bicharacteristic strips. On the other hand, we can give examples of hyperbolic but not-stlongly hyperbolic differential operator whose singularity of solutions propagates along the bicharacteristic strip(Corollary 9.3).

1 Fundamental solutions of hyperbolic equations.

Let $\boldsymbol{V} := \mathbb{R}^m$ be an m-dimensional real vector space with a linear coordinate $x = (x_1, \dots, x_m)$. We denote by ∂_i the partial derivative $\frac{\partial}{\partial x_i}$ with respect to the variable x_i. We define a monomial of ∂_i's by $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$ with $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_{\geq 0}^m$. We define the degrees of multi-index by $|\alpha| := \alpha_1 + \cdots + \alpha_m$. A differential operator of constant coefficients on $\boldsymbol{V}$ is a polynomial of ∂_i's, i.e., a linear combination of monomials of ∂_i's. We say that $P(\partial)$ is *homogeneous* if all the monomials in $P(\partial)$ have the same degree. The degree is called the homogeneous degree of $P(\partial)$. We denote $\xi = (\xi_1, \dots, \xi_m)$ the dual coordinate of the dual vector space $\boldsymbol{V}^*$ and $P(\xi)$ is a polynomial on $\boldsymbol{V}^*$. For a differential operator $P(\partial)$, we say that a distribution $E(x)$ is a *fundamental solution* of $P(\partial)$ if it satisfies $P(\partial)E(x) = \delta(x)$. Here $\delta(x)$ denotes the Dirac's delta function on $\boldsymbol{V}$ with respect to the coordinate $(x_1, \dots, x_m)$.

Definition 1.1 (homogeneous hyperbolic differential operator). Let $P(\partial)$ be a homogeneous differential operator with constant coefficients. Let $N_\vartheta := \{x \in \boldsymbol{V} \mid \langle x, \vartheta \rangle = 0\}$ for $\vartheta \in \boldsymbol{V}^*$. We say that

$P(\partial)$ is *hyperbolic* with respect to N_ϑ if $P(\vartheta) \neq 0$ and the algebraic equation $P(\xi + \tau\vartheta) = 0$ in τ has only real roots for all $\xi \in \boldsymbol{V}^*$. In particular, we say that $P(\partial)$ is *strongly hyperbolic* if all the roots of $P(\xi + \tau\vartheta) = 0$ are distinct for any $\xi \in \boldsymbol{V}^*$ satisfying $\xi \neq c\vartheta$ with a constant c.

For a homogeneous hyperbolic differential operator $P(\partial)$, we denote by $\Gamma(P, \vartheta)$ the connected component of $\{\xi \in \boldsymbol{V}^* \mid P(\xi) \neq 0\}$ containing ϑ. This becomes a convex cone in $\boldsymbol{V}^*$ (see Hörmander[6, Page 120]). We define the dual cone of $\Gamma(P, \vartheta)$ by $\{x \in \boldsymbol{V} \mid \langle x, \theta \rangle \geq 0 \text{ for all } \theta \in \Gamma(P, \vartheta)\}$ and denote it by $\Gamma^\circ(P, \vartheta)$. The dual cone $\Gamma^\circ(P, \vartheta)$ is a closed convex cone in $\boldsymbol{V}$.

When a hyperfunction $E(x)$ satisfies the differential equation $P(\partial)E(x) = \delta(x)$ for a given differential operator $P(\partial)$, we call $E(x)$ *a fundamental solution* of $P(\partial)$. The following proposition about the support of the fundamental solution of hyperbolic differential operators is well known. See Hörmander[6, Theorem 12.5.1 in Page 120].

Proposition 1.1 (Unique fundamental solutions of hyperbolic equation). *Let $P(\partial)$ be a homogeneous hyperbolic differential operator with respect to N_ϑ. Then there exists one and only one fundamental solution $E(x)$ of $P(\partial)$ with support in the half space $H_\vartheta := \{x \in \boldsymbol{V} \mid \langle x, \vartheta \rangle \geq 0\}$. The support of this fundamental solution is contained in the dual cone $\Gamma^\circ(P, \vartheta)$.*

However, the support of $E(x)$ does not always coincide with $\Gamma^\circ(P, \vartheta)$. We often observe that the support of $E(x)$ is contained in the boundary set of $\Gamma^\circ(P, \vartheta)$ and the dimension of the support is sometimes very small. In such cases we say that $P(\partial)$ satisfies the *Huygens principle*. In particular, if the dimension of $\mathbf{Supp}(E(x))$ is strictly less than $m - 1$, we say that $P(\partial)$ satisfies the *strong Huygens principle*.

If $P(\partial)$ is hyperbolic with respect to the initial plane N_ϑ, then it is also hyperbolic with respect to $N_{-\vartheta} = N_\vartheta$. Therefore there also exists one and only one fundamental solution $E'(x)$ of $P(\partial)$ with support in the half space $H_{-\vartheta} := \{x \in \boldsymbol{V} \mid \langle x, -\vartheta \rangle \geq 0\}$. In particular, let $P(\partial)$ be a homogeneous hyperbolic operator of degree n on $\mathbb{R}^m$ and let $E(x)$ be the unique fundamental solution supported in H_ϑ. Then the unique fundamental solution $E'(x)$ supported in $H_{-\vartheta}$ is given by $E'(x) = (-1)^{n+m}E(-x)$ since $P(\partial)(-1)^{m+n}E(-x) = (-1)^m P(-\partial)E(-x) = (-1)^m \delta(-x) = \delta(x)$.

2 Singularity spectrum of the fundamental solution.

Let $\mathcal{B}_{\boldsymbol{V}}$ be the sheaf of hyperfunctions on $\boldsymbol{V}$ and let $\mathcal{C}_{\boldsymbol{V}}$ be the sheaf of microfunctions on the cotangent bundle $T^*\boldsymbol{V}$ of $\boldsymbol{V}$. We have a natural isomorphism $\mathbf{sp}$:

$$\mathbf{sp} : \mathcal{B}_{\boldsymbol{V}} \xrightarrow{\sim} \pi_*(\mathcal{C}_{\boldsymbol{V}}) \tag{1}$$

and an exact sequence

$$0 \longrightarrow \mathcal{A}_{\boldsymbol{V}} \longrightarrow \mathcal{B}_{\boldsymbol{V}} \longrightarrow \pi_*(\mathcal{C}_{\boldsymbol{V}}|_{T^*\boldsymbol{V} - \boldsymbol{V} \times \{0\}}) \longrightarrow 0. \tag{2}$$

Here, π is the projection map from the cotangent bundle $T^*\boldsymbol{V}$ to $\boldsymbol{V}$ and $\mathcal{A}_{\boldsymbol{V}}$ is the sheaf of real analytic functions on $\boldsymbol{V}$. By the isomorphism (1), we can regard a hyperfunction $f(x)$ on $\boldsymbol{V}$ as a microfunction $\mathbf{sp}(f(x))$ on $T^*\boldsymbol{V}$. In this article, we often identify the hyperfunction $f(x)$ on $\boldsymbol{V}$ with the microfunction $\mathbf{sp}(f(x))$ on $T^*\boldsymbol{V}$ through the isomorphism (1). In particular, we call the set $\mathbf{Supp}(\mathbf{sp}(f(x))) - \boldsymbol{V} \times \{0\}$ the *singularity spectrum* of the hyperfunction $f(x)$ and denote it by $S.S.(f(x))$.

Hörmander[6, Theorem 12.6.2 in Page 125] gave an estimate of the singularity spectrum (analytic wave front set) of the fundamental solution of a hyperbolic differential operator. Let $P(\partial)$ be a homogeneous hyperbolic differential operator with respect to N_ϑ. For a fixed $\xi \neq 0$ in $\boldsymbol{V}^*$, we denote by P_ξ the lowest order homogeneous part in the Taylor expansion $\eta \to P(\xi + \eta)$. Then P_ξ is also hyperbolic with respect to N_ϑ (Hörmander[7, Theorem 8.7.2]). Then we have the following estimate of the singularity spectrum of the fundamental solution $E(x)$.

Theorem 2.1 (Hörmander[6]). *Let $E(x)$ be a fundamental solution of a hyperbolic differential operator $P(\partial)$ with support in $\Gamma^\circ(P,\vartheta)$. Then we have*

$$S.S.(E(x)) \subset \{(x,\xi) \in T^*\boldsymbol{V} \mid \xi \neq 0 \text{ and } x \in \Gamma^\circ(P_\xi,\vartheta)\}.$$

3 Cauchy Problems for hyperbolic equation.

In this section we denote $x_1 = \langle x, \vartheta\rangle$ and by ∂_1 the partial derivative with respect to x_1. We denote by $x' := (x_2, \dots, x_m)$ another coordinate and by $\partial' := (\partial_2, \dots, \partial_m)$ the partial derivatives with respect to $x' := (x_2, \dots, x_m)$. Let $P(\partial)$ be a hyperbolic differential operator with respect to the initial plane $N_\vartheta := \{x \in \boldsymbol{V} \mid \langle x, \vartheta\rangle = 0\}$. Let l be the order of $P(\partial)$. Then we can write

$$P(\partial) = p_0\partial_1^l + p_1(\partial')\partial_1^{l-1} + \cdots + p_{l-1}(\partial')\partial_1 + p_l(\partial'),$$

where p_0 is a non-zero constant and $p_1(\partial'), \dots, p_l(\partial')$ are differential operators in ∂'. The *Cauchy problem for $P(\partial)$ with respect to N_ϑ* is the following problem: for a given initial data of hyperfunctions with compact support $v_0(x'), \dots, v_{l-1}(x')$ on N_ϑ, construct a hyperfunction solution to the differential equation

$$\begin{aligned} &P(\partial)u(x) = 0 \\ &u(x)|_{x_1=0} = v_0(x'), \partial_1 u(x)|_{x_1=0} = v_1(x'), \dots, \partial_1^{l-1}u(x)|_{x_1=0} = v_{l-1}(x'). \end{aligned} \tag{3}$$

For a hyperbolic differential operator $P(\partial)$, there exists a unique local hyperfunction solution to the Cauchy problem (3). To prove this, we have only to construct a fundamental solution to the Cauchy problem of hyperbolic differential operator $P(\partial)$.

We define the *fundamental solution* for the Cauchy problem. Let $F_0(x)$ be a hyperfunction solution to the Cauchy problem

$$\begin{aligned} &P(\partial)F_0(x) = 0 \\ &F_0(x)|_{x_1=0} = \partial_1 F_0(x)|_{x_1=0} = \cdots = \partial_1^{l-2}F_0(x)|_{x_1=0} = 0, \\ &\partial_1^{l-1}F_0(x)|_{x_1=0} = \delta(x'). \end{aligned} \tag{4}$$

Then $F_0(x)$ is uniquely determined by virtue of the Holmgren's uniqueness theorem if it exists. We put

$$F_k(x) := \frac{1}{p_0}(p_0\partial_1^k + p_1(\partial')\partial_1^{k-1} + \cdots + p_k(\partial'))F_0(x) \tag{5}$$

for $k = 0, 1, \dots, l-1$.

Definition 3.1 (fundamental solution to the Cauchy problem). The l-tuple of hyperfunctions

$$(F_0(x), F_1(x), \dots, F_{l-1}(x)) \in \mathcal{B}(\boldsymbol{V})^l \tag{6}$$

is called the *fundamental solution to the Cauchy problem* (3). In fact,

$$u(x) = \sum_{k=0}^{l-1} \int_{N_\vartheta} F_{l-k-1}(x_1, x' - y')v_k(y')dy' \tag{7}$$

satisfies (3).

The fundamental solution to the Cauchy problem can be constructed by using the fundamental solutions supported in the convex cones in the following way. Let $\theta(x_1)$ be the Heaviside function in x_1. Then it is proved that $\frac{1}{p_0}\theta(x_1)F_0(x)$ is a fundamental solution of $P(\partial)$ supported in H_ϑ, which is uniquely determined, and hence we have $\frac{1}{p_0}\theta(x_1)F_0(x) = E(x)$. On the other hand, we can prove that the solution $F_0(x)$ is given by

$$F_0(x) := p_0(E(x) - E'(x)) \tag{8}$$

where $E(x)$ and $E'(x)$ are fundamental solutions of $P(\partial)$ whose supports are contained in H_ϑ and $H_{-\vartheta}$, respectively. Since $\mathbf{Supp}(E(x)) \subset \Gamma^\circ(P,\vartheta)$ and $\mathbf{Supp}(E'(x)) \subset \Gamma^\circ(P,-\vartheta)$, we have

$$\mathbf{Supp}(F_k(x)) \subset \Gamma^\circ(P,\vartheta) \cup \Gamma^\circ(P,-\vartheta)$$

for $k =, 1, \dots, l-1$. This means that the support of the fundamental solution $\{F_0(x), F_1(x), \dots, F_{l-1}(x)\}$ to the Cauchy problem for $P(\partial)$ with respect to the initial plane N_ϑ is contained in $\Gamma^\circ(P,\vartheta)\cup\Gamma^\circ(P,-\vartheta)$.

4 Prehomogeneous vector spaces of commutative parabolic type and their properties.

The prehomogeneous vector spaces we are considering here are the following ones.

1. **real symmetric matrix space** Let $\boldsymbol{V} := \mathrm{Sym}_n(\mathbb{R})$ be the space of $n \times n$ symmetric matrices over the real field $\mathbb{R}$ and let $\boldsymbol{G} := \mathbf{GL}_n(\mathbb{R})$ be the general linear group over $\mathbb{R}$ of degree n. Then the group $\mathbf{GL}_n(\mathbb{R})$ acts on the vector space $\boldsymbol{V}$ by the representation

$$\rho(g) : x \longmapsto g \cdot x \cdot {}^t g, \tag{9}$$

with $x \in \boldsymbol{V}$ and $g \in \mathbf{GL}_n(\mathbb{R})$. Then the subgroup

$$\boldsymbol{G}^1 := \{g \in \mathbf{GL}_n(\mathbb{R}) \mid \det(g \cdot {}^t g) = 1\} \tag{10}$$

acts on $\boldsymbol{V}$ naturally. Here ${}^t g$ means the transposed matrix of g. In the case of symmetric matrix space, we define the coordinate x of $\mathrm{Sym}_n(\mathbb{R})$ by $x = (x_{ij})_{n\geq i,j\geq 1} \in \boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R})$ with $x_{ij} = x_{ji}$. The derivation with respect to the coordinate is defined by

$$\partial := (\partial_{ij})_{n\geq i,j\geq 1} = \left(\frac{\partial}{\partial x_{ij}}\right)_{n\geq i,j\geq 1}$$

and

$$\partial^* = (\partial^*_{ij}) = \left(\epsilon_{ij}\frac{\partial}{\partial x_{ij}}\right), \text{ with } \epsilon_{ij} := \begin{cases} 1 & i = j \\ 1/2 & i \neq j \end{cases} \tag{11}$$

The dual coordinate is given by $\xi = (\xi_{ij})_{n\geq i,j\geq 1} \in \boldsymbol{V}^* = \mathrm{Sym}_n(\mathbb{R})$ and we denote $\xi^* = (\xi^*_{ij}) = (\epsilon_{ij}\xi_{ij})$. The canonical bilinear form on $(x,\xi) \in \boldsymbol{V}\times\boldsymbol{V}^*$ is given by $\langle x,\xi\rangle := \mathrm{tr}(x\xi^*) = \sum_{n\geq j\geq i\geq 1} x_{ij}\xi_{ij}$.

2. **complex Hermitian matrix space** Let $\boldsymbol{V} := \mathrm{Her}_n(\mathbb{C})$ be the space of $n \times n$ Hermitian matrices over the complex field $\mathbb{C}$ and let $\boldsymbol{G} := \mathbf{GL}_n(\mathbb{C})$ be the special linear group over $\mathbb{R}$ of degree n. Then the group $\mathbf{GL}_n(\mathbb{C})$ acts on the vector space $\boldsymbol{V}$ by the representation

$$\rho(g) : x \longmapsto g \cdot x \cdot {}^t\overline{g}, \tag{12}$$

with $x \in \boldsymbol{V}$ and $g \in \mathbf{GL}_n(\mathbb{C})$. Then the subgroup

$$\boldsymbol{G}^1 := \{g \in \mathbf{GL}_n(\mathbb{C}) \mid \det(g \cdot {}^t\overline{g}) = 1\} \tag{13}$$

acts on $\boldsymbol{V}$ naturally. Here ${}^t\overline{g}$ means the transposed matrix of the complex conjugate of g. We define the coordinate x of $\mathrm{Her}_n(\mathbb{C})$ by $x = (x_{ij})_{n\geq i,j\geq 1} \in \boldsymbol{V} = \mathrm{Her}_n(\mathbb{C})$ with $x_{ij} = x^1_{ij} + \sqrt{-1}x^2_{ij}$ and

$x_{ji} = \overline{x_{ij}} = x^1_{ij} - \sqrt{-1}x^2_{ij}$. In particular, $x^2_{ii} = 0$ and hence $x_{ii} = x^1_{ii}$. The derivation with respect to the coordinate is defined by

$$\partial := (\partial_{ij})_{n \geq i,j \geq 1} = \left(\frac{\partial}{\partial x^1_{ij}} + \sqrt{-1}\frac{\partial}{\partial x^2_{ij}} \right)_{n \geq i,j \geq 1}$$

with $\partial_{ji} = \overline{\partial_{ij}} = \frac{\partial}{\partial x^1_{ij}} - \sqrt{-1}\frac{\partial}{\partial x^2_{ij}}$. We denote $\partial^* = (\partial^*_{ij}) = \left(\epsilon_{ij}\frac{\partial}{\partial x_{ij}}\right)$, by using ϵ_{ij} defined in (11). The dual coordinate is given by $\xi = (\xi_{ij})_{n \geq i,j \geq 1} \in \boldsymbol{V}^* = \mathrm{Her}_n(\mathbb{C})$ with $\xi_{ij} = \xi^1_{ij} + \sqrt{-1}\xi^2_{ij}$ and $\xi_{ji} = \overline{\xi_{ij}}$. We denote $\xi^* = (\xi^*_{ij}) = (\epsilon_{ij}\xi_{ij})$. The canonical bilinear form on $(x, \xi) \in \boldsymbol{V} \times \boldsymbol{V}^*$ is given by $\langle x, \xi \rangle := \Re(\mathrm{tr}(x\xi^*)) = \sum_{n \geq i \geq 1} x^1_{ii}\xi^1_{ii} + \sum_{n \geq j > i \geq 1} x^1_{ij}\xi^1_{ij} + x^2_{ij}\xi^2_{ij}$.

3. **quaternion Hermitian matrix space** Let $\boldsymbol{V} := \mathrm{Her}_n(\mathbb{H})$ be the space of $n \times n$ Hermitian matrices over the quaternion field $\mathbb{H}$ and let $\boldsymbol{G} := \mathbf{GL}_n(\mathbb{H})$ be the general linear group over $\mathbb{H}$ of degree n. Then the group $\mathbf{GL}_n(\mathbb{H})$ acts on the vector space $\boldsymbol{V}$ by the representation

$$\rho(g) : x \longmapsto g \cdot x \cdot {}^t\overline{g}, \tag{14}$$

with $x \in \boldsymbol{V}$ and $g \in \mathbf{GL}_n(\mathbb{H})$. Then the subgroup

$$\boldsymbol{G}^1 := \{g \in \mathbf{GL}_n(\mathbb{H}) \mid \det(g \cdot {}^t\overline{g}) = 1\} \tag{15}$$

acts on $\boldsymbol{V}$ naturally. Here ${}^t\overline{g}$ means the transposed matrix of the quaternion conjugate of g. We define the coordinate x of $\mathrm{Her}_n(\mathbb{H})$ by $x = (x_{ij})_{n \geq i,j \geq 1} \in \boldsymbol{V} = \mathrm{Her}_n(\mathbb{H})$ with $x_{ij} = x^1_{ij} + \mathbf{i}x^2_{ij} + \mathbf{j}x^3_{ij} + \mathbf{k}x^4_{ij} \in \mathbb{H}$ and $x_{ji} = \overline{x_{ij}} = x^1_{ij} - \mathbf{i}x^2_{ij} - \mathbf{j}x^3_{ij} - \mathbf{k}x^4_{ij}$. Here $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are the imaginary units of the quaternion and satisfy the relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ijk} = -1$. In particular, $x^2_{ii} = x^3_{ii} = x^4_{ii} = 0$ and hence $x_{ii} = x^1_{ii}$. The derivation with respect to the coordinate is defined by

$$\partial := (\partial_{ij})_{n \geq i,j \geq 1} = \left(\frac{\partial}{\partial x^1_{ij}} + \mathbf{i}\frac{\partial}{\partial x^2_{ij}} + \mathbf{j}\frac{\partial}{\partial x^3_{ij}} + \mathbf{k}\frac{\partial}{\partial x^4_{ij}} \right)_{n \geq i,j \geq 1}$$

and $\partial_{ji} = \overline{\partial_{ij}} = \frac{\partial}{\partial x^1_{ij}} - \mathbf{i}\frac{\partial}{\partial x^2_{ij}} - \mathbf{j}\frac{\partial}{\partial x^3_{ij}} - \mathbf{k}\frac{\partial}{\partial x^4_{ij}}$. We denote $\partial^* = (\partial^*_{ij}) = \left(\epsilon_{ij}\frac{\partial}{\partial x_{ij}}\right)$, by using ϵ_{ij} defined in (11). The dual coordinate is given by $\xi = (\xi_{ij})_{n \geq i,j \geq 1} \in \boldsymbol{V}^* = \mathrm{Her}_n(\mathbb{H})$ with $\xi_{ij} = \xi^1_{ij} + \mathbf{i}\xi^2_{ij} + \mathbf{j}\xi^3_{ij} + \mathbf{k}\xi^4_{ij}$ and $\xi_{ji} = \overline{\xi_{ij}}$. We denote $\xi^* = (\xi^*_{ij}) = (\epsilon_{ij}\xi_{ij})$. The canonical bilinear form on $(x, \xi) \in \boldsymbol{V} \times \boldsymbol{V}^*$ is given by $\langle x, \xi \rangle := \Re(\mathrm{tr}(x\xi^*)) = \sum_{n \geq i \geq 1} x^1_{ii}\xi^1_{ii} + \sum_{n \geq j > i \geq 1} x^1_{ij}\xi^1_{ij} + x^2_{ij}\xi^2_{ij} + x^3_{ij}\xi^3_{ij} + x^4_{ij}\xi^4_{ij}$.

We can define the determinant of a symmetric matrix or a complex Hermitian matrix but the determinant of a quaternion Hermitian matrix is not well defined since $\mathbb{H}$ is not commutative. It is defined in the following way. Note that we can write

$$z = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d = (a + \mathbf{i}b) + \mathbf{j}(c + \mathbf{i}d) = \alpha + \mathbf{j}\beta$$

with $\alpha = a + \mathbf{i}b$ and $\beta = c + \mathbf{i}d$. Then we can regard $\mathbb{H}$ as the algebra $\mathbb{C} \oplus \mathbf{j}\mathbb{C}$. Consider the algebra homomorphism ι from $\mathbb{H}$ to $\mathrm{M}_2(\mathbb{C})$ by

$$\iota \ : \ z = \alpha + \mathbf{j}\beta \ \longmapsto \begin{bmatrix} \alpha, & \overline{\beta} \\ -\beta, & \overline{\alpha} \end{bmatrix} \tag{16}$$

Let $X = (z_{i,j}) \in \mathrm{Her}_n(\mathbb{H})$ be an $n \times n$ quaternion Hermitian matrix. By the homomorphism ι in (16), X is mapped in $\mathrm{M}_{2n}(\mathbb{C})$ by

$$X \ \longmapsto \ \iota(X \cdot \mathbf{j}) = (\iota(z_{i,j} \cdot \mathbf{j})) \tag{17}$$

Since $-{}^t(\iota(X \cdot \mathbf{j})) = \iota(X \cdot \mathbf{j})$, we see that $\iota(X \cdot \mathbf{j})$ is an alternating matrix. Then by putting

$$\det(X) = \mathrm{Pf}(\iota(X \cdot \mathbf{j})) \tag{18}$$

we can define the determinant for the quaternion Hermitian matrix X. Here $\mathrm{Pf}(A)$ means the Pfaffian of an alternating matrix A.

We denote $P(x) := \det(x)$ and we put $\boldsymbol{S} := \{x \in \boldsymbol{V} | \det(x) = 0\}$. We call the set $\boldsymbol{S}$ the *singular set* of $\boldsymbol{V}$. The subset $\boldsymbol{V} - \boldsymbol{S}$ decomposes into $n+1$ connected components,

$$\boldsymbol{V}_i := \begin{cases} \{x \in \mathrm{Sym}_n(\mathbb{R}) \mid \mathrm{sgn}(x) = (i, n-i)\} & \text{if } \boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R}), \\ \{x \in \mathrm{Her}_n(\mathbb{C}) \mid \mathrm{sgn}(x) = (2i, 2(n-i))\} & \text{if } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{C}), \\ \{x \in \mathrm{Her}_n(\mathbb{H}) \mid \mathrm{sgn}(x) = (4i, 4(n-i))\} & \text{if } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{H}), \end{cases} \tag{19}$$

with $i = 0, 1, \ldots, n$. The vector space $\boldsymbol{V}$ decomposes into a finite number of $\boldsymbol{G}$-orbits;

$$\boldsymbol{V} := \bigsqcup_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n-i}} \boldsymbol{S}_i^j \tag{20}$$

where

$$\boldsymbol{S}_i^j := \begin{cases} \{x \in \mathrm{Sym}_n(\mathbb{R}) \mid \mathrm{sgn}(x) = (j, n-i-j)\} & \text{if } \boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R}) \\ \{x \in \mathrm{Her}_n(\mathbb{C}) \mid \mathrm{sgn}(x) = (2j, 2(n-i-j))\} & \text{if } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{C}) \\ \{x \in \mathrm{Her}_n(\mathbb{H}) \mid \mathrm{sgn}(x) = (4j, 4(n-i-j))\} & \text{if } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{H}) \end{cases} \tag{21}$$

with integers $0 \leq i \leq n$ and $0 \leq j \leq n-i$. Here, $\mathrm{sgn}(x)$ for $x \in \mathrm{Sym}_n(\mathbb{R})$ is the signature of the quadratic form $q_x(\vec{v}) := {}^t\vec{v} \cdot x \cdot \vec{v}$ on $\vec{v} \in \mathbb{R}^n$ and $\mathrm{sgn}(x)$ for $x \in \mathrm{Her}_n(\mathbb{C})$ (resp. $x \in \mathrm{Her}_n(\mathbb{H})$) is the signature of the quadratic form $q_x(\vec{v}) := {}^t\bar{\vec{v}} \cdot x \cdot \vec{v}$ on $\vec{v} \in \mathbb{C}^n$ (resp. $\vec{v} \in \mathbb{H}^n$). It is clear that $\boldsymbol{V}_i = \boldsymbol{S}_0^i$ from the definition. All orbits in $\boldsymbol{S}$ are $\boldsymbol{G}^1$-orbits. A $\boldsymbol{G}^1$-orbit in $\boldsymbol{S}$ is called a *singular orbit*. The subset $\boldsymbol{S}_i := \{x \in \boldsymbol{V} \mid \mathrm{rank}(x) = n-i\}$ is the set of elements of rank $n-i$. It is easily seen that $\boldsymbol{S} := \bigsqcup_{1 \leq i \leq n} \boldsymbol{S}_i$ and $\boldsymbol{S}_i = \bigsqcup_{0 \leq j \leq n-i} \boldsymbol{S}_i^j$.

The strata $\{\boldsymbol{S}_i^j\}_{0 \leq i \leq n, 0 \leq j \leq n-i}$ have the following closure inclusion relation

$$\overline{\boldsymbol{S}_i^j} \supset \boldsymbol{S}_{i+1}^{j-1} \cup \boldsymbol{S}_{i+1}^j, \tag{22}$$

where $\overline{\boldsymbol{S}_i^j}$ is the *closure* of the stratum $\boldsymbol{S}_i^j$. In particular, we have

$$\begin{aligned} \overline{\boldsymbol{V}_0} &= \overline{\boldsymbol{S}_0^0} = \boldsymbol{S}_0^0 \sqcup \boldsymbol{S}_1^0 \sqcup \cdots \sqcup \boldsymbol{S}_n^0 \\ \overline{\boldsymbol{V}_n} &= \overline{\boldsymbol{S}_0^n} = \boldsymbol{S}_0^n \sqcup \boldsymbol{S}_1^{n-1} \sqcup \cdots \sqcup \boldsymbol{S}_n^0 \end{aligned} \tag{23}$$

and

$$\begin{aligned} \overline{\boldsymbol{S}_i^0} &= \boldsymbol{S}_i^0 \sqcup \boldsymbol{S}_{i+1}^0 \sqcup \cdots \sqcup \boldsymbol{S}_n^0 \\ \overline{\boldsymbol{S}_i^{n-i}} &= \boldsymbol{S}_i^{n-i} \sqcup \boldsymbol{S}_{i+1}^{n-i-1} \sqcup \cdots \sqcup \boldsymbol{S}_n^0 \end{aligned} \tag{24}$$

We denote by $\boldsymbol{V}^*$ the dual vector space of $\boldsymbol{V}$. We define the inner product $\langle x, y \rangle$ on $(x, y) \in \boldsymbol{V} \times \boldsymbol{V}$ by $\langle x, y \rangle := \Re(\mathrm{tr}(xy))$ where $\Re$ and tr denote the real part and the trace, respectively. Then we can identify $\boldsymbol{V}$ and $\boldsymbol{V}^*$. The group $\boldsymbol{G}$ operates on $\boldsymbol{V}^*$ by the contragredient action and the $\boldsymbol{G}$-orbits in $\boldsymbol{V}^*$ are the same as the ones in $\boldsymbol{V}$. The cotangent bundle $T^*\boldsymbol{V}$ of $\boldsymbol{V}$ can be identified with $\boldsymbol{V} \times \boldsymbol{V}^*$.

5 Invariant differential operators on prehomogeneous vector spaces.

Proposition 5.1 (hyperbolic operator). *Let $\boldsymbol{V}$ be one of the vector spaces* $\mathrm{Sym}_n(\mathbb{R})$, $\mathrm{Her}_n(\mathbb{C})$ *and* $\mathrm{Her}_n(\mathbb{C})$. *Then*

1. *Every non-trivial homogeneous $\boldsymbol{G}^1$-invariant differential operator $P(\partial)$ with constant coefficients is written as a constant multiple of* $\det(\partial^*)^k$ *with some positive integer k.*
2. *The differential operator $P(\partial)$ is hyperbolic with respect to the initial plane $N_\vartheta := \{x \in \boldsymbol{V} \mid \langle x, \vartheta \rangle = 0\}$ if and only if $\vartheta \in \boldsymbol{V}^*$ is a positive definite matrix or a negative definite matrix.*

3. *In the case above, $P(\partial)$ is strongly hyperbolic if and only if $n = 2$ and $k = 1$. This is the wave operator of space dimension 2.*

Let $P(\partial) := \det(\partial^*)^k$ and let ϑ_+ (resp. ϑ_-) be a positive (resp. negative) definite matrix. Then the connected component $\Gamma(P, \vartheta_+) \subset \boldsymbol{V}^*$ (resp. $\Gamma(P, \vartheta_-) \subset \boldsymbol{V}^*$) is the set of positive (resp. negative) definite matrices in $\boldsymbol{V}^*$. On the other hand, the dual cone $\Gamma^\circ(P, \vartheta_+)$ (resp. $\Gamma^\circ(P, \vartheta_-)$) is the set of semi-positive (resp. semi-negative) definite matrices in $\boldsymbol{V}$. Therefore we have

$$\begin{aligned} \Gamma^\circ(P, \vartheta_+) &= \overline{\boldsymbol{V}_n} = \bigsqcup_{0 \le i \le n} \boldsymbol{S}_i^{n-i} \\ \Gamma^\circ(P, \vartheta_-) &= \overline{\boldsymbol{V}_0} = \bigsqcup_{0 \le i \le n} \boldsymbol{S}_i^{0} \end{aligned} \tag{25}$$

By Proposition 1.1 and Proposition 5.1, there exist unique fundamental solutions supported in $\overline{\boldsymbol{V}_n}$ and $\overline{\boldsymbol{V}_0}$.

6 Complex powers of relative invariants.

We construct the fundamental solutions of the differential operator $P(\partial)$ which are supported in $\overline{\boldsymbol{V}_n}$ or $\overline{\boldsymbol{V}_0}$ by using the complex powers of $P(x)$. We define the complex powers $|P(x)|_i^s$ $(i = 0, 1, \dots, n)$ of $P(x)$ by

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & \text{if } x \in \boldsymbol{V}_i, \\ 0 & \text{if } x \notin \boldsymbol{V}_i, \end{cases} \tag{26}$$

for a complex number $s \in \mathbb{C}$. Let $\mathcal{S}(\boldsymbol{V})$ be the space of rapidly decreasing smooth functions on $\boldsymbol{V}$. For $f(x) \in \mathcal{S}(\boldsymbol{V})$, the integral

$$Z_i(f, s) := \int_{\boldsymbol{V}} |P(x)|_i^s f(x) dx, \tag{27}$$

is convergent if the real part $\Re(s)$ of s is sufficiently large and is meromorphically extended to the whole complex plane. Thus we can regard $|P(x)|_i^s$ as a tempered distribution — and hence a hyperfunction — with a meromorphic parameter $s \in \mathbb{C}$. We call each $|P(x)|_i^s$ the *complex power of* $P(x)$. We consider a linear combination of the hyperfunctions $|P(x)|_i^s$

$$P^{[\vec{a}, s]}(x) := \sum_{i=0}^{n} a_i \cdot |P(x)|_i^s \tag{28}$$

with $s \in \mathbb{C}$ and $\vec{a} := (a_0, a_1, \dots, a_n) \in \mathbb{C}^{n+1}$. Then $P^{[\vec{a}, s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \mathbb{C}$, and depends on $\vec{a} \in \mathbb{C}^{n+1}$ linearly.

Since $P^{[\vec{a}, s]}(x)$ is meromorphic with respect to $s \in \mathbb{C}$, we can expand $P^{[\vec{a}, s]}(x)$ to a Laurent series. Let

$$P^{[\vec{a}, s]}(x) = \sum_{j \in \mathbb{Z}} P_j^{[\vec{a}, s_0]}(x)(s - s_0)^j$$

be the Laurent expansion of $P^{[\vec{a}, s]}(x)$ at $s = s_0$. Then each Laurent expansion coefficient $P_j^{[\vec{a}, s_0]}(x)$ is a linear function on $\vec{a} \in \mathbb{C}^{n+1}$.

In particular, let $\vec{e_n} := (0, \dots, 0, 1) \in \mathbb{C}^{n+1}$ and let $\vec{e_0} := (1, 0, \dots, 0) \in \mathbb{C}^{n+1}$. Then we have $P^{[\vec{e_n}, s]}(x) = |P(x)|_n^s$ and $P^{[\vec{e_0}, s]}(x) = |P(x)|_0^s$ and hence

$$\begin{aligned} \mathbf{Supp}(P^{[\vec{e_n}, s]}(x)) &\subset \Gamma^\circ(P, \vartheta_+) \\ \mathbf{Supp}(P^{[\vec{e_0}, s]}(x)) &\subset \Gamma^\circ(P, \vartheta_-) \end{aligned}$$

Therefore, every Laurent expansion coefficient has the same property:

$$\mathbf{Supp}(P_j^{[\vec{e_n},s_0]}(x)) \subset \Gamma^\circ(P,\vartheta_+)$$
$$\mathbf{Supp}(P_j^{[\vec{e_0},s_0]}(x)) \subset \Gamma^\circ(P,\vartheta_-)$$

for each $j \in \mathbb{Z}$ and $s_0 \in \mathbb{C}$.

We can construct the fundamental solutions satisfying the property in Proposition 1.1 as a constant multiple of the Laurent expansion coefficients $P_j^{[\vec{e_n},s_0]}(x)$ and $P_0^{[\vec{e_n},s_0]}(x)$. The exact supports of them are given in the following proposition.

Proposition 6.1. *The hyperfunctions $P^{[\vec{e_n},s]}(x)$ and $P^{[\vec{e_n},s]}(x)$ have the following properties.*

1. *They have poles of order*

$$\begin{cases} -\lfloor s_0 \rfloor \ \text{at}\ s_0 = -1, -\frac{3}{2}, \dots, -\frac{n+1}{2} & \text{when } \boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R}), \\ -s_0 \ \text{at}\ s_0 = -1, -2, \dots, -n & \text{when } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{C}), \\ -\lfloor s_0/2 \rfloor \ \text{at}\ s_0 = -1, -2, \dots, -2n+1 & \text{when } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{H}). \end{cases} \tag{29}$$

2. (a) *When $\boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R})$, we have*

$$\mathbf{Supp}(P_{-\lfloor (k+1)/2 \rfloor}^{[\vec{e_0},-(k+1)/2]}(x)) = \overline{\boldsymbol{S}_k^0}$$

and

$$\mathbf{Supp}(P_{-\lfloor (k+1)/2 \rfloor}^{[\vec{e_n},-(k+1)/2]}(x)) = \overline{\boldsymbol{S}_k^{n-k}}$$

for $k = 1, 2, \dots, n$.

(b) *When $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{C})$, we have*

$$\mathbf{Supp}(P_{-k}^{[\vec{e_0},-k]}(x)) = \overline{\boldsymbol{S}_k^0}$$

and

$$\mathbf{Supp}(P_{-k}^{[\vec{e_n},-k]}(x)) = \overline{\boldsymbol{S}_k^{n-k}}$$

for $k = 1, 2, \dots, n$.

(c) *When $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{H})$, we have*

$$\mathbf{Supp}(P_{-\lfloor (k+1)/2 \rfloor}^{[\vec{e_0},-k]}(x)) = \overline{\boldsymbol{S}_{\lfloor (k+1)/2 \rfloor}^0}$$

and

$$\mathbf{Supp}(P_{-\lfloor (k+1)/2 \rfloor}^{[\vec{e_n},-k]}(x)) = \overline{\boldsymbol{S}_{\lfloor (k+1)/2 \rfloor}^{n-\lfloor (k+1)/2 \rfloor}}$$

for $k = 1, 2, \dots, 2n-1$.

7 Construction of fundamental solutions.

Theorem 7.1 (fundamental solution). *Fundamental solutions for the differential operator $P(\partial) = (\det(\partial)^*)^k$ $(k = 1, 2, \dots)$ is given as a Laurent expansion coefficient of $P^{[\vec{a},s]}(x)$. Let k be a positive integer.*

1. *When* $\boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R})$, *we put*

$$P_{+,k}(x) := P^{[\vec{e_n}, -(n+1-2k)/2]}_{\min\{0,-\lfloor (n+1-2k)/2\rfloor\}}(x),$$
$$P_{-,k}(x) := P^{[\vec{e_0}, -(n+1-2k)/2]}_{\min\{0,-\lfloor (n+1-2k)/2\rfloor\}}(x).$$

Then we have

$$\det(\partial^*)^k P_{+,k}(x) = c_{+,k}\delta(x),$$
$$\det(\partial^*)^k P_{-,k}(x) = c_{-,k}\delta(x),$$

with certain non-zero constants $c_{+,k}$ *and* $c_{-,k}$. *Therefore* $F_{+,k}(x) := c_{+,k}^{-1}P_{+,k}(k)$ *and* $F_{-,k}(x) := c_{+,k}^{-1}P_{-,k}(k)$ *are unique fundamental solutions whose supports are contained in the half spaces* H_ϑ *and* $H_{-\vartheta}$, *respectively. The exact supports of* $F_{+,k}(x)$ *and* $F_{-,k}(x)$ *are given by*

$$\mathbf{Supp}(F_{+,k}(x)) := \begin{cases} \overline{\boldsymbol{S}^{2k}_{n-2k}} & \text{if } k = 1,2,\dots,\lfloor (n-1)/2\rfloor \\ \overline{\boldsymbol{S}^{n}_{0}} = \overline{\boldsymbol{V}_n} & \text{if } k = \lfloor (n+1)/2\rfloor, \lfloor (n+1)/2\rfloor + 1, \dots \end{cases}$$
$$\mathbf{Supp}(F_{-,k}(x)) := \begin{cases} \overline{\boldsymbol{S}^{0}_{n-2k}} & \text{if } k = 1,2,\dots,\lfloor (n-1)/2\rfloor \\ \overline{\boldsymbol{S}^{0}_{0}} = \overline{\boldsymbol{V}_0} & \text{if } k = \lfloor (n+1)/2\rfloor, \lfloor (n+1)/2\rfloor + 1, \dots \end{cases}$$

2. *When* $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{C})$, *we put*

$$P_{+,k}(x) := P^{[\vec{e_n}, -(n-k)]}_{\min\{0,-(n-k)\}}(x),$$
$$P_{-,k}(x) := P^{[\vec{e_0}, -(n-k)]}_{\min\{0,-(n-k)\}}(x).$$

Then we have

$$\det(\partial^*)^k P_{+,k}(x) = c_{+,k}\delta(x),$$
$$\det(\partial^*)^k P_{-,k}(x) = c_{-,k}\delta(x),$$

with certain non-zero constants $c_{+,k}$ *and* $c_{-,k}$. *Therefore* $F_{+,k}(x) := c_{+,k}^{-1}P_{+,k}(k)$ *and* $F_{-,k}(x) := c_{+,k}^{-1}P_{-,k}(k)$ *are unique fundamental solutions whose supports are contained in the half spaces* H_ϑ *and* $H_{-\vartheta}$, *respectively. The exact supports of* $F_{+,k}(x)$ *and* $F_{-,k}(x)$ *are given by*

$$\mathbf{Supp}(F_{+,k}(x)) := \begin{cases} \overline{\boldsymbol{S}^{k}_{n-k}} & \text{if } k = 1,2,\dots,n-1 \\ \overline{\boldsymbol{S}^{n}_{0}} = \overline{\boldsymbol{V}_n} & \text{if } k = n, n+1, \dots \end{cases}$$
$$\mathbf{Supp}(F_{-,k}(x)) := \begin{cases} \overline{\boldsymbol{S}^{0}_{n-k}} & \text{if } k = 1,2,\dots,n-1 \\ \overline{\boldsymbol{S}^{0}_{0}} = \overline{\boldsymbol{V}_0} & \text{if } k = n, n+1, \dots \end{cases}$$

3. *When* $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{H})$, *we put*

$$P_{+,k}(x) := P^{[\vec{e_n}, -2n+k+1]}_{\min\{0,-n-\lfloor -k/2\rfloor\}}(x),$$
$$P_{-,k}(x) := P^{[\vec{e_0}, -2n+k+1]}_{\min\{0,-n-\lfloor -k/2\rfloor\}}(x).$$

Then we have

$$\det(\partial^*)^k P_{+,k}(x) = c_{+,k}\delta(x),$$
$$\det(\partial^*)^k P_{-,k}(x) = c_{-,k}\delta(x),$$

with certain non-zero constants $c_{+,k}$ *and* $c_{-,k}$. *Therefore* $F_{+,k}(x) := c_{+,k}^{-1}P_{+,k}(k)$ *and* $F_{-,k}(x) := c_{+,k}^{-1}P_{-,k}(k)$ *are unique fundamental solutions whose supports are contained in the half spaces* H_ϑ *and* $H_{-\vartheta}$, *respectively. The exact supports of* $F_{+,k}(x)$ *and* $F_{-,k}(x)$ *are given by*

$$\mathbf{Supp}(F_{+,k}(x)) := \begin{cases} \overline{\boldsymbol{S}^{-\lfloor -k/2\rfloor}_{n+\lfloor -k/2\rfloor}} & \text{if } k = 1,2,\dots,2(n-1) \\ \overline{\boldsymbol{S}^{n}_{0}} = \overline{\boldsymbol{V}_n} & \text{if } k = 2(n-1)+1, 2(n-1)+2, \dots \end{cases}$$
$$\mathbf{Supp}(F_{-,k}(x)) := \begin{cases} \overline{\boldsymbol{S}^{0}_{n+\lfloor -k/2\rfloor}} & \text{if } k = 1,2,\dots,2(n-1) \\ \overline{\boldsymbol{S}^{0}_{0}} = \overline{\boldsymbol{V}_0} & \text{if } k = 2(n-1)+1, 2(n-1)+2, \dots \end{cases}$$

Corollary 7.2 (Huygens principle). *The hyperbolic operator* $\det(\partial^*)^k$ *satisfies the Huygens principle if and only if*

$$\begin{aligned} k &= 1,2,\dots,\lfloor (n-1)/2 \rfloor \text{ when } \boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R}), \\ k &= 1,2,\dots,(n-1) \text{ when } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{C}), \\ k &= 1,2,\dots,2(n-1) \text{ when } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{H}). \end{aligned} \tag{30}$$

In particular, it satisfies the strong Huygens principle except for the case that $k = \lfloor (n-1)/2 \rfloor$ *and* n *is odd in* $\boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R})$, *the case that* $k = (n-1)$ *in* $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{C})$ *or the case that* $k = 2(n-1), 2(n-1)-1$ *in* $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{H})$.

Remark 7.1. The exact supports of the fundamental solutions have been partly determined in some preceding papers. For example, see Gindikin[5, p. 112, Example 2] and Atiyah, Bott, and Gårding[1, p. 181, Example 8.8]. However, in both of the papers, they mentioned only that the support of the fundamental solution of $\det(\partial^*)^k$ coincides with the set of positive semi-definite matrices of rank $\leq k$ in the case when $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{C})$ while we have determined in this paper the exact support of the fundamental solutions in the case of $\boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R})$ and $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{H})$. Instead of the precise calculation of the support of fundamental solutions in the specific examples, they gave a theory to handle a wide range of examples. For example, Gindikin's theory can also be applied to a certain kind of parabolic differential operators.

8 Singularity spectrum of fundamental solutions.

Definition 8.1 (conormal bundle of a subvariety). Let A be a non-singular subvariety in $\boldsymbol{V}$. We define the *conormal bundle* $T^*_A\boldsymbol{V}$ *of* A to be $T^*_A\boldsymbol{V} := \cup_{x\in A}(T^*_A\boldsymbol{V})_x$ where $(T^*_A\boldsymbol{V})_x := \{(x,\xi) \in T^*\boldsymbol{V} \mid \xi \in (T^*\boldsymbol{V})_x$ that satisfies $\langle \zeta, \xi \rangle = 0$ for all $\zeta \in (TA)_x\}$. Here $(T^*\boldsymbol{V})_x$ and $(T\boldsymbol{V})_x$ are tangent or cotangent vector spaces of $\boldsymbol{V}$ at $x \in \boldsymbol{V}$, respectively, and $(TA)_x$ is the tangent vector space of A at $x \in A$.

Theorem 8.1 (singularity spectrum). *Let* $F_{+,k}(x)$ *and* $F_{-,k}(x)$ *be the fundamental solutions of* $(\det \partial^*)^k$ *defined in Theorem 7.1. The singularity spectrum of them are given in the following formulas.*

1. *When* $\boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R})$, *we have*

$$\begin{aligned} S.S.(F_{+,k}(x)) &= \overline{\bigcup_{i=\max\{n-2k,1\}}^{n} T^*_{S_i^{n-i}}\boldsymbol{V}} \\ S.S.(F_{-,k}(x)) &= \overline{\bigcup_{i=\max\{n-2k,1\}}^{n} T^*_{S_i^0}\boldsymbol{V}} \end{aligned} \tag{31}$$

2. *When* $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{C})$, *we have*

$$\begin{aligned} S.S.(F_{+,k}(x)) &= \overline{\bigcup_{i=\max\{n-k,1\}}^{n} T^*_{S_i^{n-i}}\boldsymbol{V}} \\ S.S.(F_{-,k}(x)) &= \overline{\bigcup_{i=\max\{n-k,1\}}^{n} T^*_{S_i^0}\boldsymbol{V}} \end{aligned} \tag{32}$$

3. *When* $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{H})$, *we have*

$$\begin{aligned} S.S.(F_{+,k}(x)) &= \overline{\bigcup_{i=\max\{n+\lfloor -k/2 \rfloor,1\}}^{n} T^*_{S_i^{n-i}}\boldsymbol{V}} \\ S.S.(F_{-,k}(x)) &= \overline{\bigcup_{i=\max\{n+\lfloor -k/2 \rfloor,1\}}^{n} T^*_{S_i^0}\boldsymbol{V}} \end{aligned} \tag{33}$$

Remark 8.1. We have $S.S.(F_{+,k}(x)) \subset \bigcup_{i=1}^{n-1} \overline{T^*_{S_i^{n-i}} V}$ and $S.S.(F_{-,k}(x)) \subset \bigcup_{i=1}^{n-1} \overline{T^*_{S_i^0} V}$ in all cases by applying Hörmander's Theorem 2.1.

9 The Cauchy problem and the propagation of singularity.

Let $\boldsymbol{V}$ be one of $\mathrm{Sym}_n(\mathbb{R})$, $\mathrm{Her}_n(\mathbb{C})$ and $\mathrm{Her}_n(\mathbb{H})$ and let $P(\partial) = (\det(\partial^*))^k$ be the differential operator on $\boldsymbol{V}$. For an non-zero element $\vartheta \in \boldsymbol{V}^*$ we put $N_\vartheta := \{x \in \boldsymbol{V} \mid \langle x, \vartheta \rangle = 0\}$. Let $x_1 := \langle x, \vartheta \rangle$ and let $(x_1, x') = (x_1, x_2, \dots, x_m)$ be a coordinate of $\boldsymbol{V}$. We denote by $(\partial_1, \partial') = (\partial_1, \partial_2, \dots, \partial_m)$ the partial derivatives with respect to the coordinate $(x_1, x') = (x_1, x_2, \dots, x_m)$. Here m is the dimension of $\boldsymbol{V}$.

We denote by $l = kn$ the order of the differential operator $P(\partial) = (\det(\partial^*))^k$. Then $P(\partial)$ can be written as

$$P(\partial) = p_0 \partial_1^l + p_1(\partial')\partial_1^{l-1} + \dots + p_{l-1}(\partial')\partial_1 + p_l(\partial') \tag{34}$$

We consider the Cauchy problem

$$\begin{aligned} &P(\partial)u(x) = 0 \\ &\partial_j u(x)|_{x_1=0} = v_j(x') \quad (j = 0, 1, \dots, l-1). \end{aligned} \tag{35}$$

for a given initial data $\mathbf{v} := (v_0(x'), \dots, v_{l-1}(x')) \in \mathcal{B}(N_\vartheta)^l$ consisting of compact supported hyperfunctions on N_ϑ. The unique solution to the Cauchy problem (35) is given by

$$u(x) = \sum_{j=0}^{l-1} \int_{N_\vartheta} F_{l-j-1}(x_1, x' - y')v_j(y')dy'$$

by using the fundamental solution

$$FS_\vartheta := (F_0, \dots, F_{l-1}) \in \mathcal{B}(\boldsymbol{V})^l \tag{36}$$

where

$$\begin{aligned} F_0(x) &:= p_0(F_{+,k} - F_{-,k}) \\ F_j(x) &:= \frac{1}{p_0}(p_0\partial_1^j + p_1(\partial')\partial_1^{j-1} + \dots + p_j(\partial'))F_0(x) \quad (\text{for } j = 1, \dots, l-1) \end{aligned}$$

The support and the singularity spectrum of the initial data are defined by

$$\begin{aligned} \mathbf{Supp}(\mathbf{v}(x)) &= \bigcup_{j=0}^{l-1} \mathbf{Supp}(v_j(x')) \subset N_\vartheta, \\ S.S.(\mathbf{v}(x)) &= \bigcup_{j=0}^{l-1} S.S.(v_j(x')) \subset T^*N_\vartheta, \end{aligned} \tag{37}$$

and those of the fundamental solution FS_ϑ are defined by

$$\begin{aligned} \mathbf{Supp}(FS_\vartheta) &= \bigcup_{j=0}^{l-1} \mathbf{Supp}(F_j(x)) \subset \boldsymbol{V}, \\ S.S.(FS_\vartheta) &= \bigcup_{j=0}^{l-1} S.S.(F_j(x)) \subset T^*\boldsymbol{V}. \end{aligned} \tag{38}$$

The support and the singularity spectrum of the fundamental solution FS_ϑ can be computed explicitly in the following theorem.

Theorem 9.1 (support and singular spectrum). *The exact support and the exact singularity spectrum of the fundamental solutions FS_ϑ to the Cauchy problems* (35) *are given by* (39) *and* (40), *respectively.*

$$\mathbf{Supp}(FS_\vartheta) = \begin{cases} \displaystyle\bigsqcup_{i=\max\{n-2k,0\}}^{n} (\boldsymbol{S}_i^0 \sqcup \boldsymbol{S}_i^{n-i}) & \text{if } \boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R}) \\ \displaystyle\bigsqcup_{i=\max\{n-k,0\}}^{n} (\boldsymbol{S}_i^0 \sqcup \boldsymbol{S}_i^{n-i}) & \text{if } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{C}) \\ \displaystyle\bigsqcup_{i=\max\{n+\lfloor -k/2\rfloor,0\}}^{n} (\boldsymbol{S}_i^0 \sqcup \boldsymbol{S}_i^{n-i}) & \text{if } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{H}) \end{cases} \tag{39}$$

$$S.S.(FS_\vartheta) = \begin{cases} \displaystyle\bigcup_{i=\max\{n-2k,1\}}^{n-1} \overline{(T^*_{\boldsymbol{S}_i^0}\boldsymbol{V} \cup T^*_{\boldsymbol{S}_i^{n-i}}\boldsymbol{V})} & \text{if } \boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R}) \\ \displaystyle\bigcup_{i=\max\{n-k,1\}}^{n-1} \overline{(T^*_{\boldsymbol{S}_i^0}\boldsymbol{V} \cup T^*_{\boldsymbol{S}_i^{n-i}}\boldsymbol{V})} & \text{if } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{C}) \\ \displaystyle\bigcup_{i=\max\{n+\lfloor -k/2\rfloor,1\}}^{n-1} \overline{(T^*_{\boldsymbol{S}_i^0}\boldsymbol{V} \cup T^*_{\boldsymbol{S}_i^{n-i}}\boldsymbol{V})} & \text{if } \boldsymbol{V} = \mathrm{Her}_n(\mathbb{H}) \end{cases} \tag{40}$$

Theorem 9.2 (propagation of singularity). *Let $u(x)$ be the unique hyperfunction solution to the Cauchy problem* (35). *Then we have:*

1. *If $x_0 \in \mathbf{Supp}(u(x))$, then*

$$x_0 \in \{x_0 = y_0 + z_0 \mid y_0 \in \mathbf{Supp}(\mathbf{v}(x)) \text{ and } z_0 \in \mathbf{Supp}(FS_\vartheta)\}. \tag{41}$$

2. *If $(x_0, \xi_0) \in S.S.(u(x))$ and $x_0 \notin N_\vartheta$, then there exists $y_0 \in N_\vartheta$ satisfying the following conditions:*
   (a) *$x_0 - y_0 \in \mathbf{Supp}(FS_\vartheta)$.*
   (b) *Let $\boldsymbol{S}_i^p$ ($p = 0$ or $p = n-i$) be a $\boldsymbol{G}^1$-orbit in $\mathbf{Supp}(FS_\vartheta)$ that $x_0 - y_0$ belongs to. Then $(x_0 - y_0, \xi_0) \in T^*_{\boldsymbol{S}_i^p}\boldsymbol{V}$.*
   (c) *$(y_0, \overline{\xi_0}) \in S.S.(\mathbf{v}(x))$. Here $\overline{\xi_0}$ means the projection of $\xi_0 \in \boldsymbol{V}^*$ onto N_ϑ^*.*

Corollary 9.3. *Let $P(\partial) = \det(\partial^*)^k$. The singularity spectrum of the hyperfunction solution of the Cauchy problem for $P(\partial)$ propagates along $T^*_{\boldsymbol{S}_{n-1}^0}\boldsymbol{V}$ and $T^*_{\boldsymbol{S}_{n-1}^1}\boldsymbol{V}$ if and only if $k = 1$ and $n = 2$ in $\boldsymbol{V} = \mathrm{Sym}_n(\mathbb{R})$ or $k = 1$ in $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{C})$ or in $\boldsymbol{V} = \mathrm{Her}_n(\mathbb{H})$.*

In particular, $T^*_{\boldsymbol{S}_{n-1}^0}\boldsymbol{V}$ and $T^*_{\boldsymbol{S}_{n-1}^1}\boldsymbol{V}$ are subvarieties consisting of bicharacteristic strips of $P(\partial) = \det(\partial^*)$. The differential operator $\det(\partial^*)$ is not strongly hyperbolic except for the case of $n = 2$. Therefore, for $n \geq 3$ in $\mathrm{Her}_n(\mathbb{C})$ or in $\mathrm{Her}_n(\mathbb{H})$, $\det(\partial^*)$ is an example of a non-strongly hyperbolic differential operator whose singularity spectrum of solution propagates only along bicharacteristic strips.

References

[1] M. F. Atiyah, R. Bott, and L. Gårding, *Lacunas for hyperbolic differential operators with constant coefficients I*, Acta Math. **124** (1970), 109–189.

[2] J.J. Duistermaat, *Fourier Integral Operators*, Progress in Mathematics, vol. 130, Birkhäuser, 1996.

[3] L. Gårding, *The solution of Cauchy's problem for two totally hyperbolic differential equations by means of Riesz integrals*, Ann. of Math. **48** (1947), 785–826.

[4] I.M. Gelfand and G.E. Shilov, *Generalized Functions — properties and operations*, Generalized Functions, vol. 1, Academic Press, New York and London, 1964.

[5] S.G. Gindikin, *Tube domains and the Cauchy problem*, Translation of Mathematical Monographs, vol. 111, American Mathematical Society, 1992.

[6] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Grundlehren Der Mathematischen Wissenschaften, vol. 257, Springer-Verlag, 1983.

[7] ———, *The Analysis of Linear Partial Differential Operators I*, second ed., Springer Study Edition., Springer-Verlag, 1990.

[8] A. Kaneko, *Linear partial differential equations with constant coeffcients*, Iwanami Shoten Publishers, Tokyo, 1976 (Japanese).

[9] M. Kashiwara, T. Kawai, and T. Kimura, *Daisuukaisekigaku no Kiso (Foundations of Algebraic Analysis)*, Kinokuniya, Tokyo, 1980 (Japanese), The English translation was published by Princeton UP in 1985.

[10] M. Muro, *Microlocal analysis and calculations on some relatively invariant hyperfunctions related to zeta functions associated with the vector spaces of quadratic forms*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **22** (1986), no. 3, 395–463.

[11] ———, *Singular invariant tempered distributions on regular prehomogeneous vector spaces*, J. Funct. Anal. **76** (1988), no. 2, 317 – 345.

[12] ———, *Invariant hyperfunctions on regular prehomogeneous vector spaces of commutative parabolic type*, Tôhoku Math. J. (2) **42** (1990), no. 2, 163–193.

[13] ———, *Singular invariant hyperfunctions on the space of real symmetric matrices*, Tôhoku Math. J. (2) **51** (1999), 329–364.

[14] ———, *Singular invariant hyperfunctions on the space of complex and quaternion Hermitian matrices*, J. Math. Soc. Japan **53** (2001), no. 3, 589–602.

[15] ———, *Invariant hyperfunction solutions to invariant differential equations on the space of real symmetric matrices*, J. Funct. Anal. **193** (2002), no. 2, 346–384.

[16] ———, *Algorithmic construction of hyperfunction solutions to invariant differential equations on the space of real symmetric matrices*, to appear in J. Lie Theory, 2003.

[17] ———, *Singular invariant hyperfunctions on the square matrix space and the alternating matrix space*, Nagoya Math. J. **169** (2003), 19–75.

[18] M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math. **81** (1949), 1–223.

[19] M. Sato and T. Shintani, *On zeta functions associated with prehomogeneous vector spaces*, Ann. of Math. (2) **100** (1974), 131–170.

[20] T. Shintani, *On zeta functions associated with the vector spaces of quadratic forms*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), 25–65.