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On Quantum Capacity and Quantum Communication Gate

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1 Introduction

In communication process, a channel has an activity to communicate information of input system to the output system. The mutual entropy denotes an amount of information correctly transmitted to the output system from the input system through a channel. The (semi-classical) mutual entropies for classical input and quantum output were defined by several researchers [7, 6, 9]. The fully quantum mutual entropy for quantum input and output by means of the relative entropy of Umegaki [24] was defined by Ohya [14] in 1983, and he extended it [16] to general quantum systems by using the relative entropy of Araki [1] and Uhlmann [25]. Capacity is one of the most fundamental tools to measure the efficiency of information transmission. The channel capacity is defined by taking the supremum of the quantum mutual entropy over all input states in a certain state space.

In order to construct an idealistic logical gate, Fredkin and Toffoli [4] proposed a logical conservative gate. Based on this logical gate, Milburn constructed a quantum logical gate [11] using a Mach - Zender interferometer with a Kerr medium. We call this gate a Fredkin - Toffoli - Milburn (FTM) gate in this paper.

In this talk, we briefly review quantum channels for several models and we briefly explain the quantum mutual entropy and the quantum capacity for quantum channels. We concretely calculate the quantum capacity for the quantum channels. We construct a quantum channel for the FTM gate and discuss the information conservation by computing the quantum mutual entropy.

2 Quantum Channels

In development of quantum information theory, the concept of channel has been played an important role. In particular, an attenuation channel introduced in [14] has been paid much attention in optical communication. A quantum channel is a map describing the state change from an initial system to a final system, mathematically. Let us consider the construction of the quantum channels.

Let $\mathcal{H}_1, \mathcal{H}_2$ be the separable Hilbert spaces of an input and an output systems, respectively, and let $\mathbf{B}(\mathcal{H}_k)$ be the set of all bounded linear operators on $\mathcal{H}_k$. $\mathfrak{S}(\mathcal{H}_k)$ is the set of all density operators on $\mathcal{H}_k$ $(k = 1, 2)$: $\mathfrak{S}(\mathcal{H}_k) \equiv \{\rho \in \mathbf{B}(\mathcal{H}_k)\,; \rho \geq 0,\ \rho = \rho^*,\ tr\rho = 1\}$

A map Λ^* from the input system to the output system is called a (purely) quantum channel. The quantum channel Λ^* satisfying the affine property (i.e., $\sum_k \lambda_k = 1\ (\forall \lambda_k \geq 0) \Rightarrow \Lambda^*(\sum_k \lambda_k \rho_k) = \sum_k \lambda_k\ \Lambda^*(\rho_k),\ \forall \rho_k \in \mathfrak{S}(\mathcal{H}_1)$) is called a linear channel. A map Λ from $\mathbf{B}(\mathcal{H}_2)$ to $\mathbf{B}(\mathcal{H}_1)$ is called the dual map of $\Lambda^* : \mathfrak{S}(\mathcal{H}_1) \to \mathfrak{S}(\mathcal{H}_2)$ if Λ satisfies

$$tr\rho\Lambda(A) = tr\Lambda^*(\rho)A$$

for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$ and any $A \in \mathbf{B}(\mathcal{H}_2)$. Λ^* from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_2)$ is called a completely positive (CP) channel if its dual map Λ satisfies

$$\sum_{j,k=1}^{n} B_j^* \Lambda\left(A_j^* A_k\right) B_k \geq 0$$

for any $n \in \mathbf{N}$, any $B_j \in \mathbf{B}(\mathcal{H}_1)$ and any $A_k \in \mathbf{B}(\mathcal{H}_2)$.

A channel transmitted from a probability measure to a quantum state is called a classical-quantum (CQ) channel, and a channel from a quantum state to a probability measure is called a quantum-classical (QC) channel. The capacity of both CQ and QC channels have been discussed in several papers [7], [17], [21].

2.1 Noisy quantum channel

In order to discuss the communication system using the laser signal mathematically, it is necessary to formulate a (quantum) communication theory being able to treat the quantum effects of signals and channels. In order to discuss influences of noise and loss in communication processes, one needs the following two systems [14]. Let $\mathcal{K}_1$, $\mathcal{K}_2$ be the separable Hilbert spaces for the noise and the loss systems, respectively. A quantum channel Λ^* is given by the composition of three mappings a^*, π^*, γ^* such as

$$\Lambda^* = a^* \circ \pi^* \circ \gamma^*.$$

a^* is a CP channel from $\mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$ to $\mathfrak{S}(\mathcal{H}_2)$ defined by

$$a^*(\sigma) = tr_{\mathcal{K}_2}\sigma$$

for any $\sigma \in \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$, where $tr_{\mathcal{K}_2}$ is a partial trace with respect to $\mathcal{K}_2$. π^* is the CP channel from $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$ to $\mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$ depending on the physical property of the device. γ^* is the CP channel from $\mathfrak{S}(\mathcal{H}_2)$ to $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$ with a certain noise state $\xi \in \mathfrak{S}(\mathcal{K}_2)$ defined by

$$\gamma^*(\rho) = \rho \otimes \xi$$

for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$. The quantum channel Λ^* with the noise ξ is written by

$$\Lambda^*(\rho) = tr_{\mathcal{K}_2}\pi^*(\rho \otimes \xi)$$

for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$.

Here we briefly review noisy quantum channel [22]. A channel Λ^* is called a noisy quantum channel if π^* and ξ above are given by

$$\xi \equiv |m\rangle\langle m| \text{ and } \pi^*(\cdot) \equiv V(\cdot)V^*,$$

where $|m\rangle\langle m|$ is m photon number state in $\mathcal{H}_1$ and V is a linear mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ given by

$$\begin{aligned} V(|n\rangle \otimes |m\rangle) &\equiv \sum_{j=0}^{n+m} C_j^{n,m} |j\rangle \otimes |n+m-j\rangle, \\ C_j^{n,m} &\equiv \sum_{r=L}^{K} (-1)^{n-r} \frac{\sqrt{n!m!j!(n+m-j)!}}{r!(n-r)!(j-r)!(m-j+r)!} \alpha^{m-j+2r} \left(-\bar{\beta}\right)^{n+j-2r} \end{aligned}$$

for any $|n\rangle$ in $\mathcal{H}_1$ and $K \equiv \min\{j, n\}$, $L \equiv \max\{j-m, 0\}$, where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$, and $\eta = |\alpha|^2$ is the transmission rate of the channel. In particular, $\rho \otimes \xi$ is given by the tensor products of two coherent states $|\theta\rangle\langle\theta| \otimes |\kappa\rangle\langle\kappa|$, then $\pi^*(\rho\otimes\xi)$ is obtained by

$$\begin{aligned}\pi^*(\rho\otimes\xi) &= |\alpha\theta+\beta\kappa\rangle\langle\alpha\theta+\beta\kappa| \\ &\quad\otimes\left|-\bar{\beta}\theta+\bar{\alpha}\kappa\right\rangle\left\langle-\bar{\beta}\theta+\bar{\alpha}\kappa\right|.\end{aligned}$$

Here we remark that an attenuation channel Λ_0^* [14] is derived from the noisy quantum channel with $m=0$.

3 Quantum Mutual Entropy

The quantum entropy was introduced by von Neumann around 1932, which is defined by

$$S(\rho)\equiv -tr\rho\log\rho$$

for any density operators ρ in $\mathfrak{S}(\mathcal{H}_1)$. For the density operator ρ, the decomposion into one dimensional projections

$$\rho=\sum_n \lambda_n E_n.$$

is called a Schatten decomposition of ρ. If these exists a degenerated eigenvalue in the spectral decomposition of ρ, the Schatten decomposition is not unique. For a quantum channel Λ^*, the compound state σ_E representing the correlation between the input state ρ and the output state $\Lambda^*\rho$ was defined in [14] by

$$\sigma_E=\sum_n \lambda_n E_n\otimes\Lambda^* E_n,$$

where the subscript E of σ means a certain Schatten decomposition of ρ. The compound state σ_E depends on a Schatten decomposition of an input state ρ.

The classical mutual entropy is determied by an input state and a channel, so that we denote the quantum mutual entropy with respect to the input state ρ and the quantum channel Λ^* by $I(\rho;\Lambda^*)$. This quantum mutual entropy $I(\rho;\Lambda^*)$ should satisfy the following three conditions:

(1) If the channel Λ^* is identity map, then the quantum mutual entropy equals to the von Neumann entropy of the input state, that is, $I(\rho;id)=S(\rho)$.

(2) If the system is classical, then the quantum mutual entropy equals to the classical mutual entropy.

(3) The following fundamental inequalities are satisfied:

$$0 \leq I(\rho; \Lambda^*) \leq S(\rho).$$

To define such a quantum mutual entropy extending Shannon's and Gelefand-Yaglom's classical mutual entropy, we need the quantum relative entropy and the joint state (it is called "compound state" in the sequel) describing the correlation between an input state ρ and the output state $\Lambda^*\rho$ through a channel Λ^*. A finite partition of measurable space in classical case corresponds to an orthogonal decomposition $\{E_k\}$ of the identity operator I of $\mathcal{H}$ in quantum case because the set of all orthogonal projections is considered to make an event system in a quantum system. It is known [18] that the following equality holds

$$\sup\left\{-\sum_k tr\rho E_k \log tr\rho E_k; \{E_k\}\right\} = -tr\rho\log\rho,$$

and the supremum is attained when $\{E_k\}$ is a Schatten decomposition of ρ. Therefore the Schatten decomposition is used to define the compound state and the quantum mutual entropy following the formulation of the classical mutual entropy by Kolmogorov , Gelfand and Yaglom [5].

The compound state σ_E (corresponding to joint state in CS) of ρ and $\Lambda^*\rho$ was introduced in [16, 17], which is given by

$$\sigma_E = \sum_k \lambda_k E_k \otimes \Lambda^* E_k, \tag{3.1}$$

where E stands for a Schatten decomposition $\{E_k\}$ of ρ, so that the compound state depends on how we decompose the state ρ into basic states (elementary events), in other words, how to see the input state.

The relative entropy for two states ρ and σ is defined by Umegaki [24] and Lindblad [10], which is written as

$$S(\rho,\sigma) = \begin{cases} tr\rho(\log\rho - \log\sigma) & (\text{when } \overline{ran\rho} \subset \overline{ran\sigma}) \\ \infty & (\text{otherwise}) \end{cases} \tag{3.2}$$

Then we can define the mutual entropy by means of the compound state and the relative entropy [14], that is,

$$I(\rho;\Lambda^*) = \sup\left\{S\left(\sigma_E, \rho\otimes\Lambda^*\rho\right); E=\{E_k\}\right\}, \tag{3.3}$$

where the supremum is taken over all Schatten decompositions because this decomposition is not unique unless every eigenvalue is not degenerated. The following lemma was proved in [13]:

For a Schatten decomposition $\rho = \sum_n \lambda_n E_n$, the relative entropy $S(\sigma_E, \sigma_0)$ with respect to σ_E and σ_0 is written by

$$S(\sigma_E, \rho\otimes\Lambda^*\rho) = \sum_n \lambda_n S(\Lambda^* E_n, \Lambda^*\rho).$$

This lemma reduces it to the following form:

$$I(\rho;\Lambda^*) = \sup\left\{\sum_k \lambda_k S(\Lambda^* E_k, \Lambda^*\rho); E=\{E_k\}\right\}. \tag{3.4}$$

This mutual entropy satisfies all conditions (i)$\sim$(iii) mentioned above.

We will briefly review more general case. If $\Lambda : \mathcal{B} \to \mathcal{A}$ is a unitial completely positive mapping between the algebras $\mathcal{A}$ and $\mathcal{B}$, that is, the dual Λ^* is a channeling transformation from the state space of $\mathcal{A}$ into that of $\mathcal{B}$, then

$$S(\Lambda^*\varphi_1, \Lambda^*\varphi_2) \le S(\varphi_1, \varphi_2). \tag{3.5}$$

Let $\Lambda : \mathcal{B} \to \mathcal{A}$ be completely positive unitial mapping and φ be a state of $\mathcal{B}$. So φ is an initial state of the channel Λ^*. The quantum mutual entropy is defined after [14] as

$$I(\varphi;\Lambda^*) = \sup\{\textstyle\sum_j \lambda_j S(\Lambda^*\varphi_j, \Lambda^*\varphi) : \sum_j \lambda_j\varphi_j = \varphi\}, \tag{3.6}$$

where the least upper bound is over all orthogonal extremal decompositions.

Note that the definition (3.3) of the mutual entropy is written as

$$I(\rho;\Lambda^*) = \sup\left\{\sum_k \lambda_k S(\Lambda^*\rho_k, \Lambda^*\rho); \rho = \sum_k \lambda_k\rho_k \in F_o(\rho)\right\},$$

where $F_o(\rho)$ is the set of all orthogonal finite decompositions of ρ. The proof of the above equality is given in [?] by means of fundamental properties of the quantum relative entropy. For the case of probability distribution

$\rho = \sum_k \lambda_k \delta_k$ and classical -quantum channel γ^*, the mutual entropy can be denoted by

$$I\left(\rho;\gamma^*\right) = \sum_k \lambda_k S\left(\gamma^*\delta_k, \gamma^*\rho\right), \tag{3.7}$$

where δ_k is the delta measure. When the minus is well-defined, it equals to

$$I\left(\rho;\gamma^*\right) = S\left(\gamma^*\rho\right) - \sum_k \lambda_k S\left(\gamma^*\delta_k\right), \tag{3.8}$$

which has been taken as the definition of the semi-classical mutual entropy for a classical-quantum channel [7, 6, 9].

Holevo proved the following inequality in 1973 [7].

When $\mathcal{A}$ =$\mathbf{C}^k$ and $\mathcal{B}$ =$\mathbf{C}^m$ of the above notation

$$I_{cl} = \sum_{i,j} p_{ji} \log \frac{p_{ji}}{p_i q_j} \leq S(\gamma^*\varphi) - \sum_k \lambda_i S(\gamma^*\varphi_k)$$

holds.

Holevo's upper bound can now be expressed by

$$S(\gamma^*\varphi) - \sum_i \lambda_i S(\gamma^*\varphi_i) = \sum_i \lambda i S(\gamma^*\varphi_i, \gamma^*\varphi). \tag{3.9}$$

Yuen and Ozawa [26] propose to call Theorem 1 the fundamental theorem of quantum communication. The theorem bounds the performance of the detecting scheme. For general quantum case, we have the following inequality according to the lemma of [14].

When the Schatten decomposition (i.e., one dimensional spectral decomposition) $\varphi = \sum_i \lambda i \varphi_i$ is unique,

$$I_{cl} \leq I = \sum_i p_i S(\Lambda^*\varphi_i, \Lambda^*\varphi).$$

for any quantum channel Λ^*.

We see that in most cases the bound can not be achieved. Namely, the bound may be achieved in the only case when the output states $\Lambda^*\varphi_i$ have commuting densities.

If the states $\Lambda^*\varphi_i$, $1 \leq i \leq m$, do not commute, then

$$I_{cl} = \sum_{i,j} p_{ji} \log \frac{p_{ji}}{p_i q_j} < S(\Lambda^*\varphi) - \sum_i \lambda_i S(\Lambda^*\varphi_i)$$

is a strict inequality.

4 Quantum capacity

The capacity of purely quantum channel was studied in [19], [20] , [23].

Let $\mathcal{S}$ be the set of all input states satisfying some physical conditions. Let us consider the ability of information transmition for the quantum channel Λ^*. The answer of this question is the capacity of quantum channel Λ^* for a certain set $\mathcal{S} \subset \mathfrak{S}(\mathcal{H}_1)$ defined by

$$C_q^{\mathcal{S}}(\Lambda^*) \equiv \sup\{I(\rho;\Lambda^*);\rho\in\mathcal{S}\}.$$

When $\mathcal{S} = \mathfrak{S}(\mathcal{H}_1)$, the capacity of quantum channel Λ^* is denoted by $C_q(\Lambda^*)$. Then the following theorem for an attenuation channel was proved in [19]. We here give a proof for a noisy quantum channel.

For a subset $\mathcal{S}_n \equiv \{\rho \in \mathfrak{S}(\mathcal{H}_1);\dim s(\rho) = n\}$, the capacity of the noisy quantum channel Λ^* satisfies

$$C_q^{\mathcal{S}_n}(\Lambda^*) = \log n,$$

where $s(\rho)$ is the support projection of ρ.

When the mean energy of the input state vectors $\{|\tau\theta_k\rangle\}$ can be taken infinite, i.e.,

$$\lim_{\tau\to\infty} |\tau\theta_k|^2 = \left|x^{(\infty)}\right|^2 = \infty$$

the above theorem tells that the quantum capacity for the noisy quantum channel Λ^* with respect to $\mathcal{S}_n$ becomes $\log n$. It is a natural result, however it is impossible to take the mean energy of input state vector infinite. Therefore we have to compute the quantum capacity

$$C_q^{\mathcal{S}_e}(\Lambda^*) = \sup\{I(\rho;\Lambda^*);\rho\in\mathcal{S}_e\}$$

under some constraint $\mathcal{S}_e \equiv \{\rho\in\mathcal{S}; E(\rho) < e\}$ on the mean energy $E(\rho)$ of the input state ρ.

In [16, 19, ?], we also considered the pseudo-quantum capacity $C_p(\Gamma^*)$ defined by (??) with the pseudo-mutual entropy $I_p(\rho;\Gamma^*)$ where the supremum is taken over all finite decompositions instead of all orthogonal pure decompositions:

$$I_p(\rho;\Gamma^*) = \sup\left\{\sum_k \lambda_k S(\Gamma^*\rho_k,\Gamma^*\rho);\rho = \sum_k \lambda_k\rho_k,\ \text{finite decomposition}\right\}. \tag{4.1}$$

However the pseudo-mutual entropy is not well-matched to the conditions explained in Sec.2, and it is difficult to compute numerically [20]. From the monotonicity of the mutual entropy [18], we have

$$0 \leq C^{\mathcal{S}_0}(\Gamma^*) \leq C_p^{\mathcal{S}_0}(\Gamma^*) \leq \sup\left\{S(\rho); \rho \in \mathcal{S}_0\right\}.$$

5 Numerical Computation for Capacity of Noisy Quantum Channel

In this section, we compute the capacity of the noisy quantum channel for input coherent states with a coherent noise state.

First we prove the following theorem.

For any states ρ given by $\rho = \lambda|x\rangle\langle x| + (1-\lambda)|y\rangle\langle y|$ with any nonorthogonal pair $x, y \in \mathcal{H}$ and any $\lambda \in [0,1]$, the Schatten decomposition of ρ is uniquely determined by

$$\rho = \lambda_0 E_0 + \lambda_1 E_1,$$

where two eigenvalues λ_0 and λ_1 of ρ are

$$\begin{aligned} \lambda_0 &= \frac{1}{2}\left\{1 + \sqrt{1 - 4\lambda(1-\lambda)\left(1 - |\langle x, y\rangle|^2\right)}\right\} = \|\rho\|, \\ \lambda_1 &= \frac{1}{2}\left\{1 - \sqrt{1 - 4\lambda(1-\lambda)\left(1 - |\langle x, y\rangle|^2\right)}\right\} = 1 - \|\rho\|. \end{aligned}$$

Moreover two projections E_0, E_1 are constructed by the eigenvectors $|e_j\rangle$ with respect to λ_j $(j = 0, 1)$

$$\begin{aligned} E_0 &= |e_0\rangle\langle e_0| = (a|x\rangle + b|y\rangle)\left(\bar{a}\langle x| + \bar{b}\langle y|\right), \\ E_1 &= |e_1\rangle\langle e_1| = (c|x\rangle + d|y\rangle)\left(\bar{c}\langle x| + \bar{d}\langle y|\right), \end{aligned}$$

where the constants a, b, c, d are given as follows:

$$
\begin{aligned}
|a|^2 &= \frac{\tau^2}{\tau^2+2\,|\langle x,y\rangle|\,\tau+1}, \\
|b|^2 &= \frac{1}{\tau^2+2\,|\langle x,y\rangle|\,\tau+1}, \\
a\bar{b} &= \bar{a}b = \frac{\tau}{\tau^2+2\,|\langle x,y\rangle|\,\tau+1}, \\
\tau &= \frac{-(1-2\lambda)+\sqrt{1-4\lambda(1-\lambda)(1-|\langle x,y\rangle|^2)}}{2(1-\lambda)\,|\langle x,y\rangle|}, \\
|c|^2 &= \frac{t^2}{t^2+2\,|\langle x,y\rangle|\,t+1}, \\
|d|^2 &= \frac{1}{t^2+2\,|\langle x,y\rangle|\,t+1}, \\
c\bar{d} &= \bar{c}d = \frac{t}{t^2+2\,|\langle x,y\rangle|\,t+1}, \\
t &= -\frac{1+|\langle x,y\rangle|\tau}{\tau+|\langle x,y\rangle|} = \frac{-(1-2\lambda)-\sqrt{1-4\lambda(1-\lambda)(1-|\langle x,y\rangle|^2)}}{2(1-\lambda)\,|\langle x,y\rangle|}.
\end{aligned}
$$

Let ρ be an input coherent state given by

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|$$

where $|0\rangle$ is a vacuum state vector in $\mathcal{H}$ and $|\theta\rangle$ is a coherent state vector in $\mathcal{H}$. From the above proposition, the Schatten decomposition of ρ is obtained by

$$\rho = \lambda_0 E_0^{0,\theta} + \lambda_1 E_1^{0,\theta},$$

where the eigenvalues λ_0 and λ_1 of ρ are

$$
\begin{aligned}
\lambda_0 &= \frac{1}{2}\left\{1+\sqrt{1-4\lambda(1-\lambda)\left(1-\exp(-|\theta|^2)\right)}\right\}, \\
\lambda_1 &= \frac{1}{2}\left\{1-\sqrt{1-4\lambda(1-\lambda)\left(1-\exp(-|\theta|^2)\right)}\right\}
\end{aligned}
$$

and the two projections $E_0^{0,\theta}$, $E_1^{0,\theta}$ are

$$
\begin{aligned}
E_0^{0,\theta} &= \left|e_0^{0,\theta}\right\rangle\left\langle e_0^{0,\theta}\right|, \\
E_1^{0,\theta} &= \left|e_1^{0,\theta}\right\rangle\left\langle e_1^{0,\theta}\right|.
\end{aligned}
$$

The eigenvector $|e_0^{0,\theta}\rangle$ with respect to λ_0 is

$$|e_0^{0,\theta}\rangle = a_{0,\theta}|0\rangle + b_{0,\theta}|\theta\rangle,$$

where

$$\begin{aligned}
|a_{0,\theta}|^2 &= \frac{\tau_{0,\theta}^2}{\tau_{0,\theta}^2 + 2\exp(-\frac{1}{2}|\theta|^2)\tau_{0,\theta} + 1} \\
|b_{0,\theta}|^2 &= \frac{1}{\tau_{0,\theta}^2 + 2\exp(-\frac{1}{2}|\theta|^2)\tau_{0,\theta} + 1} \\
a_{0,\theta}\bar{b}_{0,\theta} = \bar{a}_{0,\theta}b_{0,\theta} &= \frac{\tau_{0,\theta}}{\tau_{0,\theta}^2 + 2\exp(-\frac{1}{2}|\theta|^2)\tau_{0,\theta} + 1} \\
\tau_{0,\theta} &= \frac{-(1-2\lambda) + \sqrt{1 - 4\lambda(1-\lambda)(1-\exp(-|\theta|^2))}}{2(1-\lambda)\exp(-\frac{1}{2}|\theta|^2)}
\end{aligned}$$

The eigenvector $|e_1^{0,\theta}\rangle$ with respect to λ_1 is

$$|e_1^{0,\theta}\rangle = c_{0,\theta}|0\rangle + d_{0,\theta}|\theta\rangle,$$

where

$$\begin{aligned}
|c_{0,\theta}|^2 &= \frac{t_{0,\theta}^2}{t_{0,\theta}^2 + 2\exp(-\frac{1}{2}|\theta|^2)t_{0,\theta} + 1} \\
|d_{0,\theta}|^2 &= \frac{1}{t_{0,\theta}^2 + 2\exp(-\frac{1}{2}|\theta|^2)t_{0,\theta} + 1} \\
c_{0,\theta}\bar{d}_{0,\theta} = \bar{c}_{0,\theta}d_{0,\theta} &= \frac{t_{0,\theta}}{t_{0,\theta}^2 + 2\exp(-\frac{1}{2}|\theta|^2)t_{0,\theta} + 1} \\
t_{0,\theta} &= -\frac{1 + \exp(-\frac{1}{2}|\theta|^2)\tau_{0,\theta}}{\tau_{0,\theta} + \exp(-\frac{1}{2}|\theta|^2)}.
\end{aligned}$$

In order to compute the quantum capacity, we use the following two subsets of $\mathfrak{S}(\mathcal{H}_1)$ according to the energy constraint:

$$\begin{aligned}
\mathcal{S}_e &\equiv \left\{\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta| \in \mathfrak{S}(\mathcal{H}_1)\,;\lambda \in [0,1]\,,\theta \in \mathbf{C}, \mathbf{E}(\rho) = |\theta|^2 \le e\right\}, \\
\mathcal{S}_e' &\equiv \left\{\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta| \in \mathfrak{S}(\mathcal{H}_1)\,;\lambda \in [0,1]\,,\theta \in \mathbf{C}, \mathbf{E}'(\rho) = (1-\lambda)|\theta|^2 \le e\right\}.
\end{aligned}$$

5.1 Noisy quantum channel:

When Λ^* is the noisy quantum channel with the transmission rate η and the coherent noise state $|\kappa\rangle\langle\kappa|$, the output state $\Lambda^*\rho$ is represented by

$$\Lambda^*\rho = \lambda|\sqrt{1-\eta}\kappa\rangle\langle\sqrt{1-\eta}\kappa| + (1-\lambda)|\sqrt{\eta}\theta + \sqrt{1-\eta}\kappa\rangle\langle\sqrt{\eta}\theta + \sqrt{1-\eta}\kappa|.$$

From the above proposition, the eigenvalues of $\Lambda^*\rho$ are given by

$$\begin{aligned}
\|\Lambda^*\rho\| &= \frac{1}{2}\left\{1+\sqrt{1-4\lambda(1-\lambda)\left(1-\left|\left\langle\sqrt{1-\eta}\kappa, \sqrt{\eta}\theta+\sqrt{1-\eta}\kappa\right\rangle\right|^2\right)}\right\}, \\
&= \frac{1}{2}\left\{1+\sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-\left|\sqrt{\eta}\theta\right|^2\right)\right)}\right\}, \\
1-\|\Lambda^*\rho\| &= \frac{1}{2}\left\{1-\sqrt{1-4\lambda(1-\lambda)\left(1-\left|\left\langle\sqrt{1-\eta}\kappa, \sqrt{\eta}\theta+\sqrt{1-\eta}\kappa\right\rangle\right|^2\right)}\right\}, \\
&= \frac{1}{2}\left\{1-\sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-\left|\sqrt{\eta}\theta\right|^2\right)\right)}\right\}.
\end{aligned}$$

$\Lambda^* E_j^{0,\theta}$ can be written by

$$\Lambda^* E_j^{0,\theta} = \bar{\lambda}_j \bar{E}_{j0} + (1-\bar{\lambda}_j)\bar{E}_{j1},$$

where $\bar{\lambda}_j$ $(j = 0, 1)$ are given by

$$\begin{aligned}
\bar{\lambda}_0 &= \frac{1}{2}(1+\exp(-\frac{1}{2}(1-\eta)|\theta|^2)) \\
&\quad\times\frac{\tau_{0,\theta}^2 + 2\exp\left(-\frac{1}{2}\left|\sqrt{\eta}\theta\right|^2\right)\tau_{0,\theta}+1}{\tau_{0,\theta}^2 + 2\exp(-\frac{1}{2}|\theta|^2)\tau_{0,\theta}+1} \\
\bar{\lambda}_1 &= \frac{1}{2}(1-\exp(-\frac{1}{2}(1-\eta)|\theta|^2)) \\
&\quad\times\frac{\tau_{0,\theta}^2 + 2\exp\left(-\frac{1}{2}\left|\sqrt{\eta}\theta\right|^2\right)\tau_{0,\theta}+1}{t_{0,\theta}^2 + 2\exp(-\frac{1}{2}|\theta|^2)t_{0,\theta}+1}
\end{aligned}$$

and each projection $\bar{E}_{jk}$ is constructed by each state vector $|\bar{x}_{jk}\rangle$ as

$$\bar{E}_{jk} = |\bar{x}_{jk}\rangle\langle\bar{x}_{jk}|\,(j,k=0,1)$$

satisfying the following conditions:

$$
\begin{aligned}
\langle \bar{x}_{jk}, \bar{x}_{jk} \rangle &= 1 \quad (j, k = 0, 1), \\
\langle \bar{x}_{00}, \bar{x}_{01} \rangle &= \frac{\tau_{0,\theta}^2 - 1}{\sqrt{(\tau_{0,\theta}^2 + 1)^2 - 4\exp(-|\theta_\eta|^2)\tau_{0,\theta}^2}} \neq 0, \\
\langle \bar{x}_{10}, \bar{x}_{11} \rangle &= \frac{t_{0,\theta}^2 - 1}{\sqrt{(t_{0,\theta}^2 + 1)^2 - 4\exp(-|\theta_\eta|^2)t_{0,\theta}^2}} \neq 0.
\end{aligned}
$$

From the above proposition, the eigenvalues $\bar{\lambda}_{ji}$ $(j, i = 0, 1)$ are obtained as

$$
\begin{aligned}
\bar{\lambda}_{j0}^{\bar{x}_{j0}, \bar{x}_{j1}} &= \frac{1}{2}\left\{1 + \sqrt{1 - 4\bar{\lambda}_j(1 - \bar{\lambda}_j)(1 - |\langle \bar{x}_{j0}, \bar{x}_{j1} \rangle|^2)}\right\} \\
\bar{\lambda}_{j1}^{\bar{x}_{j0}, \bar{x}_{j1}} &= \frac{1}{2}\left\{1 - \sqrt{1 - 4\bar{\lambda}_j(1 - \bar{\lambda}_j)(1 - |\langle \bar{x}_{j0}, \bar{x}_{j1} \rangle|^2)}\right\}.
\end{aligned}
$$

The quantum mutual entropy (3.6) with respect to the input coherent states ρ and the noisy quantum channel Λ^* is uniquely obtained as

$$
I(\rho; \Lambda^*) = S(\Lambda^*\rho) - \|\rho\| S(\Lambda^* E_0^{0,\theta}) - (1 - \|\rho\|) S(\Lambda^* E_1^{0,\theta})
$$

for j,k = 0,1. Moreover

$$
\begin{aligned}
S(\Lambda^*\rho) &= -\|\Lambda^*\rho\| \log \|\Lambda^*\rho\| - (1 - \|\Lambda^*\rho\|) \log (1 - \|\Lambda^*\rho\|), \\
S(\Lambda^* E_j^{0,\theta}) &= -\sum_{i=0}^{1} \bar{\lambda}_{ji}^{\bar{x}_{j0}, \bar{x}_{j1}} \log \bar{\lambda}_{ji}^{\bar{x}_{j0}, \bar{x}_{j1}} \quad (j, k = 0, 1).
\end{aligned}
$$

From the above result of the quantum mutual entropy $I(\rho; \Lambda^*)$, we expricitly compute the quantum capacity for the noisy quantum channel Λ^* with the coherent noise state $|\kappa\rangle\langle\kappa|$ such as

$$
C_q^{\mathcal{S}_e}(\Lambda^*) = \sup\{I(\rho; \Lambda^*); \rho \in \mathcal{S}_e\}.
$$

$$
C_q^{\mathcal{S}_e}(\Lambda^*) < C_q^{\mathcal{S}_e'}(\Lambda^*).
$$

Next we discuss what is the most suitable modulation in OOK, PPM, PWM, PSK for the noisy quantum channel. The subsets with respect to the optical modulations OOK, PPM, PWK, PSK are given by

$$\begin{aligned}
\mathcal{S}_e^{OOK} &\equiv \left\{\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta| \in \mathfrak{S}(\mathcal{H}_1); \lambda \in [0,1], \theta \in \mathbb{C}, |\theta|^2 \leq e\right\} \\
\mathcal{S}_e^{PPM} &\equiv \{\rho = \lambda|0\rangle\langle 0| \otimes |\theta\rangle\langle\theta| + (1-\lambda)|\theta\rangle\langle\theta| \otimes |0\rangle\langle 0| \in \mathfrak{S}(\mathcal{H}_1) \otimes \mathfrak{S}(\mathcal{H}_1); \\
&\qquad \lambda \in [0,1], \theta \in \mathbb{C}, |\theta|^2 \leq e\} \\
\mathcal{S}_e^{PWM} &\equiv \{\rho = \lambda|0\rangle\langle 0| \otimes |\theta\rangle\langle\theta| + (1-\lambda)|\theta\rangle\langle\theta| \otimes |\theta\rangle\langle\theta| \in \mathfrak{S}(\mathcal{H}_1) \otimes \mathfrak{S}(\mathcal{H}_1); \\
&\qquad \lambda \in [0,1], \theta \in \mathbb{C}, |\theta|^2 \leq e\} \\
\mathcal{S}_e^{PSK} &\equiv \left\{\rho = \lambda|\theta\rangle\langle\theta| + (1-\lambda)|-\theta\rangle\langle-\theta| \in \mathfrak{S}(\mathcal{H}_1); \lambda \in [0,1], \theta \in \mathbb{C}, |\theta|^2 \leq e\right\}
\end{aligned}$$

Calculating the capacity of the noisy quantum channel for the above subsets consisted by the optical modulations, we have the following theorem.

The capacities of the noisy quantum channel for the subsets $\mathcal{S}_e^{OOK}$, $\mathcal{S}_e^{PPM}$ and $\mathcal{S}_e^{PSK}$ satisfy the following inequalities

$$C_q^{\mathcal{S}_e^{OOK}}(\Lambda^*) \leq C_q^{\mathcal{S}_e^{PPM}}(\Lambda^*) = C_q^{\mathcal{S}_e^{PSK}}(\Lambda^*) \leq C_q^{\mathcal{S}_e^{PWM}}(\Lambda^*).$$

6 Quantum channel for Fredkin-Toffoli-Milburn gate

Fredkin and Toffoli [4] proposed a conservative gate, by which any logical gate is realized and it is shown to be a reversible gate in the sense that there is no loss of information. This gate was developed by Milburn [11] as a quantum gate with quantum input and output. We call this gate Fredkin-Toffoli-Milburn (FTM) gate here. In this section, we first formulate the FTM gate by means of quantum channels and discuss the information conservation using the quantum mutual entropy in the next section.

The FTM gate is composed of two input gates I_1, I_2 and one control gate C. Two inputs come to the first beam splitter and one spliting input passes through the control gate made from an optical Kerr device, then two spliting inputs come in the second beam splitter and appear as two outputs (Fig.2.1). We construct quantum channels to express the beam splitters and the optical Kerr medium and discuss the works of the above gate, in particular, conservation of information.

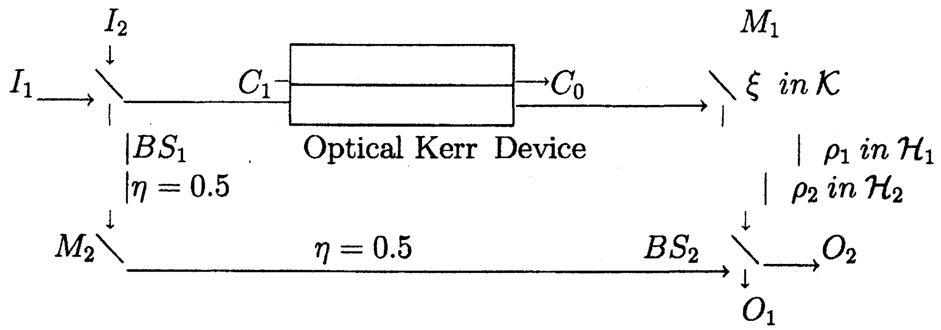

Fig 2.1 FTM gate

(1) Beam splitters: (a) Let V_1 be a mapping from $\mathcal{H}_1 \otimes \mathcal{H}_2$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with transmission rate η_1 given by

$$V_1\left(|n_1\rangle \otimes |n_2\rangle\right) \equiv \sum_{j=0}^{n_1+n_2} C_j^{n_1,n_2} |j\rangle \otimes |n_1+n_2-j\rangle \tag{6.1}$$

for any photon number state vectors $|n_1\rangle \otimes |n_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. The quantum channel Π_{BS1}^* expressing the first beam splitter (beam splitter 1) is defined by

$$\Pi_{BS1}^*\left(\rho_1 \otimes \rho_2\right) \equiv V_1\left(\rho_1 \otimes \rho_2\right) V_1^* \tag{6.2}$$

for any states $\rho_1 \otimes \rho_2 \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. In particular, for an input state in two gates I_1 and I_2 given by the tensor product of two coherent states $\rho_1 \otimes \rho_2 = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2|$, $\Pi_{BS1}^*(\rho_1 \otimes \rho_2)$ is written as

$$\begin{aligned}\Pi_{BS1}^*(\rho_1 \otimes \rho_2) &= \left|\sqrt{\eta_1}\theta_1 + \sqrt{1-\eta_1}\theta_2\right\rangle\left\langle\sqrt{\eta_1}\theta_1 + \sqrt{1-\eta_1}\theta_2\right| \\ &\quad \otimes\left|-\sqrt{1-\eta_1}\theta_1 + \sqrt{\eta_1}\theta_2\right\rangle\left\langle-\sqrt{1-\eta_1}\theta_1 + \sqrt{\eta_1}\theta_2\right| \end{aligned} \tag{6.3}$$

(b) Let V_2 be a mapping from $\mathcal{H}_1 \otimes \mathcal{H}_2$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with transmission rate η_2 given by

$$V_2\left(|n_1\rangle \otimes |n_2\rangle\right) \equiv \sum_{j=0}^{n_1+n_2} C_j^{n_2,n_1} |n_1+n_2-j\rangle \otimes |j\rangle \tag{6.4}$$

for any photon number state vectors $|n_1\rangle \otimes |n_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. The quantum channel Π_{BS2}^* expressing the second beam splitter (beam splitter 2) is defined by

$$\Pi^*_{BS2}\left(\rho_1 \otimes \rho_2\right) \equiv V_2\left(\rho_1 \otimes \rho_2\right) V_2^* \tag{6.5}$$

for any states $\rho_1 \otimes \rho_2 \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. In particular, for coherent input states $\rho_1 \otimes \rho_2 = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2|$, $\Pi^*_{BS2}(\rho_1 \otimes \rho_2)$ is written as

$$\begin{aligned}\Pi^*_{BS2}(\rho_1 \otimes \rho_2) &= \left|\sqrt{\eta_2}\theta_1 - \sqrt{1-\eta_2}\theta_2\right\rangle\left\langle\sqrt{\eta_2}\theta_1 - \sqrt{1-\eta_2}\theta_2\right| \\ &\otimes \left|\sqrt{1-\eta_2}\theta_1 + \sqrt{\eta_2}\theta_2\right\rangle\left\langle\sqrt{1-\eta_2}\theta_1 + \sqrt{\eta_2}\theta_2\right|. \end{aligned} \tag{6.6}$$

(2) Optical Kerr medium: The interaction Hamiltonian in the optical Kerr medium is given in [11] by the number operators N_1 and N_c for the input system 1 and the Kerr medium, respectively, such as

$$H_{int} = \hbar\chi\left(N_1 \otimes I_2 \otimes N_c\right), \tag{6.7}$$

where $\hbar$ is the Plank constant divided by 2π, χ is a constant proportional to the susceptibility of the medium and I_2 is the identity operator on $\mathcal{H}_2$. Let T be the passing time of a beam through the Kerr medium and put $\sqrt{F} = \hbar\chi T$, a parameter exhibiting the power of the Kerr effect. Then the unitary operator U_K describing the evolution for time T in the Kerr medium is given by

$$U_K = \exp\left(-i\sqrt{F}\left(N_1 \otimes I_2 \otimes N_c\right)\right). \tag{6.8}$$

We assume that an initial (input) state of the control gate is a number state $\xi = |n\rangle\langle n|$, a quantum channel Λ^*_K representing the optical Kerr effect is given by

$$\Lambda^*_K(\rho_1 \otimes \rho_2 \otimes \xi) \equiv U_K(\rho_1 \otimes \rho_2 \otimes \xi)U_K^* \tag{6.9}$$

for any state $\rho_1 \otimes \rho_2 \otimes \xi \in \mathfrak{S}\left(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K}\right)$. In particular, for an initial state $\rho_1 \otimes \rho_2 \otimes \xi = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2| \otimes |n\rangle\langle n|$, $\Lambda^*_K(\rho_1 \otimes \rho_2 \otimes \xi)$ is denoted by

$$\begin{aligned}&\Lambda^*_K(\rho_1 \otimes \rho_2 \otimes \xi) \\ &= \left|\exp\left(-i\sqrt{F}n\right)\theta_1\right\rangle\left\langle\exp\left(-i\sqrt{F}n\right)\theta_1\right| \otimes |\theta_2\rangle\langle\theta_2| \otimes |n\rangle\langle n|. \end{aligned} \tag{6.10}$$

Using the above channels, the quantum channel for the whole FTM gate is constructed as follows: Let both one input and output gates be described by $\mathcal{H}_1$, another input and output gates be described by $\mathcal{H}_2$ and the control gate be done by $\mathcal{K}$, all of which are Fock spaces. For a total state $\rho_1 \otimes \rho_2 \otimes \xi$ of two input states and a control state, the quantum channels $\Lambda^*_{BS1}, \Lambda^*_{BS2}$ from $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$ to $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$ are written by

$$\Lambda^*_{BSk}(\rho_1 \otimes \rho_2 \otimes \xi) = \Pi^*_{BSk}(\rho_1 \otimes \rho_2) \otimes \xi \quad (k = 1, 2) \tag{6.11}$$

Therefore, the whole quantum channel Λ^*_{FTM} of the FTM gate is defined by

$$\Lambda^*_{FTM} \equiv \Lambda^*_{BS2} \circ \Lambda^*_K \circ \Lambda^*_{BS1}. \tag{6.12}$$

In particular, for an initial state $\rho_1 \otimes \rho_2 \otimes \xi = |\theta_1\rangle \langle\theta_1| \otimes |\theta_2\rangle \langle\theta_2| \otimes |n\rangle \langle n|$, $\Lambda^*_{FTM}(\rho_1 \otimes \rho_2 \otimes \xi)$ is obtained by

$$\begin{aligned} & \Lambda^*_{FTM}(\rho_1 \otimes \rho_2 \otimes \xi) \\ = \ & |\mu\theta_1 + \nu\theta_2\rangle \langle\mu\theta_1 + \nu\theta_2| \otimes |\nu\theta_1 + \mu\theta_2\rangle \langle\nu\theta_1 + \mu\theta_2| \otimes |n\rangle \langle n| \end{aligned} \tag{6.13}$$

where

$$\mu = \frac{1}{2}\left\{\exp\left(-i\sqrt{F}n\right) + 1\right\}, \tag{6.14}$$

$$\nu = \frac{1}{2}\left\{\exp\left(-i\sqrt{F}n\right) - 1\right\}. \tag{6.15}$$

7 Information change in optical Fredkin-Toffoli-Milburn gate

In this section, we examine information conservation in the FTM gate by computing the mutual entropy.

Although the control gate, hence the Hilbert space $\mathcal{K}$, is necessary to make the truth table, the original information is carried by the input states, so it is interesting to study conservation of the information from the input to the output. For this purpose, we need the quantum channel Λ^* describing the change of states from the input gate to the output gate, which is defined as

$$\Lambda^*(\rho_1 \otimes \rho_2) \equiv tr_{\mathcal{K}} \Lambda^*_{FTM}(\rho_1 \otimes \rho_2 \otimes \xi) \tag{7.1}$$

for any input states $\rho_1 \otimes \rho_2$.

The total channel Λ^*_{FTM} is obviously unitarily implemented from the construction discussed in the previous section, but the channel Λ^* is not so as seen below:

When Λ^* is unitarily implemented, that is $\Lambda^*(\rho) = U\rho U^*$, $\rho \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with a certain unitary operator U, the dual Λ is written as $\Lambda(A) = U^*AU$ for any $A \in \mathbf{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Therefore for the CONS (complete orthonormal system) consisting of number vector states, namely, $\{|n_1\rangle\}$ in $\mathcal{H}_1$, $\{|n_2\rangle\}$ in $\mathcal{H}_2$, an equality

$$tr\Lambda(|n_1\rangle\langle k_1| \otimes |n_2\rangle\langle k_2|) = \delta_{n_1k_1}\delta_{n_2k_2}$$

should be satisfied. However the direct computation according to the definition of Λ^* implies the equality

$$\begin{aligned}
& tr\Lambda(|n_1\rangle\langle k_1| \otimes |n_2\rangle\langle k_2|) \\
= & \sum_{m_1}\sum_{m_2} tr\Lambda^*(|m_1\rangle\langle m_1| \otimes |m_2\rangle\langle m_2|)|n_1\rangle\langle k_1| \otimes |n_2\rangle \otimes \langle k_2| \\
= & \sum_{m_1}\sum_{m_2}\sum_{j=0}^{m_1+m_2}\sum_{j'=0}^{m_1+m_2} C_j^{m_1,m_2}\overline{C_{j'}^{m_1,m_2}}\exp(-i\sqrt{F}n(j-j')) \\
& \times \sum_{i=0}^{m_1+m_2}\sum_{i'=0}^{m_1+m_2} C_i^{m_1+m_2-j,j}\overline{C_{i'}^{m_1+m_2-j',j'}}\delta_{k_1,m_1+m_2-i}\delta_{k_2,i}\delta_{m_1+m_2-i',n_1}\delta_{i',n_2},
\end{aligned}$$

where $\sum_{m_j}|m_j\rangle\langle m_j| = I_j$, identity operator on $\mathcal{H}_j$ $(j = 1, 2)$. The above equality is not zero if and only if

$$n_1 + n_2 = k_1 + k_2.$$

Thus Λ^* is not unitarily implemented.

The next question is whether the information carried by two input states is preserved after passing through the whole gate, that is, whether the following equality is held or not for a certain class of input states $\rho = \rho_1 \otimes \rho_2$.

$$S(\rho) = S(\rho_1) + S(\rho_2) = I(\rho; \Lambda^*)$$

This equality means that all information carried by $\rho = \rho_1 \otimes \rho_2$ is completely transmitted to the output gates. If the channel Λ^* is unitarily implemented as Λ^*_{FTM}, then the above equality is satisfied [18]. However, our Λ^* is not, so it is important to check the above equality.

Let us consider any state ρ_i given by

$$\rho_i = \lambda_i|0\rangle\langle 0| + (1-\lambda_i)|\theta_i\rangle\langle\theta_i| \ , (i = 1, 2) \tag{7.2}$$

with $\lambda_i \in [0,1]$. Such a state is often used to send information expressed by two symbols 0 and 1. In order to compute quantum entropy and mutual entropy, we need the Schatten decomposition of $\rho = \rho_1 \otimes \rho_2$, which is uniquely given in [19] such that

$$\rho_i = \|\rho_i\| E_0^i + (1 - \|\rho_i\|) E_1^i, (i = 1, 2) \tag{7.3}$$

where$\|\rho_i\|$ is one of the eigenvalues of ρ_i and E_0^i is its associated one dimensional projection;

$$\|\rho_i\| = \frac{1 + \sqrt{1 - 4\lambda_i(1-\lambda_i)(1 - \exp(-(|\theta_i|^2)))}}{2} \tag{7.4}$$

The Schatten decomposition of $\rho = \rho_1 \otimes \rho_2$ is written by

$$\rho = \sum_{j=0}^{1} \sum_{k=0}^{1} \mu_j^1 \mu_k^2 E_j^1 \otimes E_k^2,$$

where $\mu_0^i = \|\rho_i\|$ and $\mu_1^i = 1 - \|\rho_i\|$ $(i = 1, 2)$. Then von Neumann entropy of ρ becomes

$$S(\rho) = -\sum_{i=1}^{2} \sum_{j=0}^{1} \mu_j^i \log \mu_j^i.$$

We assume $\xi = |n\rangle \langle n|$ $(n \neq 0)$ and $\sqrt{F}n = (2m+1)\pi$ $(m = 0, 1, 2, \cdots)$. For the input state $\rho = \rho_1 \otimes \rho_2$, the output state $\Lambda^* \rho$ is given by

$$\Lambda^* \rho = \sigma_2 \otimes \sigma_1,$$

where $\sigma_i = \lambda_i |0\rangle\langle 0| + (1 - \lambda_i)| - \theta_i\rangle\langle -\theta_i|$, $(i = 1, 2)$. Then von Neumann entropy of $\Lambda^* \rho$ is

$$S(\Lambda^* \rho) = S(\sigma_2) + S(\sigma_1) = S(\rho). \tag{7.5}$$

Since $\Lambda^* \left(E_j^1 \otimes E_k^2\right)$ is pure state, .$S\left(\Lambda^* \left(E_j^1 \otimes E_k^2\right)\right) = 0$ for each j,k. Thus the quantum mutual entropy is

$$\begin{aligned} I(\rho; \Lambda^*) &= S(\Lambda^* \rho) - \{\sum_{j=0}^{1} \sum_{k=0}^{1} \mu_j^1 \mu_k^2 S\left(\Lambda^* \left(E_j^1 \otimes E_k^2\right)\right)\} \\ &= S(\Lambda^* \rho) = S(\rho). \end{aligned} \tag{7.6}$$

This equalities means that there does not exist the loss of information for the quantum channel of the FTM gate. Therefore the information is preserved for Λ^* through the FTM gate. From this result, the FTM gate is considered to be an idealistic logical gate for quantum computer.

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