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# Topics on random fields

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**Abstract**

First we review our previous work on finding innovation of Gaussian processes and Gaussian random fields. Next stochastic processes and random fields which can be expressed by the representation in terms of Poisson noises are discussed. From the characteristic functional of a given compound Poisson process, a single Poisson process is deduced by computing its characteristic function . The computability of jump finding is shown for Poisson paths with fixed height of jumps and different heights of jumps.

## 1 Gaussian white noise; innovations of some linear processes

We have so far discussed the innovation of Lévy's Brownian motion, Gaussian processes, Gaussian random fields particularly for canonical cases. We shall first deal with a simpler case. Let $\{X(t)\}$ be an ordinary Gaussian process with one dimensional parameter $t \in T \subset R^1$. Assume, in particular, that the $X(t)$ has a representation in terms of a white noise $\dot{B}(t)$ as a Wiener integral of the form

$$X(t) = \int^t F(t,u)\dot{B}(u)du, \quad t \in T, \tag{1.1}$$

where the kernel $F(t,u)$ is assumed to be smooth enough in both variables. Then, its variation over an infinitesimal time interval $[t, t+dt)$ is given by

$$\delta X(t) = F(t,t)\dot{B}(t)dt + dt \int^t F_t(t,u)\dot{B}(u)du + o(dt), \tag{1.2}$$

where $F_t(t,u) = \frac{\partial}{\partial t}F(t,u)$.

As is well known, the representation of the form (1.1) is not unique for a given $X(t)$. Let us take the cannonical representation, which gives some advantage to our innovation approach. With such a choice of the representation, it satisfies the condition

$$E(X(t)|\mathbf{B}_s(X)) = \int^s F(t,u)\dot{B}(u)du, \quad \text{for any } s < t, \tag{1.3}$$

where $\mathbf{B}_s(X)$ is the smallest $\sigma-$field with respect to which all the $X(u), u \le s$, are measurable.

**Proposition 1.1** *If a Gaussian process has a representation of the form* (1.1), *the function $F(t,t)^2$ is uniquely determined regardless the representation is canonical or not.*

**Proof.** The variance of $X(t+dt)-X(t)$ is $F(t,t)^2dt+o(dt)$, which is independent of the way of representation. Hence, the assertion is proved.

We have a freedom to choose the sign of $F(t,t)$, but we do not care the sign, since $\dot{B}(t)$, which is to be associated to $dt$, has symmetric probability distribution.

Assume that

$$\delta X(t) \text{ is of order } \sqrt{dt}. \tag{1.4}$$

This means that $X(t)$ is not differentiable and $\delta X(t)$ is non-trivial. Then, the first term of (1.2) is non-vanishing. Then $F(t,t)$ is not zero and it may be taken to be positive and continuous. With this assumption and with the note that $X(t)$ has unit multiplicity (which is equivalent to the existence of the canonocal representation), we can prove the following theorem.

**Theorem 1.1** *The limit*

$$\lim_{dt\to 0+} \frac{\delta X(t) - E[\delta X(t)|\mathbf{B}_t(X)]}{F(t,t)dt} \tag{1.5}$$

*gives the innovation.*

**Remark.** The innovation obtained above will be denoted by the same symbol $\dot{B}(t)$ as was used in (1.1). However, we should note that it may be different from the original one, if the representation (1.1) is not a cannonical representation.

Once the $\dot{B}(t)$ is given for every $t$, we can define the differential operator given by

$$\partial_u = \frac{\partial}{\partial \dot{B}(u)}, \; u \le t. \tag{1.6}$$

Apply $\partial_u$ to $X(t)$ to have $F(t,u)$ :

$$\partial_u X(t) = F(t,u), u \le t.$$

It is the canonical kernel that we are looking for. Noting that $\dot{B}(t)$ is the innovation, we can establish the following proposition.

**Proposition 1.2** *The exact value of the canonical kernel $F(t,u)$ is obtained by applying the operator $\partial_u$, $u \le t$, to the $X(t)$ .*

Thus we can see that the expression (1.1) for the canonical representation can be completely determined through the determination of the innovation,and hence the structure of the given Gaussian process $X(t)$ can be known.

We now consider the non-canonical case. Let $F(t,u)$ be a non-canonical kernel $F(t,u)$ in (1.1).

We can see that the first term of (1.2) is of $O(\sqrt{dt})$ and the rest are of $O(dt)$.

$$\sum_i (\Delta_i X(t))^2 \to \int_\Delta F(t,t)^2 dt,$$

Hence we obtain

$$\dot{\overline{B}}(t) = Y(t)\dot{B}(t),$$

where $\dot{B}(t)$ is the original white noise and $Y(t) = \pm 1$.
We may choose $Y(t)$ to be non-random or independent of $\dot{B}(t)$.

**Proposition 1.3** *The characteristic functional of $\dot{\overline{B}}(t)$ is the same as $\dot{B}(t)$.*

Here we claim that $\dot{\overline{B}}(t)$ can be considered to be equivalent to the innovation.

**Theorem 1.2** *In the case of non-canonical representation, one can find an innovation which is equivalent to the original input $\dot{B}(t)$.*

**Remarks**

1. In the construction of white noise using interpolation, we may replace $X_i$ with $Y_i X_i$ where $Y_i = \pm 1$ which is independent of $\{X_i\}$.
2. This is a path-wise theory (cf. Hida-Si Si IDAQP).

## 2 Poisson noise : path-wise theory

Define the process $X(t)$ and $\tilde{X}(t)$, with the kernels $F(t,u)$ and $G(t,u)$ which are smooth in $u$, as follows.

$$X(t) = \int F(t,u)\dot{P}(u)du,$$

where $\dot{P}(u)$ ia a Poisson noise and

$$\tilde{X}(t) = \int G(t,u)\dot{Z}_\alpha(u)du,$$

where $\dot{Z}_\alpha(u)$is a symmetric stable process with exponent $\alpha$.

We can see that $X(t)$ and $\tilde{X}(t)$ are always canonical, provided that $F(t,t) \neq 0,\ G(t,t) \neq 0$.

**Problem : Jump finding for $\tilde{X}(t)$.**

**1) The characteristic functional**

We use the method of reducing characteristic functional to characteristic function. The characteristic functional of $Z_\alpha$ is

$$C_\alpha(\xi) = \exp\left[\int\int (e^{itzu} - 1)\frac{du}{|u|^{\alpha+1}}\xi(t)dt\right].$$

Take $\xi(t) = \delta_1$, then it becomes

$$\int (e^{izu} - 1)\frac{du}{|u|^{\alpha+1}}.$$

$$\int (e^{izu} - 1)\frac{du}{|u|^{\alpha+1}} - \int (e^{iz'u} - 1)\frac{du}{|u|^{\alpha+1}} = \int (e^{iz''u}(e^{izu} - 1)\frac{du}{|u|^{\alpha+1}}$$

Let $z'$ vary to get $(e^{izu} - 1)\frac{1}{|u|^{\alpha+1}}du$ which corresponds to a single Poisson with jump $u$.

**2) The case of the same height $u$ of jumps; i.e. the case of $P(u,t)$**

Let $P(u,t)$ be a Poisson path with fixed height $u$ and countable jumps almost surely at $a_j = a_j(\omega)$, where the domain is $[0,T]$. Then it can be expressd as

$$P(u,t) = u\sum_1^N \delta_{a_j}(t), \tag{2.1}$$

and its Laplace transform is obtained as

$$\begin{aligned} f(\lambda) &= \int e^{\lambda t} P(u,t)dt \\ &= u\sum_{1}^{N} e^{a_j \lambda}, \quad 0 < a_1 < a_2 < \ldots < a_N. \end{aligned} \tag{2.2}$$

in which $N$ is finite almost surely since $T$ is finite.

By letting $\lambda \to \infty$, $u$ and $a_N$'s are obtained and again letting $\lambda \to 0$, $N$ and all $a_j$'s are obtained

### 3) The case of different heights $u_j$ of jumps

Consider a Poisson path $P(u,t)$ with different heights of jumps $\{u_i\}$, with $0 < u_1 < u_2 < \ldots < u_M$, then it can be expressed as

$$P(u,t) = \sum_{1}^{N} u_j \delta_{a_j}(t), \tag{2.3}$$

and its Laplace transform is obtained as

$$f(\lambda) = \sum_{1}^{N} u_j e^{a_j \lambda}, \quad 0 < a_1 < a_2 < \ldots < a_N. \tag{2.4}$$

in which $N$ is finite almost surely since $T$ is finite as in the above case.

Assume that the jumps $\{u_j\}$ are linearly independent over $\mathbf{Z}$.(We need any condition that guarantees finite number of $u_j$'s.)

Apply the same method as above, one recovers $u_j, a_j$ and $N$ by lettting $N$ tends to infinity and $N$ tends to zero.

## 3 Random fields

In this section Gaussian random fields and Poisson random fields are discussed. Here we recall some of our previous results on finding the innovation of Gaussian random fields to compare with the case of Poisson random fields.

Let $X(C)$ be a random field with parameter $C$ which is taken to be a smooth manifold running through the parameter space of the white noise $x(u), u \in R^d, x \in E^*$. Here, $E^*$ is the space of generalized functions on $R^d$ and the white noise measure $\mu$ is introduced on $E^*$.

To fix the idea and to avoid non-essential complex assumptions, we restrict our attention to the case where the parameter $C$ is in $\mathbf{C}$ containing smooth contours (i.e. loops) in the plane. More precisely

$$\mathbf{C} = \{C : contour; smooth, ovaloid\}.$$

## I. Gaussian random fields

Let us consider Gaussian random fields

$$\{X(C); C \in \mathbf{C}\},$$

where

$$\mathbf{C} = \{C; C \in C^2, \text{diffeomorphic to } S^1, (C) \text{ is convex}\} \text{ and}$$
$$(C): \text{ being the domain enclosed by } C.$$

If $X(C)$ is a Gaussian random field with a canonical representation

$$X(C) = \int_{(C)} F(C,u)x(u)du, \tag{3.1}$$

where x(u) is a white noise, we can easily find the innovation.

Let $Y(C)$ be a Gaussian random field which is expressed by *a sample functionwise stochastic integral* with non-canonical kernel:

$$Y(C) = \int_{(C)} G(C,u)x(u)du, \tag{3.2}$$

where $G(C,u) \neq 0$ for all $u \in R^2$.

Then

$$\delta Y(C) = \int_C G(C,s)x(s)\delta n(s)ds + o(\delta). \tag{3.3}$$

Change $\delta n(s)$ so as $\{\delta n(s)\}$ to be densed in $L^2(C)$. Then a generalized functional $\overline{x}(s), s \in C$, which is eqivalent to $x(s)$, is obtained.

Thus $x(u)$ is determined as a generalized functional.

**Note** This is a big advantage to discuss variations for random field case.

**II. Homogeneous Chaos** (Hida,Si Si 1998)

Assume that

1. $X(C)$ *is causal* in terms of white noise. This means that $X(C)$ is a function only of the $x(u)$, $u \in (C)$, $(C)$ being the domain enclosed by $C$, $x \in E^*$.
2. $X(C) = X(C, x)$ is in $(S)^*$ and homogeneous in $x$. Here homogeneity means that the $S$-transform $U(C, \xi)$ is a homogeneous polynomial in $\xi$ of degree $n$ in the sense of P. Lévy.
3. $X(C, x)$ is a regular functional of $x$.

This assumption means that the kernel function which is given by the following proposition is an ordinary $L^2(R^n)-$ function.

**Proposition** *Under the above assumptions there is a positive integer $n$ such that $X(C)$ can be expressed in the form*

$$X(C) = \int_{(C)^n} F(C; u_1, \ldots, u_n) : x(u_1)x(u_2)\ldots x(u_n) : du^n, \tag{3.4}$$

*where $F(C, u_1, u_2, \ldots u_n)$ is a symmetric $L^2(R^n)-$function and where $:\ :$ is the Wick product.*

**Theorem** *The innovation for the random field $X(C)$ given by* (4.4) *is obtained as*

$$x(s) = \frac{1}{\varphi(s)}\{\frac{\delta X(C) - \hat{E}(\delta X(C)|X(C'), C' < C)}{\delta n}(s)\}. \tag{3.5}$$

**III. Poisson random field**

Let $V(u), u \in R^2$ be a Poisson white noise and

$$X(C) = \int_{(C)} V(u)du$$

be a Poisson random field.

The charactereistic functional of $X(C)$ is

$$\exp\left[\int_{(C)} (e^{i\xi(u)} - 1)du\right]$$

and it gives us countably many $\delta$-functions in $(C)$.

But $\delta X(C)$ also involves countably many $\delta$-functions on $C$. This is somewhat inconsistent wth the above fact. It means that it is not simply a marginal distribution.

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