数理解析研究所講究録 1350 巻 2004 年 107-126

Generalized sectors and macroscopic order parameters*

Izumi OJIMA
RIMS, Kyoto University

Abstract

A unified scheme for treating generalized superselection sectors is proposed on the basis of the notion of selection criteria to characterize states of relevance to each specific domain in quantum physics, ranging from the relativistic quantum fields in the vacuum situations with unbroken and spontaneously broken internal symmetries, through equilibrium and non-equilibrium states (see [1] for details). This is achieved by the help of $c \to q$ and $q \to c$ channels, the former of which determines the states to be selected and to be parametrized by the order parameters defined as the spectrum of the centre constituting the superselection sectors, and the latter of which provides, as classifying maps, the physical interpretations of selected states in terms of order parameters.

1 Introduction

The stanrdard way of treating the microscopic world on the basis of quantum field theory (QFT, for short) is to introduce first the *quantum fields* whose characterization is given by means of their behaviours under *symmetries*; e.g., the internal symmetries described by such groups as colour $SU(3)$, chiral $SU(2)$, electromagnetic $U(1)$, or any other bigger (super)groups of grand unifications (and their corresponding versions of local gauge symmetries), in combination with the spacetime symmetries with Poincaré group in Minkowski spacetime, conformal groups in massless theories, or isometry groups of curved spacetimes, and so on. In a word, the basic objects of such a system can be found in an algebra $\mathfrak{F}$ of quantum fields (called a *field algebra*, for short) acted on by two kinds of symmetries, *internal* and *spacetime*. With respect to the group of an internal symmetry denoted by G, the generators of $\mathfrak{F}$ (usually called *basic* or *fundamental fields*) are assumed (*by hand*) to belong to certain multiplet(s) transforming covariantly under the action of G, which defines mathematically an action τ of G on $\mathfrak{F}$: $G \underset{\tau}{\curvearrowright} \mathfrak{F}$.

*Talk at a RIMS-Symposium "Mathematics in Quantum Systems and its Applications to Quantum Computing" in January 2002

Contrary to this *theoretical* setting, what is to be observed in the real world is only those elements of $\mathfrak{F}$ *invariant under* G, usually called *observables* which constitute the algebra $\mathfrak{A}$ of observables: $\mathfrak{A} := \mathfrak{F}^G$, the fixed-point subalgebra of $\mathfrak{F}$ under the action τ of G. Thus, what we can directly check experimentally is only those data described in terms of $\mathfrak{A}$ (and its derived objects) and the rest of the notions appearing in our framework are just mathematical devices whose pertinence can be justified only through the data related to $\mathfrak{A}$. Except for the systematic approaches [2, 3, 4] developed in algebraic QFT, however, there have so far been no attempts to understand the basic mechanism pertaining to this point as to how a particular choice of $\mathfrak{F}$ and G can be verified, leaving aside the problems of this sort just to the heuristic arguments based on trials and errors. While a particularly chosen combination $G \underset{\tau}{\curvearrowright} \mathfrak{F}$ becomes no doubt meaningless without good agreements of its consequences with the observed data described in terms of $\mathfrak{A}$, the attained agreements support the postulated theoretical assumption only as *one of many possible candidates* of explanations, without justifying it as a *unique* inevitable solution.

Just when restricted to the cases with G of an *unbroken* global gauge symmetry (or, gauge symmetry of the first kind), a satisfactory framework in this context has been established in [5, 3], whose physical essence has, unfortunately, not been recognized widely (which may be partly due to its mathematical sophistication, but mainly due to the lack of common understanding of the importance of the above-mentioned problem). This theory enables one to *recover both* $\mathfrak{F}$ *and* G starting only from the data encoded in $\mathfrak{A}$ when supplemented by the so-called DHR *selection criterion* [6, 2] to pick up physically relevant states with localizable charges. Then, the vacuum representation of the constructed field algebra $\mathfrak{F}$ is decomposed into mutually disjoint irreducible representations of $\mathfrak{A} = \mathfrak{F}^G$, called *superselection sectors*, in one-to-one correspondence with mutually disjoint irreducible unitary representations of the internal symmetry group G which is found to be *compact Lie*. However, the traditional notion of sector structure, hinging strongly to the essential features of unbroken symmetry, has so far allowed only the *discrete sectors*, parametrized by the discrete $\hat{G}$, the dual of a compact group defined as the set of all equivalence classes of finite-dimensional continuous unitary irreducible representations of G. When we start to extend this formalism to the situations with *spontaneous symmetry breakdown* (SSB, for short), we encounter the presence of *continuous sectors* (or, "degenerate vacua" in the traditional terminology) parametrized by continuous *macroscopic order parameters* (see [1]).

In the directions from microscopic worlds to macroscopic ones, we have so far faced with so many different levels and areas ranging from the vacuum situations (the standard QFT relevant to particle physics), thermal equilibria (QFT at finite temperatures or quantum statistical mechanics),

non-equilibrium ones and so on. In [7], a general framework is proposed for defining non-equilibrium local states in relativistic QFT and for describing their thermodynamic properties in terms of the associated macroscopic observables found in the *centre* of relevant representations of observables. From the general standpoint, one notices that the thermal equilibria at different temperatures can also be seen to constitute families of continuous sectors parametrized by such thermodynamic variables as temperatures, chemical potentials and pressure, etc. In view of such roles of central observables associated with continuous sectors appearing in SSB cases as well as the above various kinds of thermal states, it seems appropriate to extend the notion of sectors so as to incorporate and to try the possibility of unified ways of treating these different cases, just regarding the traditional discrete ones as special cases; this is parallel to the extension of the traditional *eigenvalue* problems for linear operators with *discrete spectra* to the general *spectral decompositions* admitting the appearance of *continuous spectra.*

The aim of this paper is to propose a scheme to unify such a generalized notion of sectors from the viewpoint of the key roles played by the *selection criteria* at the starting point of theory in defining and choosing *physically relevant family of states* as well as in providing a systematic way for *describing* and *interpreting* the relevant physical properties. Here, we introduce the necessary ingredients for formulating the scheme through the discussions on the basic structures found in thermal situations of equilibrium (Sec.2.1) and of the extension to non-equilibrium (Sec.2.2) and in the reformulated version of DHR superselection theory (Sec.3). At the end, we explain the general mathematical meaning of the proposed scheme, in relation with the categorical adjunctions, especially with the geometric notions of classifying spaces and classifying maps. The analysis is to be continued to a systematic treatment of spontaneously broken symmetries from the viewpoint of proposed scheme, which is found in [1].

2 Thermal Situations with Continuous Sectors

2.1 Equilibrium states and thermal interpretations: roles of $c \to q$ and $q \to c$ channels

To draw a clear picture of the idea, we briefly sketch the essense of the scheme proposed in [7] for defining and describing non-equilibrium local states in a relativistic QFT. From the present standpoint, it can be reformulated as follows according to [8, 9]. To characterize an unknown state ω as a non-equilibrium local state, we prepare the following basic ingredients.

i) Candidates of such states are sought within the set E of states ω (understood as an *expectation functional*, mathematically formulated as a normalized positive linear functional on the algebra $\mathfrak{A}$ of observables of the system under consideration) with locally finite energy characterized by the

energy-bound condition

$$\omega((1+H_{\mathcal{O}})^{2m}) < \infty \tag{1}$$

valid in some spacetime local region $\mathcal{O}$ and some $m > 0$ with $H_{\mathcal{O}}$ a *local Hamiltonian* playing the role of Hamiltonian in $\mathcal{O}$ (whose definition is justified under the assumption of the nuclearity condition). This choice is so designed that the comparison is fully meaningful between an unknown state $\omega \in E$ and known reference states $\in K$ specified in the next ii) as statistical mixtures of thermal equilibria, *in infinitesimally small neighbourhoods of a spacetime point* x by means of observables $\in \mathcal{T}_x$ defined in iii).) With a given $\mathcal{O}$ we denote $E_{\mathcal{O}}$ the totality of states ω satisfying Eq.(1) with a suitable $m > 0$, $E_{\mathcal{O}} := \{\omega; \omega$: state of $\mathcal{A}$ and $\exists m > 0$ s.t. $\omega((1+H_{\mathcal{O}})^{2m}) < \infty\}$. If the local Hamiltonians $H_{\mathcal{O}}$ are positive, the family $\mathcal{O} \longmapsto E_{\mathcal{O}}$ constitutes a presheaf of state germs [10] whose stalk at x is given by $E_x (= \varinjlim_{\mathcal{O} \to x} E_{\mathcal{O}})$, the pointlike limit (projective limit) defined by the equivalence relation $\omega_1 \sim \omega_2 \overset{\text{def}}{\Longleftrightarrow} \exists \mathcal{O}$: neighbourhood of x s.t. $\omega_1 \restriction_{\mathcal{O}} = \omega_2 \restriction_{\mathcal{O}}$.

ii) The set K of *thermal reference states* consisting of all global thermal equilibrium states defined as the relativistic KMS states ω_β [11] (with inverse temperature 4-vectors $\beta = (\beta^\mu) \in V_+ := \{x \in \mathbb{R}^4; x^0 > 0, x^2 = (x^0)^2 - \vec{x}^2 > 0\}$) and of their suitable convex combinations: K plays the role of a *model space* whose analogue in the definition of a manifold can be found in a Euclidean space as the value space of local charts. Any states belonging to this set K is seen to belong to the above $E_{\mathcal{O}}$ with any arbitrary finite spacetime region $\mathcal{O}$: $K \subset E_{\mathcal{O}} \subset E$.

iii) The linear space $\mathcal{T}_x$ of *local thermal observables* is defined as linear forms on states in E_x satisfying the regularity (1) which makes meaningful the notion of *quantum fields at a point* x [7, 12]: $\mathcal{T}_x := \sum_{p,q} \mathcal{N}(\hat{\phi}_0^p)_{q,x}$,where $\hat{\phi}_0$ generically denotes the basic quantum fields defining our QFT. The notion of *normal products* $\mathcal{N}(\hat{\phi}_0^p)_{q,x}$ enters here to recover effectively the product structure of quantum fields lost through the process of pointlike limit; they arise from the operator product expansion (OPE) of $\hat{\phi}_0(x+\zeta_1)\cdots\hat{\phi}_0(x+\zeta_p)$ in the limit of $\zeta_i \to 0$, $\sum_j \zeta_j = 0$ reformulated recently by [12] in a mathematically rigorous form. What is important about $\mathcal{T}_x$ is its natural *hierarchical structure* ordered by the indices p, q related to energy bound and OPE, starting from scalar multiples of identity to higher powers $\mathcal{N}(\hat{\phi}^p)_{q,x}$ with the larger p providing the finer resolution. Along the above analogy to a manifold in differential geometry, their role is to relate our unknown state $\omega \in E$ to the known reference states in K, in parallel to the local coordinates which relate locally a generic curved space to the known Euclidean space. As explained below, the physical interpretations of local thermal observables $\hat{A}$ are given by macroscopic *thermal functions* A corresponding to $\hat{A}$, through which our unknown ω can be *compared* with thermal reference states in K.

Before going into the discussion of non-equilibrium, we need first to establish the physical roles of the above ingredients for describing the relevant thermal properties of states and quantum observables in the realm K of generalized thermal equilibria. To this end, we introduce

Definition 1 ***Thermal functions*** *are defined for each quantum observables* $\hat{A}(\in \mathcal{T}_x)$ *by the map*

$$\mathcal{C} : \hat{A} \longmapsto \mathcal{C}(\hat{A}) \in C(B_K) \quad \textit{with } \mathcal{C}(\hat{A})(\beta,\mu) := \omega_{\beta,\mu}(\hat{A}) \quad \textit{for} \quad (\beta,\mu) \in B_K,$$

where B_K *is the classifying space to parameterize thermodynamic pure phases, consisting of inverse temperature 4-vectors* $\beta \in V_+$ *in addition to thermodynamic parameters (if any) generically denoted by* μ *(e.g., chemical potentials) necessary to exhaust and discriminate all the pure phases.*

Since the map $\mathcal{C}$ is unital and positive linear, $\mathcal{C}(1) = 1, \mathcal{C}(\hat{A}^*\hat{A}) \geq 0$, it is a *completely positive* map whose dual on states defines a ***classical-quantum*** *(c→q) channel* [13] $\mathcal{C}^* : Th \ni \rho \longmapsto \mathcal{C}^*(\rho) \in K$ given by

$$\mathcal{C}^*(\rho)(\hat{A}) = \rho(\mathcal{C}(\hat{A})) = \int_{B_K} d\rho(\beta,\mu)\mathcal{C}(\hat{A})(\beta,\mu) = \int_{B_K} d\rho(\beta,\mu)\omega_{\beta,\mu}(\hat{A}),$$

$$\Longrightarrow \mathcal{C}^*(\rho) := \int_{B_K} d\rho(\beta,\mu)\omega_{\beta,\mu} = \omega_\rho \in K. \tag{2}$$

Here $Th := M_1(B_K)$ is the space of classical thermal states identified with probability measures ρ on B_K describing the mean values of thermodynamic parameters (β,μ) together with their fluctuations. One can see that thermal interpretation of local quantum thermal observables $\hat{A} \in \mathcal{T}_x$ is given in all reference states of the form $\mathcal{C}^*(\rho) = \omega_\rho \in K$ by the corresponding thermal function $\mathcal{C}(\hat{A})$ evaluated with the classical probability ρ describing the thermodynamic configurations of ω_ρ through the relation $\omega_\rho(\hat{A}) = \int_{B_K} d\rho(\beta,\mu)\,\omega_{\beta,\mu}(\hat{A}) = \rho(\mathcal{C}(\hat{A}))$. This applies to the case where ρ is known. What we need to ask in the actual situations is how to determine the unknown ρ from the given data set $\Phi \longmapsto \rho(\Phi)$ of expectation values of thermal functions Φ: this problem can be solved if $\mathcal{T}_x$ has sufficiently many local thermal observables so that the totality $\mathcal{C}(\mathcal{T}_x)$ of the corresponding thermal functions can approximate arbitrary continuous functions of $(\beta,\mu) \in B_K$. In this case ρ is given as the unique solution to a (generalized) "moment problem". Thus we see:

★ If the set $\mathcal{T}_x$ of local thermal observables is large enough to discriminate all the thermal reference states in K, then any reference state $\in K$ can be written as $\mathcal{C}^*(\rho)$ in terms of a uniquely determined probability measure ρ on B_K describing the statistical fluctuations of thermal parameters in the state in question. Then local thermal observables

$\hat{\Phi} \in \mathcal{T}_x$ provide the same information on the thermal properties of states in K as that provided by the corresponding classical macroscopic thermal functions $\Phi = C(\hat{\Phi})$ [e.g., internal energy, entropy density, etc.]: $\omega_\rho(\hat{\Phi}) = \rho(\Phi)$.

In this situation, *any continuous function* F in B_K can be approximated by thermal functions $\Phi_x = C(\hat{\Phi}(x))$ with arbitrary precision, *even if* F itself is *not* an image of C: $\overline{C(\mathcal{T}_x)}^{\|\cdot\|} = C(B_K)$.For instance, the *entropy density* $s(\beta)$ can be treated as such an *approximate* thermal function in spite of the absence of quantum observables $\hat{s}(x) \in \mathcal{T}_x$ s.t. $\omega_\beta(\hat{s}(x)) = s(\beta)$. What the above (★) says is the following equality and equivalence,

$$K = \mathcal{C}^*(Th); \quad \omega_{\rho_1} \underset{\mathcal{T}_x}{\equiv} \omega_{\rho_2} \Longleftrightarrow \rho_1 \underset{\mathcal{C}(\mathcal{T}_x)}{\equiv} \rho_2, \tag{3}$$

for $\rho_i \in Th$, $\omega_{\rho_i} = \mathcal{C}^*(\rho_i) = \int_{B_K} d\rho_i(\beta,\mu)\omega_{\beta,\mu} \in K$, where $\underset{\mathcal{T}_x}{\equiv}$ and $\underset{\mathcal{C}(\mathcal{T}_x)}{\equiv}$ denote the equivalence relations in K and Th given respectively by

$$\omega_1 \underset{\mathcal{T}_x}{\equiv} \omega_2 \Longleftrightarrow (\omega_1 - \omega_2)(\mathcal{T}_x) = \{0\}, \tag{4}$$

$$\rho_1 \underset{\mathcal{C}(\mathcal{T}_x)}{\equiv} \rho_2 \Longleftrightarrow (\rho_1 - \rho_2)(\mathcal{C}(\mathcal{T}_x)) = \{0\}. \tag{5}$$

So, it ensures the existence of *inverse of c→q channel* $\mathcal{C}^*$ on K:

$$K \ni \omega_\rho = \mathcal{C}^*(\rho) \longleftrightarrow (\mathcal{C}^*)^{-1}(\omega_\rho) = \rho \in Th, \tag{6}$$

and the thermal interpretation of thermal reference states $\in K$ is just given by this ***q →c channel*** $(\mathcal{C}^*)^{-1} : K \ni \omega \longmapsto \rho \in Th$ s.t. $\omega = \mathcal{C}^*(\rho)$ [8]. In the parallelism between the integral representation in Eq.(2) and the Fourier decomposition of a function, we note that $(\mathcal{C}^*)^{-1}$ acting on $\omega_\rho \in K$ corresponds to the Fourier transform.

To adapt to our discussion of local thermal situations, we summarize the above points in such a form of *adjunction* [14] as

$$K/\mathcal{T}_x(\omega, \mathcal{C}^*(\rho)) \overset{q \rightleftarrows c}{\simeq} Th/\mathcal{C}(\mathcal{T}_x)((\mathcal{C}^*)^{-1}(\omega), \rho), \tag{7}$$

with a quantum state $\omega \in E$ and a probability measure $\rho \in Th$. While the general notion of adjunction is convenient in formulating a unified scheme for attaining simultaneously the selection of relevant objects (on the left) and interpreting the selected objects (on the right), it can be understood in the present context as a mutual relation between two groupoids denoted respectively by $K/\mathcal{T}_x$ and $Th/\mathcal{C}(\mathcal{T}_x)$ arising from the equivalence relations $\underset{\mathcal{T}_x}{\equiv}$ and $\underset{\mathcal{C}(\mathcal{T}_x)}{\equiv}$ given by Eqs.(4) and (5) on K and Th. Then

Proposition 2 *Under the condition of (★), the groupoids* $K/\mathcal{T}_x$ *and* $Th/\mathcal{C}(\mathcal{T}_x)$ *are isomorphic with the c→q channel* $\mathcal{C}^* : Th \to K$ *as a groupoid isomorphism preserving the structures as in (3).*

For an arbitrary state $\omega \in E_x$ at x, the existence of a non-empty set $K/\mathcal{T}_x(\omega, \mathcal{C}^*(\rho))$ of arrows in $K/\mathcal{T}_x$ identifies it with a uniquely determined member $\mathcal{C}^*(\rho)$ of K through the relation $\omega \underset{\mathcal{T}_x}{\equiv} \mathcal{C}^*(\rho)$, which can be transmitted by the $q{\to}c$ channel $(\mathcal{C}^*)^{-1}$ meaningful on K to the right-side $Th/\mathcal{C}(\mathcal{T}_x)((\mathcal{C}^*)^{-1}(\omega), \rho)$ of Eq.(7) to provide the thermal interpretation of the selected ω by $(\mathcal{C}^*)^{-1}(\omega) \underset{\mathcal{C}(\mathcal{T}_x)}{\equiv} \rho \in Th$ in terms of a probability distribution ρ (of temperature, etc.).

From the viewpoint of adjunctions, the essence of (★) in this form (7) can easily be generalized to selection criteria for choosing states of relevance in wider contexts, where the arrows of relevant categories are not necessarily invertible and where the $c{\to}q$ and $q{\to}c$ channels need be replaced by *functors* constituting *adjoint pairs* and so on. One of the merits of the use of adjunctions is that it clearly shows the characteristic features, essence, and basic ingredients common to all the problems to *select* objects with specific properties from generic ones and to *describe, classify* and *interpret* the features of all what to be selected by comparing them with special standard reference objects. In this setup, for instance, it is evident and conceptually important that we have here *two different levels or domains*, quatum statistical mechanics with family K of mixtures of KMS states and macroscopic thermodynamics described by Th of probability measures of fluctuating thermal parameters on the parameter space B_K, which are so interrelated by the two channels, $c \to q$ $(\mathcal{C}^*)$ and $q \to c$ $((\mathcal{C}^*)^{-1})$, that the following two points are simultaneously attained:

a) characterization of thermal reference states K as image of $\mathcal{C}^*$, $\omega_\rho = \mathcal{C}^*(\rho)$: *selection criterion* for K,

b) *thermal interpretation* of selected states in K in terms of classical data, $\Phi_x = \mathcal{C}(\hat{\Phi}(x))$ and $\rho = (\mathcal{C}^*)^{-1}(\omega_\rho)$.

For implementing this sort of machineries in the actual situations, the most non-trivial steps are the pertinent choices of pair of maps (adjoint pair of functors) corresponding to (and, generalizing) the $c{\to}q$ and $q{\to}c$ channels together with the ***standard reference systems*** for comparison.

Going back to the original context, the problem is now boiled down into how to select suitable classes of *non-equilibrium states* $\omega \notin K$ in such a way that some thermal interpretations are still guaranteed. This is what to be answered in the next subsection.

2.2 Selection criterion for non-equilibrium states

Selection criterion and thermal interpretation of non-equilibrium local states based on hierarchized zeroth law of local thermodynamics [8]: To meet simultaneously the two requirements of characterizing an unknown state ω as a non-equilibrium local state and of establishing its thermal interpretations in a similar way to the above a) and b), we now compare ω with thermal refer-

ence states $\in K = \mathcal{C}^*(Th)$ by means of some local thermal observables at x whose physical meanings are exhibited by the associated thermal functions as seen above. In view of the above conclusion [$q\to c$ channel $(\mathcal{C}^*)^{-1}$ on K] = [thermal interpretation of quantum states] and also of the hierarchical structure in $\mathcal{T}_x$, we relax the requirement for ω to agree with $\exists\omega_{\rho_x} := \mathcal{C}^*(\rho_x) \in K$ up to some suitable *sub*space $\mathcal{S}_x$ of local thermal observables $\mathcal{T}_x$. Then we characterize ω as a non-equilibrium local state by

iv) a *selection criterion* for ω to be $\mathcal{S}_x$-*thermal* at x, requiring the existence of $\rho_x \in Th$ s.t.

$$\omega(\hat{\Phi}(x)) = \mathcal{C}^*(\rho_x)(\hat{\Phi}(x)) \quad \text{for } \forall\hat{\Phi}(x) \in \mathcal{S}_x, \tag{8}$$

or, $\omega \underset{\mathcal{S}_x}{\equiv} \mathcal{C}^*(\rho_x)$, for short. In terms of thermal functions $\Phi := \mathcal{C}(\hat{\Phi}(x)) \in \mathcal{C}(\mathcal{S}_x)$, this can be rewritten as $\omega(\Phi)(x) := \omega(\hat{\Phi}(x)) = \rho_x(\Phi)$ for $\Phi \in \mathcal{C}(\mathcal{S}_x)$. So, ω: $\mathcal{S}_x$-*thermal* implies that the selection criterion $\omega \underset{\mathcal{S}_x}{\equiv} \mathcal{C}^*(\rho_x)$ can be "solved" conditionally in favour of ρ_x as "$(\mathcal{C}^*)^{-1}$"$(\omega) \underset{\mathcal{C}(\mathcal{S}_x)}{\equiv} \rho_x$, which provides the local thermal interpretation of ω [8]. Physically this means the state ω looks like a statistical mixture $\mathcal{C}^*(\rho_x)$ of thermal equilibria *locally* at x to within a level controlled by a subset $\mathcal{S}_x$ of thermal observables.

To be precise mathematically, we should be careful here about the meaning of the heuristic expression "$(\mathcal{C}^*)^{-1}$"(ω) for $\omega \notin K$ in relation to the above fact: $\omega \notin K = \mathcal{C}^*(Th)$. Physically this is related to the deviations of ω from $\mathcal{C}^*(\rho_x)$ revealed by the finer resolutions which exhibit the extent of ω being *away from equilibrium* even locally. As seen below, "$(\mathcal{C}^*)^{-1}$" outside of K is certainly *not* a $q\to c$ channel preserving the positivity, whereas it can be seen to be still definable on the states ω selected out by the above criterion Eq.(8), by means of its equivalent reformulation given by:

Proposition 3 *[7] For a subspace $\mathcal{S}_x$ of $\mathcal{T}_x$ containing* **1**, *a state $\omega \in E_x$ is $\mathcal{S}_x$-thermal iff there is a compact set $B \subset V_+$ of inverse temperatures s.t.*

$$\begin{aligned} |\omega(\hat{\Phi}(x))| \le \tau_B(\hat{\Phi}(x)) &:= \sup_{(\beta,\mu)\in B_K, \beta\in B} |\omega_{\beta,\mu}(\hat{\Phi}(x))| \\ &= \left|\left|\mathcal{C}(\hat{\Phi}(x))\right|\right|_B, \quad \textit{for } \hat{\Phi}(x) \in \mathcal{S}_x. \end{aligned} \tag{9}$$

(For the above semi-norm to be well-defined, $B_K \ni (\beta,\mu) \longmapsto \omega_{\beta,\mu} \in K$ should be (weakly) continuous, which requires singularities of critical points to be excluded from our considerations.)

Since the requirement for "$(\mathcal{C}^*)^{-1}$"(ω) to be a probability measure forces ω to belong to K, it is incompatible with our premise $\omega \notin K$. However, the above inequality (9) combined with the Hahn-Banach extension theorem (under the assumption for τ_B to be a norm) allows us to extend $\mathcal{C}(\mathcal{S}_x) \ni \mathcal{C}(\hat{\Phi}(x)) \longmapsto \omega(\hat{\Phi}(x))$ as a *linear functional* defined on $\mathcal{C}(\mathcal{S}_x)$ to one ν defined

on $\overline{\mathcal{C}(\mathcal{T}_x)} = C(B_K)$, which should *not* be a positive-definite measure but is allowed to be a *signed* measure: $\nu = \nu_+ - \nu_-$, $0 \leq \nu_\pm \in C(B_K)^*_+$, $\nu_- \neq 0$, $\nu_- \restriction_{\mathcal{C}(\mathcal{S}_x)} = 0$, $\mathcal{C}^*(\nu_+) \restriction_{\mathcal{S}_x} = \omega \restriction_{\mathcal{S}_x}$. Thus, understanding $(\mathcal{C}^*)^{-1}(\omega)$ as the set of inverse images of ω under $\mathcal{C}^*$ in the space $C(B_K)^*$ of linear functionals, $(\mathcal{C}^*)^{-1}(\omega) := \{\nu \in C(B_K)^*; \nu = \nu_+ - \nu_-, \nu_\pm \geq 0, \nu_- \restriction_{\mathcal{C}(\mathcal{S}_x)} = 0, \mathcal{C}^*(\nu_+) \restriction_{\mathcal{S}_x} = \omega \restriction_{\mathcal{S}_x}\}$, we can put Eq.(8) into the similar form to Eq.(7) as

v) The characterization and local thermal interpretation of a non-equilibrium local state:

Proposition 4 *[9] The following isomorphism holds for $\omega \in E_x$, $\rho_x \in Th$ and a subspace $\mathcal{S}_x \subset \mathcal{T}_x$,*

$$E_x/\mathcal{S}_x(\omega, \mathcal{C}^*(\rho_x)) \overset{q \rightleftarrows c}{\simeq} Th/\mathcal{C}(\mathcal{S}_x)((\mathcal{C}^*)^{-1}(\omega), [\rho_x]), \tag{10}$$

where $[\rho_x] := \{\sigma \in Th; \sigma \restriction_{\mathcal{C}(\mathcal{S}_x)} = \rho_x \restriction_{\mathcal{C}(\mathcal{S}_x)}\}$. The existence of ρ_x making the arrows sets non-empty is equivalent to the $\mathcal{S}_x$-thermality of ω.

This relation can be viewed as a form of "*hierarchized zeroth law of local thermodynamics*"; the reason for mentioning the "zeroth law" here is due to the implicit relevance of measuring processes of local thermal observables validating the above equalities, which require the *contacts of two bodies*, measured object(s) and measuring device(s), in a local thermal equilibrium, conditional on the chosen $\mathcal{S}_x$. The transitivity of this contact relation just corresponds to the localized and hierarchized version of the standard zeroth law of thermodynamics. In use of the relation

$$\exists \nu = \nu_+ - \nu_- \in (\mathcal{C}^*)^{-1}(\omega) \text{ with } \nu_- = 0 \Longleftrightarrow (\mathcal{C}^*)^{-1}(\omega) = \{\nu\} \subset Th$$
$$\Longleftrightarrow \omega \in K \Longleftrightarrow [\text{maximal choice of } \mathcal{S}'_x \text{ s.t. } \mathcal{C}^*(\nu_+) \restriction_{\mathcal{S}'_x} = \omega \restriction_{\mathcal{S}'_x}] = \mathcal{T}_x,$$

we can specify by the *failure of positivity* ($\nu_- \neq 0$) the extent to which a non-equilibrium $\mathcal{S}_x$-thermal ω deviates from equilibria belonging to K, and can also measure it by the *maximal size* of $\mathcal{S}'_x$ within the hierarchy of subspaces $\mathcal{S}'_x$ in $\mathcal{T}_x$ such that $\mathcal{S}'_x \supset \mathcal{S}_x$, $\nu_- \restriction_{\mathcal{C}(\mathcal{S}'_x)} = 0$ with all the possible choices of $\nu \in (\mathcal{C}^*)^{-1}(\omega)$: owing to the presence of ν_-, ω ceases to be $\mathcal{S}'_x$-thermal when $\mathcal{S}'_x$ is so enlarged that $\nu_- \restriction_{\mathcal{C}(\mathcal{S}'_x)} = 0$ is invalidated, which shows that ω shares with reference states in K only gross thermal properties described by smaller $\mathcal{S}'_x$. In this sense, the hierarchy of $\mathcal{S}'_x$ in $\mathcal{T}_x$ should have a close relationship with the thermodynamic hierarchy at various scales appearing in the transitions between non-equilibrium and equilibrium controlled by certain family of *coarse graining* procedures. Thus, we see that our selection criterion can give a characterization of states identifiable as non-equilibrium ones and, at the same time, provide associated relevant physical interpretations of the selected states in a systematic way. Since the presence of the positivity-violating term $\nu_- \neq 0$ is a signal of the fact that the thermal parameters belonging to B_K are *not* suitable for characterizing a non-equilibrium state

$\omega \notin K$, however, it may be more natural to expect the emergence of *new kinds of order parameters* characteristic to non-equilibrium states which requires to transform $\nu_- \neq 0$ into a form without violating positivity.

The two goals of identifying non-equilibrium local states admitting local thermal interpretation and of describing their specific thermodynamic properties are solved simultaneously by the above selection criterion based upon a *localized and hierarchized form of the zeroth law* of thermodynamics. In this framework, we can identify at least three different kinds of sources of derivations of an $\mathcal{S}_x$-thermal non-equilibrium local state $\omega \in E_x$ from the genuine equilibrium states ω_β as

a) *spacetime dependence* of thermal parameters such as temperature distributions $x \longmapsto \beta(x)$,

b) *statistical fluctuations* of thermal parameters at x described by probability distributions $d\rho_x(\beta) \in Th$,

c) essential deviations of local states $\omega \in E_x$ from states in K expressed by the *positivity-violating* term $\nu_- \neq 0$ in $\nu = \nu_+ - \nu_- \in (\mathcal{C}^*)^{-1}(\omega) \subset C(B_K)^*$ with $\nu_- \restriction_{\mathcal{C}(\mathcal{S}_x)} = 0, \mathcal{C}^*(\nu_+) \restriction_{\mathcal{S}_x} = \omega \restriction_{\mathcal{S}_x}$.

3 Reformulation of DHR-DR superselection theory

According to the above viewpoint, we can now reformulate the essence of the DHR- and DR-superselection theory whose essence can be summarized as follows. Here the basic ingredients of the theory with localizable charges [2, 4] are a *net* $\mathcal{K} \ni \mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$ of local W*-subalgebras $\mathfrak{A}(\mathcal{O})$ of *local observables*, defined on the set, $\mathcal{K} := \{(a+V_+) \cap (b-V_+); a, b \in \mathbb{R}^4\}$, of all double cones in the Minkowski spacetime $\mathbb{R}^4$; it is assumed to satisfy the isotony $\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$, allowing the global (or, quasi-local) algebra of observables $\mathfrak{A} := C^*\text{-} \varinjlim_{\mathcal{K} \ni \mathcal{O} \to \mathbb{R}^4} \mathfrak{A}(\mathcal{O})$ to be defined as the C*-inductive limit, to transform covariantly under the action of the Poincaré group $\mathcal{P}_+^\uparrow := \mathbb{R}^4 \rtimes L_+^\uparrow \ni (a, \Lambda) \longmapsto \alpha_{(a,\Lambda)} \in Aut(\mathfrak{A})$, $\alpha_{(a,\Lambda)}(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\Lambda(\mathcal{O}) + a)$, and to satisfy the local commutativity, $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$ for $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ spacelike separated: $\forall x \in \mathcal{O}_1, \forall y \in \mathcal{O}_2$, $(x-y)^2 < 0$.

- **DHR criterion**: A physically relevant state $\omega \in E_{\mathfrak{A}}$(: the set of all states of $\mathfrak{A}$ defined as normalized positive linear functionals on $\mathfrak{A}$) around a pure vacuum $\omega_0 \in E_{\mathfrak{A}}$ is selected by the Doplicher-Haag-Roberts (DHR) criterion which requires the GNS reprepresentation π_ω corresponding to ω to be unitarily equivalent to the vacuum representation $\pi_{\omega_0} =: \pi_0$ in spacelike distance; i.e., $\exists \mathcal{O} \in \mathcal{K}$ s.t. for $\forall a \in \mathbb{R}^4$

with $\mathcal{O}_a := \mathcal{O} + a \in \mathcal{K}$

$$\pi_\omega \restriction_{\mathfrak{A}(\mathcal{O}'_a)} \cong \pi_0 \restriction_{\mathfrak{A}(\mathcal{O}'_a)}, \tag{11}$$

where $\mathcal{O}' := \{x \in \mathbb{R}^4;\ (x-y)^2 < 0 \text{ for } \forall y \in \mathcal{O}\}$ is the causal complement of $\mathcal{O}$ and $\mathfrak{A}(\mathcal{O}') := C^*\text{-}\varinjlim_{\mathcal{K} \ni \mathcal{O}_1 \subset \mathcal{O}'} \mathfrak{A}(\mathcal{O}_1)$.

- **Local endomorphisms**: In the GNS representation $(\pi_0, \mathfrak{H}_0)$ corresponding to ω_0, the validity of Haag duality, $\pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O}))''$, is assumed. On the basis of the standard postulates [2], the DHR selection criterion (11) can be shown to be equivalent to the existence of a *local endomorphism* $\rho \in End(\mathfrak{A})$ such that $\pi_\omega = \pi_0 \circ \rho$, localized in some $\mathcal{O} \in \mathcal{K}$ in the sense of $\rho(A) = A$ for $\forall A \in \mathfrak{A}(\mathcal{O}')$. In this situation, we say (in a rather sloppy way) that the *support of* ρ is (contained in) $\mathcal{O}$: $\mathrm{supp}(\rho) \subset \mathcal{O}$.

- **Transportability** (of charges associated with an *internal* symmetry): The above spacetime dependence of ρ coming from its localization region $\mathcal{O}$ can be absorbed into its *transportability*, namely, for any translation $a \in \mathbb{R}^4$, there exists $\rho_a \in End(\mathfrak{A})$ with support in $\mathcal{O} + a$ and $\rho \cong \rho_a = Ad(u_a) \circ \rho$ with a unitary $u_a \in \mathfrak{A}$. We denote $\Delta(\mathcal{O}) := \{\rho \in End(\mathfrak{A});\ \rho$: transportable and localized in $\mathcal{O}\}$.

- **DR-category** [5]: Then a **C*-tensor category** $\mathcal{T}$ which we call here a DR-category is defined as a full subcategory of $End(\mathfrak{A})$ consisting of *objects* $\rho \in \Delta := \cup_{\mathcal{O} \in \mathcal{K}} \Delta(\mathcal{O})$ and with *morphisms* (or, *arrows*) given by intertwiners $T \in \mathfrak{A}$ between $\rho, \sigma \in \Delta$ s.t. $T\rho(A) = \sigma(A)T$. $\mathcal{T}$ has the *permutation symmetry* due to the locality, and is closed under *direct sums* and *subobjects* [3] (due to the Property B [6] following from the spectrum condition, locality and weak additivity).

Up to the technical details, the essential contents can be summarized in the following basic results due to the structure of $\mathcal{T}$ as a C*-tensor category having the *permutation symmetry*, *direct sums*, *subobjects* and *conjugates*:

- Unique existence of an *internal symmetry group* G such that $\mathcal{T} \simeq Rep_G \overset{\text{Tannaka-Krein duality}}{\longleftrightarrow} G = End_\otimes(V)$,where $End_\otimes(V)$ is the group of natural unitary transformations $g = (g_\rho)_{\rho \in \mathcal{T}} : V \dot{\to} V$ from the C*-tensor functor $V : \mathcal{T} \hookrightarrow Hilb$ to itself [5, 14] characterized by $g_{\rho\sigma} = g_\rho \otimes g_\sigma$ and the commutativity of the diagram:

$$\begin{array}{ccccc} \rho_1 & V_{\rho_1} & \overset{g_{\rho_1}}{\to} & V_{\rho_1} & \\ T\downarrow & T\downarrow & \circlearrowleft & \downarrow T & . \\ \rho_2 & V_{\rho_2} & \overset{g_{\rho_2}}{\to} & V_{\rho_2} & \end{array} \tag{12}$$

V embeds $\mathcal{T}$ into the category *Hilb* of Hilbert spaces and its image turns out to be just the category Rep_G of unitary representations (γ, V_γ) of a compact Lie group $G \subset SU(d)$ (owing to the presence of *conjugates* in $\mathcal{T}$), where the dimensionality d is intrinsically defined in $\mathcal{T}$ by the generating element $\rho \in \mathcal{T}$ [5]. In this formulation, the essence of Tannaka-Krein duality [15] is found in the one-to-one correspondence, ${}_{\mathcal{I}}\backslash^{\Delta} \ni [\rho] = [\rho_\gamma] \longleftrightarrow \gamma = \gamma_\rho \in \hat{G}$, with $\rho \in \Delta$ satisfying $\rho(\mathfrak{A})' \cap \mathfrak{A} = \mathbb{C}\mathbf{1}$ (corresponding to the *irreducibility* of γ_ρ), and the identification $g_\rho = \gamma_\rho(g)$ for $g \in G$, where ${}_{\mathcal{I}}\backslash^{\Delta}$ is the set of equivalence classes $\{Ad(v) \circ \rho; \mathcal{O} \in \mathcal{K}, v \in \mathfrak{A}(\mathcal{O})\} \subset \Delta$ of ρ w.r.t. the action of inner automorphism group $\mathcal{I} = \{Ad(v); \mathcal{O} \in \mathcal{K}, v \in \mathcal{U}(\mathfrak{A}(\mathcal{O}))$: unitary operators$\}$ and the group dual $\hat{G}$ is defined by the totality of equivalence classes of unitary irreducible representations of G. The relation $g_{\rho\sigma} = g_\rho \otimes g_\sigma$ can be understood as representating the tensor structure of representations γ_ρ of G (i.e., a representation of representations) and the relation $Tg_{\rho_1} = g_{\rho_2}T$ rewritten by $T\gamma_{\rho_1}(g) = \gamma_{\rho_2}(g)T$ shows that the intertwiner T from ρ_1 to ρ_2 is just the one from γ_{ρ_1} to γ_{ρ_2} in the context of group representations.

- Unique existence of a *field algebra* such that

$$\mathfrak{F} := \mathfrak{A} \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d \qquad \curvearrowleft G = Aut_{\mathfrak{A}}(\mathfrak{F}) = Gal(\mathfrak{F}/\mathfrak{A}) \tag{13}$$

$$:= \{\tau \in Aut(\mathfrak{F}); \tau(A) = A,\ \forall A \in \mathfrak{A}\}\ (:\ \text{Galois group}),$$

with $\mathfrak{A} = \mathfrak{F}^G$, where $\mathcal{O}_d$ is the Cuntz algebra [16] generated by d isometries ψ_i, $i = 1, 2, \cdots, d$, $\psi_i^*\psi_j = \delta_{ij}\mathbf{1}, \sum_{i=1}^d \psi_i\psi_i^* = \mathbf{1}$, whose fixed-point subalgebra $\mathcal{O}_d^G$ is embedded into $\mathfrak{A}$, $\mu : \mathcal{O}_d^G \hookrightarrow \mathfrak{A}$, satisfying the relation $\mu \circ \sigma = \rho \circ \mu$ for the canonical endomorphism σ of $\mathcal{O}_d$: $\sigma(C) := \sum_{i=1}^d \psi_i C \psi_i^*$. $\mathfrak{F}$ is uniquely defined as a tensor product of $\mathfrak{A}$ as a right $\mathcal{O}_d^G$-module via μ and of $\mathcal{O}_d$ as a left $\mathcal{O}_d^G$-module.

- The *sector structure* in the irreducible vacuum representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$ is understood as follows: first, the group G of symmetry is **unbroken** with a unitary implementer $U : G \to \mathcal{U}(\mathfrak{H})$, $\pi(\tau_g(F)) = U(g)\pi(F)U(g)^*$ and is *global* (i.e., gauge symmetry of the 1st kind) [due to the transportability]. This representation is realized as the induced representation of $\mathfrak{F}$ from the pure vacuum representation $(\pi_0, \mathfrak{H}_0)$ of $\mathfrak{A}$ through the conditional expectation of G-average $m : \mathfrak{F} \ni F \longmapsto \int_G dg\tau_g(F) \in \mathfrak{A}$ arising from the vacuum state $\bar{\omega}$ of $\mathfrak{F}$ given by $\bar{\omega}(F) := \omega_0(m(F))$, $\pi = \pi_{\bar{\omega}}$, $\mathfrak{H} = \mathfrak{H}_{\bar{\omega}}$. Then $\mathfrak{H}$ contains the starting Hilbert space $\mathfrak{H}_0$ of the vacuum representation π_0 of $\mathfrak{A}$ as a cyclic G-fixed-point subspace, $\mathfrak{H}_0 = \mathfrak{H}^G$, $\overline{\pi(\mathfrak{F})\mathfrak{H}_0} = \mathfrak{H}$. It is decomposed into a direct

sum in the following form [2],

$$\mathfrak{H} = \bigoplus_{\gamma\in\hat{G}} (\mathfrak{H}_\gamma \otimes V_\gamma), \tag{14}$$

$$\pi(\mathfrak{A}) = \bigoplus_{\gamma\in\hat{G}} (\pi_\gamma(\mathfrak{A}) \otimes 1_{V_\gamma}), \quad U(G) = \bigoplus_{\gamma\in\hat{G}} (1_{\mathfrak{H}_\gamma} \otimes \gamma(G)), \tag{15}$$

where **superselection sectors** defined as equivalence classes of irreducible representations $(\pi_\gamma, \mathfrak{H}_\gamma)$ of $\mathfrak{A}$ are *in one-to-one correspondence*, $\pi_\gamma = \pi_0 \circ \rho_\gamma \longleftrightarrow [\rho_\gamma] \in_{\mathcal{I}} \backslash^{\Delta} \longleftrightarrow (\gamma, V_\gamma)$, with equivalence classes of irreducible unitary representations $(\gamma, V_\gamma) \in \hat{G}$ of G.

3.1 Centre and central decompositions

What is important about (15) is the *non-triviality of the centre* of $\pi(\mathfrak{A})''$,

$$\mathfrak{Z}_\pi(\mathfrak{A}) := \pi(\mathfrak{A})'' \cap \pi(\mathfrak{A})' = \mathfrak{Z}(U(G)'') = \bigoplus_{\gamma\in\hat{G}} \mathbb{C}(1_{\mathfrak{H}_\gamma} \otimes 1_{V_\gamma}) = l^\infty(\hat{G}), \tag{16}$$

which provides the (generalized) observables $(f_\gamma)_{\gamma\in\hat{G}} \in l^\infty(\hat{G})$ as G-invariant *order parameters* to distinguish among different sectors carrying different G-charges. From our viewpoint, the physical essence of the complicated mathematical story involved in the DHR-DR sector theory can be summarized as follows: a pure state $\omega \in E_{\mathfrak{A}}$ of the observable algebra $\mathfrak{A}$ is characterized as one carrying a localized charge by the DHR selection criterion, Eq.(11), for $\pi_\omega \in Rep_{\mathfrak{A}}$, which is equivalent to the existence of $\rho \in \mathcal{T}[$: DR category $(\subset End(\mathfrak{A}))]$ s.t. $\pi_\omega = \pi_0 \circ \rho$. Via the Doplicher-Roberts categorical equivalence $\mathcal{T} \simeq Rep_G$, this data is further transformed into a G-charge $\gamma = \gamma_\rho \in \hat{G} \subset Rep_G$ describing the G-behaviour of the state $\omega \circ m$ of the field algebra $\mathfrak{F}$ induced from $\mathfrak{A}$ through the conditional expectation m, as a result of which the sector structure of states of $\mathfrak{A}$ selected by the DHR-criterion (DHR-selected states for short) is parametrized and classified by $Spec(\mathfrak{Z}_\pi(\mathfrak{A})) \simeq \hat{G}$. Namely, we can draw such a flow chart:

a DHR-selected state $\omega \in E_{\mathfrak{A}} \overset{\text{GNS-rep.}}{\Longrightarrow} [\pi_\omega \in \{\pi_0 \circ \rho; \rho \in \mathcal{T}\}(\subset Rep\mathfrak{A})]$
$\overset{\text{DHR}}{\Longleftrightarrow} [\rho \in \mathcal{T}(\subset End(\mathfrak{A})) \overset{\text{DR}}{\simeq} Rep_G] \underset{\longleftarrow}{\overset{\text{many to one}}{\Longrightarrow}} [\gamma_\rho \in \hat{G}(\subset Rep_G)]$
$\Longrightarrow$ [sectors of $\mathfrak{A}$ parametrized by $Spec(\mathfrak{Z}_\pi(\mathfrak{A})) \simeq \hat{G}$ in the irreducible vacuum representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$].

While the similarity to the scheme in Sec.2 starts now to emerge, we note that the relation of the mathematical notion of *representations* to the actual physical situations is indirect in comparison to that of *states*, in view of which it is desirable to reformulate the above scheme into such a form that the parallelism with Sec.2 becomes more evident. So, we need to examine here as to how one can physically attain the information on the G-charge contents of a given state ω of $\mathfrak{A}$ encoded in $\mathfrak{Z}_\pi(\mathfrak{A})$ as in Eq.(16), which

has not been discussed in the traditional context of the sector theory. For this purpose, starting from a generic mixture ω of DHR-selected states, we aim at an expression for it of *Fourier-decomposition type* similar to Eq.(2), $\omega_\rho = \int_{B_K} d\rho(\beta,\mu)\omega_{\beta,\mu} = C^*(\rho)$, for a thermal reference state $\omega_\rho \in K$.

Consider the relation between states and representations of $\mathfrak{A}$: in the direction of [state $\Longrightarrow$ representation], the GNS construction, $E_{\mathfrak{A}} \ni \omega \stackrel{\mathrm{GNS}}{\longmapsto} (\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ induces a canonical map $E_{\mathfrak{A}} \ni \omega \longmapsto (\pi_\omega, \mathfrak{H}_\omega) \in Rep_{\mathfrak{A}}$ (well-defined up to unitary equivalence). The opposite direction is, however, many-valued, requiring to treat a suitable *set* of states, e.g., the *folium* $\mathfrak{f}(\eta)$ defined as the set of density-matrix states in $(\eta, \mathfrak{H}_\eta)$, which is related with the von Neumann algebra $\eta(\mathfrak{A})''$ by $Lin(\mathfrak{f}(\eta)) = (\eta(\mathfrak{A})'')_*$: predual. For a state ω we put $\mathfrak{f}(\omega) := \mathfrak{f}(\pi_\omega)$ with π_ω the GNS representation.

A state $\omega \in E_{\mathfrak{A}}$ of $\mathfrak{A}$ is a mixture of DHR-selected states iff $\omega \in \mathfrak{f}(\pi)$ (with π the restriction to $\mathfrak{A}$ of the vacuum representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$ induced from the vacuum representation $(\pi_0, \mathfrak{H}_0)$ of $\mathfrak{A}$), which is also equivalent to the existence of an extension $\tilde{\omega}$ of ω to the von Neumann algebra $\pi(\mathfrak{A})''$ given by $\tilde{\omega}(\tilde{A}) = Tr_{\mathfrak{H}}(\sigma_\omega \tilde{A})$ for $\tilde{A} \in \pi(\mathfrak{A})''$ with a density operator σ_ω in $\mathfrak{H}$ s.t. $\omega(A) = Tr_{\mathfrak{H}}(\sigma_\omega \pi(A))$. Through the *central decomposition* for the "simultaneous diagonalization" of centre $\mathfrak{Z}_\pi(\mathfrak{A}) = l^\infty(\hat{G})$, such a state $\omega \in \mathfrak{f}(\pi)$ can be uniquely decomposed into the sum of factor states ω_γ corresponding to $\gamma \in \hat{G}$:

$$\omega(A) = \sum_{\gamma \in \hat{G}} \mu_\omega(\gamma)\omega_\gamma(A). \tag{17}$$

Thus, we have a *q→c channel* $\omega \longmapsto \mu_\omega$ transforming quantum states into probability distributions over the spectrum $\hat{G}$ of $\mathfrak{Z}_\pi(\mathfrak{A})$, which describes G-charge contents of each such quantum state $\omega \in \mathfrak{f}(\pi)$ in terms of a probability distribution $\mu_\omega = \{\mu_\omega(\gamma)\}_{\gamma \in \hat{G}}$ over $\hat{G}$. This is in parallel with the integral decomposition Eq.(2). However, one important difference should be noted here: within a sector $(\pi_\gamma, \mathfrak{H}_\gamma)$ of the same G-charge γ, there exist *many* different states ω_γ showing different behaviours under $\mathfrak{A}$, e.g., with different localization or different energy-momentum spectrum, as energy-momentum (tensor) is invariant under G. Thus, in contrast to the thermal situation with fixed choice of $\omega_{\beta,\mu}$, each factor state ω_γ appearing on the right-hand side of Eq.(17) may vary depending upon $\omega \in \mathfrak{f}(\pi)$. In the former case, different factor KMS states $\omega_{\beta,\mu}$ are always disjoint corresponding to different order parameters (because of the uniqueness of a KMS state within its folium), whereas what is shared in common by all the pure states ω_γ within a sector is just the unitary equivalence class $[\pi_{\omega_\gamma}]$ of the corresponding GNS *representation* π_{ω_γ} of $\mathfrak{A}$ in terms of which all the above equivalent expressions starting from the DHR criterion are given. Since this point is related to the equivalence of endomorphisms $\rho \cong Ad(u) \circ \rho$ for $\rho \in \Delta$ w.r.t. $Ad(u) \in \mathcal{I} = Inn(\mathfrak{A})$ [2], we should resolve this ambiguity to extract internal

symmetry aspects of a given *state*. In view of the fact that local subalgebras $\mathfrak{A}(\mathcal{O})$ are *factor* von Neumann algebras *without centres* from which the non-trivial centre $\mathfrak{Z}_\pi(\mathfrak{A})$ arises only in the weak closure $\pi(\mathfrak{A})''$ of the *global* algebra $\mathfrak{A}$, it is also interesting to ask a related question as to how we can attain *locally* and *minimally*[1] the above solution in physical situations according to the spirit of local quantum physics [4].

This is consistently achieved in use of $\rho \in \mathcal{T}$ as follows, in parallel with the previous section. Choose a *representative* ρ_γ from each equivalence class $[\rho_\gamma] \in_{\mathcal{I}} \backslash^{\Delta}$, which amounts to a choice of a cross section $\hat{G} \ni \gamma \longmapsto \rho_\gamma \in [\rho_\gamma] \subset \Delta$ of a bundle $\Delta \twoheadrightarrow_{\mathcal{I}} \backslash^{\Delta} \simeq \hat{G} = Spec(\mathfrak{Z}_\pi(\mathfrak{A}))$. After identifying a compact Lie group G, such a choice can be achieved, e.g., by choosing one ρ_{γ_0} corresponding to the *fundamental representation* γ_0 of G; ρ_γ for arbitrary $\gamma \in \hat{G}$ can be extracted from $\rho_{\gamma_0}^n$ with suitable $n \in \mathbb{N}$ as a direct-sum component, by means of Clebsch-Gordan coefficients. In view of the physical meaning of ρ's, this choice can be interpreted as a specification of procedures to create G-charges from the vacuum.

3.2 Physical interpretation by $c \to q$ channel and its "inverse"

Then choosing an everywhere non-vanishing probability distribution μ_G over $\hat{G}$, $\mu_G = (\mu_\gamma)_{\gamma \in \hat{G}} \in (0,1)^{\hat{G}}$, $\sum_{\gamma \in \hat{G}} \mu_\gamma = 1$, we can define a *central measure* μ on $E_{\mathfrak{A}}$ with support $\{\omega_\gamma := \omega_0 \circ \rho_\gamma; \gamma \in \hat{G}\}$ in the state space $E_{\mathfrak{A}}$ whose barycentre ω_μ is given by $\omega_\mu(A) := \sum_{\gamma \in \hat{G}} \mu_\gamma \omega_0 \circ \rho_\gamma(A)$. This allows us also to define, in a similar way to the thermal situation, a *conditional expectation* $\Lambda_\mu : \mathfrak{A} \to \mathfrak{Z}_\pi(\mathfrak{A})$ as a $c \to q$ channel s.t. $\Lambda_\mu(A) := [\hat{G} \ni \gamma \longmapsto \omega_0 \circ \rho_\gamma(A)] \in \mathfrak{Z}_\pi(\mathfrak{A})$, $\Lambda_\mu^*(\nu)(A) = \sum_{\gamma \in \hat{G}} \nu_\gamma [\Lambda_\mu(A)](\gamma) = \sum_{\gamma \in \hat{G}} \nu_\gamma \omega_0 \circ \rho_\gamma(A)$. Here the definition of Λ_μ depends on the choice of a cross section $\hat{G} \ni \gamma \longmapsto \rho_\gamma \in [\rho_\gamma] \subset \Delta$ but is independent of the particular assignment of a probability weight μ_γ to each $\gamma \in \hat{G}$. In use of this freedom we see now that, similarly to the discussion in Sec.2.1, the central measure μ as a $q \to c$ *channel* allows physical interpretation w.r.t. G of all states of such forms as $\Lambda_\mu^*(\nu) = \sum_{\gamma \in \hat{G}} \nu_\gamma \omega_0 \circ \rho_\gamma \in E_{\mathfrak{A}}$ with $\nu = (\nu_\gamma)_{\gamma \in \hat{G}} \in M_1(\hat{G}) := \{(\nu'_\gamma)_{\gamma \in \hat{G}}; \nu'_\gamma \geq 0, \sum_{\gamma \in \hat{G}} \nu'_\gamma = 1\}$. Defining a map W by $W : End(\mathfrak{A}) \ni \rho \longmapsto \omega_0 \circ \rho \in E_{\mathfrak{A}}$, we see the relations

$$
\begin{aligned}
&[\Lambda_\mu(A)](\gamma) = \omega_0 \circ \rho_\gamma(A) = [W(\rho_\gamma)](A); \\
&\Lambda_\mu^*(\nu)(A) = \nu(\Lambda_\mu(A)) = \sum_{\gamma \in \hat{G}} \nu_\gamma [\Lambda_\mu(A)](\gamma) = (\sum_{\gamma \in \hat{G}} \nu_\gamma \omega_0 \circ \rho_\gamma)(A) \\
&\Longrightarrow \Lambda_\mu^*(\nu) = \sum_{\gamma \in \hat{G}} \nu_\gamma \omega_0 \circ \rho_\gamma = \sum_{\gamma \in \hat{G}} \nu_\gamma W(\rho_\gamma). \qquad (18)
\end{aligned}
$$

[1] If we are allowed to collect and to introduce *all* the information concerning $\mathfrak{A}$, then the "ambiguities" trivially disappear, because of their origins coming from the choices of states within a given sector and from that of a representative ρ among equivalent ones $Ad(u) \circ \rho$, $Ad(u) \in Inn(\mathfrak{A})$.

Therefore, the map Λ_μ^* extends W to "convex combinations" of ρ_γ's, and acts as a "*charging* map" to create from the vacuum ω_0 a state $\Lambda_\mu^*(\nu) = \sum_{\gamma\in\hat{G}} \nu_\gamma(\omega_0 \circ \rho_\gamma)$ whose charge contents are described by the charge distribution $\nu = (\nu_\gamma)_{\gamma\in\hat{G}} \in M_1(\hat{G})$ over the group dual $\hat{G}$. The role of the chosen cross section $\gamma \longmapsto \rho_\gamma$ and the state family $E_\mu := \Lambda_\mu^*(M_1(\hat{G})) = \{\sum_{\gamma\in\hat{G}} \nu_\gamma\omega_0\circ\rho_\gamma; \nu_\gamma \geq 0, \sum_{\gamma\in\hat{G}}\nu_\gamma = 1\} \subset E_{\mathfrak{A}}$ is just to make the $c\to q$ channel Λ_μ^* *invertible* on E_μ, $E_\mu \ni \omega = \Lambda_\mu^*(\nu) \longmapsto \nu \in M_1(\hat{G})$, to give a physical interpretation of ω w.r.t. G in terms of ν.

As far as the internal symmetry aspect is concerned, we see that this setup is already sufficient for providing any given state $\omega \in \mathfrak{f}(\pi)$ with its physical interpretation owing to the above observation and the simple relation between central observables and folia: any states, $\varpi_\gamma \in \mathfrak{f}(\omega_\gamma)$, in a folium of the *factorial* state $\omega_\gamma = \omega_0 \circ \rho_\gamma$ yield the same expectation value $\varpi_\gamma(f) = f_\gamma$ to each central observable $f = (f_\gamma)_{\gamma\in\hat{G}} \in l^\infty(\hat{G}) = \mathfrak{Z}_\pi(\mathfrak{A})$ which is "diagonalized" in the central decomposition. Therefore, we arrive at a similar formula to Eq.(7) in Sec.2 as

Proposition 5 *Selection and interpretation of G-charges:*

$$\begin{aligned}&(\mathfrak{f}(\pi)/\mathfrak{Z}_\pi(\mathfrak{A}))(\omega, \Lambda_\mu^*(\nu)) \simeq M_1(\hat{G})(\mu_\omega, \nu)\\ &\Longleftrightarrow \mathfrak{f}(\omega) = \mathfrak{f}(\Lambda_\mu^*(\nu)) \Longleftrightarrow \mu_\omega(\gamma) = \nu_\gamma \ (\textit{for } \forall\gamma \in \hat{G}).\end{aligned} \tag{19}$$

To obtain a formula of Fourier-decomposition type similar to Eq.(2), however, we need to exhibit the additional elements appearing in the many to one correspondence between states and representations $[E_{\mathfrak{A}} \ni \omega \overset{\text{GNS}}{\Longleftrightarrow} (\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega) \overset{\text{many to one}}{\underset{\leftarrow\!\!-}{\rightarrow}} (\pi_\omega, \mathfrak{H}_\omega) \in Rep_{\mathfrak{A}}]$, in order to relate an arbitrary state $\phi = \sum_{\gamma\in\hat{G}} \nu_\gamma\varpi_\gamma \in \mathfrak{f}(\pi)$, $\varpi_\gamma \in \mathfrak{f}(\omega\circ\rho_\gamma)$ to the family E_μ. Since each pure state belonging to $\mathfrak{f}(\omega\circ\rho_\gamma)$ is written as $\omega\circ\sigma_\gamma$ with σ_γ related to ρ_γ by $\sigma_\gamma = Ad(u_\gamma^*)\circ\rho_\gamma$ $(u_\gamma \in \mathcal{U}(\mathfrak{A}))$, we have, for $\forall\phi \in \mathfrak{f}(\pi)$ and $\forall A \in \mathfrak{A}$,

$$\phi(A) = \sum_{\gamma\in\hat{G}} \nu_\gamma \sum_{i\in I_\gamma} p_i^\gamma \omega_0 \circ Ad(u_{\gamma,i}^*) \circ \rho_\gamma(A) = \sum_{\gamma\in\hat{G}} \nu_\gamma \sum_{i\in I_\gamma} p_i^\gamma \langle u_{\gamma,i}\Omega_0 \mid \pi_0\circ\rho_\gamma(A)u_{\gamma,i}\Omega_0\rangle,$$

with $p_i^\gamma \in [0,1]$, $\sum_{i\in I_\gamma} p_i^\gamma = 1$, $u_{\gamma,i} \in \mathcal{U}(\mathfrak{A})$ for $\forall\gamma \in \hat{G}$, $\forall i \in I_\gamma$. Here ν_γ is the probability to find the sector with G-charge $\gamma \in \hat{G}$ in the state ϕ and p_i^γ is the conditional probability to find the state $\langle u_{\gamma,i}\Omega_0 \mid \pi_0\circ\rho_\gamma(-)u_{\gamma,i}\Omega_0\rangle$ associated to the vector $u_{\gamma,i}\Omega_0$, knowing that the system is already in the sector with γ.

Since the "gap" between $u_{\gamma,i}\Omega_0$ and Ω_0 is due to $u_{\gamma,i} \in \mathcal{U}(\mathfrak{A})$, its "observabality" should enable one to find some physical processes to identify it, for instance, involving *energy-momentum* (*as observables*) by some limits of taking the *lowest energy state* among $\{u_\gamma\Omega_0; u_\gamma \in \mathcal{U}(\mathfrak{A})\}$, etc. (Actually this is the same problem as discussed above concerning the choice of a section

of $\Delta \twoheadrightarrow_{\mathcal{I}} \backslash^{\Delta} \simeq \hat{G}$ in a different disguise. If we combine the data of relevant observables, such as energy-momentum, from the beginning, this can be totally absorbed into the choice of a section.) Once this is done, any other states $\phi = \sum_{\gamma\in\hat{G}} \nu_\gamma \sum_{i\in I_\gamma} p_i^\gamma \omega_0 \circ Ad(u^*_{\gamma,i}) \circ \rho_\gamma \in \mathfrak{f}(\pi)$ can be related to the corresponding $\Lambda^*_\mu(\nu) \in E_\mu$ through the measurement of relevant observables (e.g., energy momentum) and/or the limiting procedures to pick up Ω_0 as the lowest energy state among $u_\gamma\Omega_0$ with $u_\gamma \in \mathcal{U}(\mathfrak{A})$.

In this way, we attain operational interpretations of the basic results of DHR-DR theory, which provide the physical interpretation of any state $\omega \in \mathfrak{f}(\pi)$ as a mixture of the DHR-selected states, with respect to their internal-symmetry aspects, specifying its G-charge contents understood as the G-representation contents. Since the spacetime behaviours of quantum fields are expressed by the observable net $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$ and since the internal symmetry aspects are described in the above machinery also encoded in $\mathfrak{A}$, the role of the field algebra $\mathfrak{F}$ and the internal symmetry group G becomes now quite subsidiary, simply providing comprehensible vocabulary based on the covariant objects under the symmetry transformations. Thus, we have arrived at a physical and operational picture for the sector theory showing the parallelism with the previous discussion of the thermal interpretation based upon the $c\to q$ channel $\mathcal{C} : \mathcal{A} \to C(B_K)$. While, in the latter case, the reference system to provide the vocabulary for the interpretation is already known at the beginning, it is remarkable that the corresponding one, $\mathfrak{Z}_\pi(\mathfrak{A}) \simeq l^\infty(\hat{G})$, in the DHR-DR theory naturally emerges from the basic ingredients of the theory written in terms of the algebra $\mathfrak{A}$ of observables, through the chain of equivalence starting from the DHR criterion: [DHR-selected representations of $\mathfrak{A}$] $\Longleftrightarrow$ [Doplicher-Roberts category $\mathcal{T}$] $\Longleftrightarrow$ [Rep_G and G] $\Longrightarrow$ [$\hat{G} = Spec(\mathfrak{Z}_\pi(\mathfrak{A}))$].

4 Selection criteria as categorical adjunctions

Here we emphasize the important roles played by the *categorical adjunctions* underlying our discussions so far, in achieving the systematic organizations of various domains in physics: the essence of the three formulae (7), (10) and (19) encountered in Sec.2 and Sec.3 can be summarized as follows:

X $(=q)$: to be classified	$q \leftarrow c$	A $(=c)$: to classify
$x \underset{X}{\equiv} G(a)$	$\underset{F}{\overset{G}{\leftrightarrows}}$	$F(x) \underset{A}{\equiv} a$
selection criterion	$q \to c$	interpretation

with X a quantum domain of generic states to be characterized and classified, A a classical classifying space identified with the spectrum of centre, G the $c\to q$ *channel* and with F the $q\to c$ *channel* to provide the interpretation of X in terms of the vocabulary in A. This scheme exhibits the essential

meaning and the pertinence of the notion to our discussion of selecting, classifying and interpreting physically interesting classes of states.

These cases, however, share such special features that the relevant categories are groupoids of equivalence relations with all arrows invertible and that the mapping between quantum and classical domains are groupoid isomorphisms, and hence, the essence of adjunctions in our context is found in such quantitative form as above. Perhaps this is because the category consisting of states of C*-algebras is a rather rigid one, allowing only few meaningful morphisms among different objects, requiring strict equalities or equivalence relations. As we have seen above, however, once the contents of imposed selection criteria are paraphrased into different languages, such as thermal functions in Sec.2, the category of DHR-selected representations π_ω, the DR-category of local endomorphisms ρ and that of group representations γ_ρ in Sec.3, then the machinery stored in the category theory starts to work. In such contexts, objects are not always states, and arrows between objects (taking such forms as intertwiners among local endomorphisms or among group representations) or functors between different categories need not necessarily be invertible. We know such examples that what to be selected need not always be states but can be channels as well. What is important about categorical notions is their flexibility allowing to look at the same object in many different ways and to unify objects with different appearances in one and the same notion. So, without sticking to these special features of our examples, we should be open to the possible relevance of adjunction in its generic form [14], $X(x, G(a)) \simeq A(F(x), a)$, involving four levels of notions, objects, arrows, a pair of functors $G : A \to X$, $F : X \to A$ and a pair of natural transformations (= arrows between functors) η and ε between two functors, $\eta : 1_X \dot{\to} GF$, $\varepsilon : FG \dot{\to} 1_A$, where the meaning of $\simeq$ is specified by $\varepsilon_{F(x)} \circ F(\eta_x) = 1_{F(x)}$, $G(\varepsilon_a) \circ \eta_{G(a)} = 1_{G(a)}$. Identifying $A = Th/\mathcal{C}(\mathcal{S}_x)$, $X = E_x/\mathcal{S}_x$, $x = \omega \in E_x$, $a = \rho_x \in Th$, $G = \mathcal{C}^*$, we apply this to the case (10) of non-equilibrium local states. Then $F(\omega)$ can be understood as (the restriction to $\mathcal{S}_x$ of) the Hahn-Banach extension $\nu = \nu_+ - \nu_- \in C(B_K)^*$ of $\mathcal{C}(\mathcal{S}_x) \ni \mathcal{C}(\hat{A}) \longmapsto \omega(\hat{A})$ to $\mathcal{T}_x$ and $FG = F\mathcal{C}^* = 1_{Th}$. Then $\nu_- \neq 0$ for $\omega \notin K$ signals the deviation of $GF = \mathcal{C}^*F$ from 1_{E_x}. Therefore, we encounter the *hierarchical family of adjunctions* according to the choice of $\mathcal{S}_x (\subset \mathcal{T}_x)$, in which not only the validity of adjunctions with a suitable $\mathcal{S}_x$ but also its breakdown for a bigger $\mathcal{S}'_x (\supset \mathcal{S}_x)$ are physically meaningful.

In decoding the messages encoded in a selection criterion, what plays the decisive roles at the first stage is the identification of the *centre* of a representation containing universally all the selected quantum states; its spectrum provides us with the information on the associated sector structure, which serves as the vocabulary to be used when the interpretations of given quantum states are presented. The necessary bridge between the selected generic quantum states and the classical familar objects living on the

above centre is provided, in one direction, by the $c\to q$ *channel* which embeds all the known classical states (=probability measures) into the form of quantum states constituting the totality of the selected states by the starting selection criterion. The achieved identification between what is selected and what is embedded from the known world is nothing but the most important consequence of the adjunction formulated in the form of selection criterion. This automatically enables us to take the inverse of the $c\to q$ *channel* which brings in another most important ingredient, the $q\to c$ *channel* to decode the physical contents of selected states from the viewpoint of those aspects selected out by the starting criterion. Mathematically speaking, the spectrum of the centre is nothing but the *classifying space* universally appearing in the geometrical contexts; for instance, in Sec.3 of DR sector theory of unbroken symmetry described by a compact Lie group G, its dual $\hat{G}$ (of all the equivalence classes of irreducible unitary representations) is such a case, $\hat{G} = B_{\mathcal{T}}$ for $\mathcal{T}$ the DR category of local endomorphisms of the observable net, where our $q\to c$ *channel* $(\Lambda_\mu^*)^{-1}$ plays the role of the *classifying map* by embedding the G-representation contents of a given quantum state into the subset of $\hat{G}$ consisting of its irreducible components. For an arbitrary $(\mathfrak{A},G)$-module $E = \bigoplus_{\gamma\in M} \mathfrak{H}_\gamma$ (corresponding to a choice of state of $\mathfrak{A}$) whose G-representation structure is specified by a subset M of $\hat{G}$, we obtain the following relation in parallel with classifying maps of G-bundles:

$$\begin{array}{ccc} \bigoplus_{\gamma\in M} \mathfrak{H}_\gamma = E & & \bigoplus_{\gamma\in \hat{G}} \mathfrak{H}_\gamma : \text{ universal bundle of all the sectors} \\ \downarrow Rep_G & & \downarrow Rep_G \\ (\hat{G}\supset) M & \overset{\text{suppo}(\Lambda_\mu^*)^{-1}}{\hookrightarrow} & \hat{G} = B_{\mathcal{T}} \ : \text{classifying space} \end{array}$$

In parallel to the relevance of *homotopy* to the situations where classifying maps appear to reproduce the bundle structure up to homotopy, everything here is up to multiplicities, since the *G-charge contents* of a selected generic state ω are examined on the basis of the data coming from the centre which neglects all the information concerning the multiplicities. In this way, the present scheme can easily be related with many current topics concerning the geometric and classification aspects of commutative as well as non-commutative geometry based upon the (homotopical) notions of classifying spaces, K-theory and so on.

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