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State Extension from Subsystems to the Joint System

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1 Introduction

In algebraic approach to quantum systems, a system is described by a $\mathbf{C}^*$-algebra $\mathcal{A}$ and its state is a normalized positive linear functional φ, its value $\varphi(A)$ for $A\in \mathcal{A}$ being the expectation value of A in that state. Subsystems are described by $\mathbf{C}^*$-subalgebras $\mathcal{A}_i$ of $\mathcal{A}$, $i=1,2\cdots$. Their joint system is the total system described by $\mathcal{A}$ if the subalgebras $\mathcal{A}_i$ generate $\mathcal{A}$ as a $\mathbf{C}^*$-algebra. Restrictions φ_i of a state φ of $\mathcal{A}$ to subalgebras $\mathcal{A}_i$ are states of $\mathcal{A}_i$, $i=1,2\cdots$. Conversely, suppose that states φ_i of $\mathcal{A}_i$, $i=1,2\cdots$, are first given. Then a state φ of $\mathcal{A}$ is called a joint extension of states φ_i of $\mathcal{A}_i$, $i=1,2,\cdots$, if the restriction of φ to $\mathcal{A}_i$ is the given state φ_i for each i.

For spin or Boson lattice systems, algebras $\mathcal{A}_i$ of subsystems with mutually disjoint localization mutually commute and form a tensor product system. If $\mathcal{A}$ is the tensor product of $\mathcal{A}_i$, $i=1,2,\cdots$ and

$$\varphi(\prod_i A_i)=\prod_i \varphi(A_i), \quad A_i\in \mathcal{A}_i \tag{1}$$

holds, then φ is called the tensor product of φ_i, $i=1,2\cdots$. Otherwise φ is said to be entangled if φ is pure. Entanglement in tensor product systems is widely studied.

For Fermion lattice systems, algebras of subsystems with mutually disjoint localization do not mutually commute due to the anticommutativity of Fermion creation and annihilation operators. As electrons are Fermions, a study of Fermion systems seems to have a practical significance. Entanglement for Fermion systems is studied by one of the present authors recently [2].

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The present work studies the problem of joint extension of states from subsystems to the joint system for (discrete) Fermion systems and generalizes some results in [2].

2 The Fermion Algebra

We consider a $\mathbf{C}^*$-algebra $\mathcal{A}$, called a CAR algebra or a Fermion algebra, which is generated by its elements a_i and a_i^*, $i \in \mathbb{N}$ ($\mathbb{N} = \{1, 2, \cdots\}$) satisfying the following canonical anticommutation relations(CAR).

$$\begin{aligned} \{a_i^*, a_j\} &= \delta_{i,j} \mathbf{1} \\ \{a_i^*, a_j^*\} &= \{a_i, a_j\} = 0, \end{aligned}$$

($i, j \in \mathbb{N}$), where $\{A, B\} = AB + BA$ (anticommutator) and $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ otherwise. For finite subset I of $\mathbb{N}$, $\mathcal{A}(\mathrm{I})$ denotes the $\mathbf{C}^*$-subalgebra generated by a_i and a_i^*, $i \in \mathrm{I}$. A crucial role is played by the unique automorphism Θ of $\mathcal{A}$ characterized by

$$\Theta(a_i) = -a_i, \quad \Theta(a_i^*) = -a_i^*$$

for all $i \in \mathbb{N}$. The even and odd parts of $\mathcal{A}$ and $\mathcal{A}(\mathrm{I})$ are defined by

$$\mathcal{A}_\pm \equiv \{A \in \mathcal{A} \,|\, \Theta(A) = \pm A\},$$

For any $A \in \mathcal{A}$ (or $\mathcal{A}(\mathrm{I})$), we have the following decomposition

$$A_\pm = A_+ + A_-, \quad A_\pm = \frac{1}{2}(A \pm \Theta(A)) \in \mathcal{A}_\pm \text{ (or } \mathcal{A}(\mathrm{I})_\pm).$$

A state φ of $\mathcal{A}$ or $\mathcal{A}(\mathrm{I})$ is called even if it is Θ-invariant:

$$\varphi(\Theta(A)) = \varphi(A)$$

for all $A \in \mathcal{A}$ (or $A \in \mathcal{A}(\mathrm{I})$).

For a state φ of a $\mathbf{C}^*$-algebra $\mathcal{A}$ ($\mathcal{A}(\mathrm{I})$), $\{\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi\}$ denotes the GNS triplet of a Hilbert space $\mathcal{H}_\varphi$, a representation π_φ of $\mathcal{A}$ (of $\mathcal{A}(\mathrm{I})$), and a vector $\Omega_\varphi \in \mathcal{H}_\varphi$, which is cyclic for $\pi_\varphi(\mathcal{A})$ ($\pi_\varphi(\mathcal{A}(\mathrm{I}))$) and satisfies

$$\varphi(A) = (\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi)$$

for all $A \in \mathcal{A}$ ($\mathcal{A}(\mathrm{I})$). For any $x \in \mathcal{B}(\mathcal{H}_\varphi)$, we write

$$\overline{\varphi}(x) = (\Omega_\varphi, x\Omega_\varphi).$$

3 Product State Extension

As subsystems, we consider $\mathcal{A}(\mathrm{I})$ with mutually disjoint subsets I's. For a pair of disjoint subsets I_1 and I_2 of $\mathbb{N}$, let φ_1 and φ_2 be given states of $\mathcal{A}(\mathrm{I}_1)$ and $\mathcal{A}(\mathrm{I}_2)$, respectively. If a state φ of the joint system $\mathcal{A}(\mathrm{I}_1 \cup \mathrm{I}_2)$ (which is the same as the $\mathbf{C}^*$-subalgebra of $\mathcal{A}$ generated by $\mathcal{A}(\mathrm{I}_1)$ and $\mathcal{A}(\mathrm{I}_2)$) coincides with φ_1 on $\mathcal{A}(\mathrm{I}_1)$ and φ_2 on $\mathcal{A}(\mathrm{I}_2)$, i.e.,

$$\begin{aligned} \varphi(A_1) &= \varphi_1(A_1), \quad A_1 \in \mathcal{A}(\mathrm{I}_1), \\ \varphi(A_2) &= \varphi_2(A_2), \quad A_2 \in \mathcal{A}(\mathrm{I}_2), \end{aligned}$$

then φ is called a joint extension of φ_1 and φ_2. As a special case, if

$$\varphi(A_1 A_2) = \varphi_1(A_1)\varphi_2(A_2)$$

holds for all $A_1 \in \mathcal{A}(\mathrm{I}_1)$ and all $A_2 \in \mathcal{A}(\mathrm{I}_2)$, then φ is called a product state extension of φ_1 and φ_2. For an arbitrary (finite or infinite) number of subsystems, $\mathcal{A}(\mathrm{I}_1)$, $\mathcal{A}(\mathrm{I}_2), \cdots$ with mutually disjoint I's and a set of given states φ_i of $\mathcal{A}(\mathrm{I}_i)$, a state φ of $\mathcal{A}(\cup_i \mathrm{I}_i)$ is called a product state extension if it satisfies (1).

Theorem 1. *Let $\mathrm{I}_1, \mathrm{I}_2, \cdots$ be an arbitrary (finite or infinite) number of mutually disjoint subsets of $\mathbb{N}$ and φ_i be a given state of $\mathcal{A}(\mathrm{I}_i)$ for each i.*
(1) A product state extension of φ_i, $i = 1, 2, \cdots$, exists if and only if all states φ_i except at most one are even. It is unique if it exists. It is even if and only if all φ_i are even.
(2) Suppose that all φ_i are pure. If there exists a joint extension of φ_i, $i = 1, 2, \cdots$, then all states φ_i except at most one have to be even. If this is the case, the joint extension is uniquely given by the product state extension and is a pure state.

Remark. In Theorem 1 (2), the product state property (1) is not assumed but it is derived from the purity assumption for all φ_i.

The purity of all φ_i does not follow from that of their joint extension φ in general. For a product state extension φ, however, we have the following two theorems about consequences of purity of φ.

Theorem 2. *Let φ be the product state extension of states φ_i with disjoint I_i. Assume that all φ_i except φ_1 are even.*
(1) φ_1 is pure if φ is pure.
(2) Assume that π_{φ_1} and $\pi_{\varphi_1 \Theta}$ are not disjoint. Then φ is pure if and only if all φ_i are pure. In particular, this is the case if φ is even.

Remark. If I_1 is finite, the assumption of Theorem 2 (2) holds and hence the conclusion follows automatically.

In the case not covered by Theorem 2, the following result gives a complete analysis if we take $\cup_{i\geq 2} I_i$ in Theorem 2 as one subset of $\mathbb{N}$.

Theorem 3. *Let φ be the product state extension of states φ_1 and φ_2 of $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ with disjoint I_1 and I_2 where φ_2 is even and φ_1 is such that π_{φ_1} and $\pi_{\varphi_1\Theta}$ are disjoint.*
(1) φ is pure if and only if φ_1 and the restriction φ_{2+} of φ_2 to $\mathcal{A}(I_2)_+$ are both pure.
(2) Assume that φ is pure. φ_2 is not pure if and only if

$$\varphi_2 = \frac{1}{2}(\widehat{\varphi}_2 + \widehat{\varphi}_2\Theta)$$

where $\widehat{\varphi}_2$ is pure and $\pi_{\widehat{\varphi}_2}$ and $\pi_{\widehat{\varphi}_2\Theta}$ are disjoint.

Remark. The first two theorems are some generalization of results in [3] with the following overlap. The first part of Theorem 1 (1) is given in [3] as Theorem 5.4 (the if part and uniqueness) and a discussion after Definition 5.1 (the only if part). Theorem 1 (2) and Theorem 2 are given in Theorem 5.5 of [3] under the assumption that all φ_i are even.

4 Other State Extensions

The rest of our results concerns a joint extension of states of two subsystems, not satisfying the product state property (1). We need a few more notation. For two states φ and ψ of a $\mathbf{C}^*$-algebra $\mathcal{A}(I_1)$, consider any representation π of $\mathcal{A}(I_1)$ on a Hilbert space $\mathcal{H}$ containing vectors Φ and Ψ such that

$$\varphi(A) = (\Phi,\, \pi(A)\Phi), \quad \psi(A) = (\Psi,\, \pi(A)\Psi).$$

The transition probability between φ and ψ is defined ([4]) by

$$P(\varphi,\, \psi) \equiv \sup |(\Phi,\, \Psi)|^2$$

where the supremum is taken over all $\mathcal{H}$, π, Φ and Ψ as described above. For a state φ_1 of $\mathcal{A}(I_1)$, we need the following quantity

$$p(\varphi_1) \equiv P(\varphi_1,\, \varphi_1\Theta)^{1/2}$$

where $\varphi_1\Theta$ denotes the state $\varphi_1\Theta(A) = \varphi_1(\Theta(A))$, $A \in \mathcal{A}(I_1)$.

If φ_1 is pure, then $\varphi_1\Theta$ is also pure and the representations π_{φ_1} and $\pi_{\varphi_1\Theta}$ are both irreducible. There are two alternatives.

(α) They are mutually disjoint. In this case $p(\varphi_1) = 0$.
(β) They are unitarily equivalent.

In the case (β), there exists a self-adjoint unitary u_1 on $\mathcal{H}_{\varphi_1}$ such that

$$\begin{aligned} u_1\pi_{\varphi_1}(A)u_1 &= \pi_{\varphi_1}(\Theta(A)), \quad A \in \mathcal{A}(\mathrm{I}_1), \\ (\Omega_{\varphi_1}, u_1\Omega_{\varphi_1}) &\geq 0. \end{aligned}$$

For two states φ and ψ, we introduce

$$\lambda(\varphi, \psi) \equiv \sup\{\lambda \in \mathbb{R};\ \varphi - \lambda\psi \geq 0\}$$

Since $\varphi - \lambda_n\psi \geq 0$ and $\lim \lambda_n = \lambda$ imply $\varphi - \lambda\psi \geq 0$, we have

$$\varphi \geq \lambda(\varphi, \psi)\psi.$$

We need

$$\lambda(\varphi_2) \equiv \lambda(\varphi_2, \varphi_2\Theta).$$

The next Theorem provides a complete answer for a joint extension φ of states φ_1 and φ_2 of $\mathcal{A}(\mathrm{I}_1)$ and $\mathcal{A}(\mathrm{I}_2)$, when one of them is pure.

Theorem 4. *Let φ_1 and φ_2 be states of $\mathcal{A}(\mathrm{I}_1)$ and $\mathcal{A}(\mathrm{I}_2)$ for disjoint subsets I_1 and I_2. Assume that φ_1 is pure.*
(1) *A joint extension φ of φ_1 and φ_2 exists if and only if*

$$\lambda(\varphi_2) \geq \frac{1 - p(\varphi_1)}{1 + p(\varphi_1)}. \tag{2}$$

(2) *If eq. (2) holds and if $p(\varphi_1) \neq 0$, then a joint extension φ is unique and satisfies*

$$\begin{aligned} \varphi(A_1A_2) &= \varphi_1(A_1)\varphi_2(A_{2+}) + \frac{1}{p(\varphi_1)}f(A_1)\varphi_2(A_{2-}), \\ f(A_1) &\equiv \overline{\varphi_1}(\pi_{\varphi_1}(A_1)u_1) \end{aligned}$$

for $A_1 \in \mathcal{A}(\mathrm{I}_1)$ and $A_2 = A_{2+} + A_{2-}$, $A_{2\pm} \in \mathcal{A}(\mathrm{I}_2)_\pm$.
(3) *If $p(\varphi_1) = 0$, (2) is equivalent to evenness of φ_2. If this is the case, at least a product state extension of Theorem 1 exists.*
(4) *Assume that $p(\varphi_1) = 0$ and φ_2 is even. There exists a joint extension*

of φ_1 and φ_2 other than the unique product state extension if and only if φ_1 and φ_2 satisfy the following pair of conditions:

(4-i) *π_{φ_1} and $\pi_{\varphi_1\Theta}$ are unitarily equivalent.*

(4-ii) *There exists a state $\widetilde{\varphi}_2$ of $\mathcal{A}(\mathrm{I}_2)$ such that $\widetilde{\varphi}_2 \neq \widetilde{\varphi}_2\Theta$ and*

$$\varphi_2 = \frac{1}{2}(\widetilde{\varphi}_2 + \widetilde{\varphi}_2\Theta).$$

(5) *If $p(\varphi_1) = 0$, then corresponding to each $\widetilde{\varphi}_2$ above, there exists a joint extension φ which satisfies*

$$\varphi(A_1A_2) = \varphi_1(A_1)\varphi_2(A_{2+}) + \overline{\varphi_1}(\pi_{\varphi_1}(A_1)u_1)\widetilde{\varphi}_2(A_{2-}). \tag{3}$$

Such extensions along with the unique product state extension (which satisfies eq. (3) for $\widetilde{\varphi}_2 = \varphi_2$) exhaust all joint extensions of φ_1 and φ_2 when $p(\varphi_1) = 0$.

Remark. The eq.(2) is sufficient for the existence of a joint extension also for general states φ_1 and φ_2.

We have a necessary and sufficient condition for the existence of a joint extension of states φ_1 and φ_2 under a specific condition on φ_1.

Theorem 5. *Let φ_1 and φ_2 be states of $\mathcal{A}(\mathrm{I}_1)$ and $\mathcal{A}(\mathrm{I}_2)$ for disjoint subsets I_1 and I_2. Assume that π_{φ_1} and $\pi_{\varphi_1\Theta}$ are disjoint. Then a joint extension of φ_1 and φ_2 exists if and only if φ_2 is even.*

5 Examples

Example 1

Let I_1 and I_2 be mutually disjoint finite subsets of $\mathbb{N}$. Let $\varrho \in \mathcal{A}(\mathrm{I}_1 \cup \mathrm{I}_2)$ be an invertible density matrix, namely $\varrho \geq \lambda \mathbf{1}$ for some $\lambda > 0$ and $\mathbf{Tr}(\varrho) = 1$, where $\mathbf{Tr}$ denotes the matrix trace on $\mathcal{A}(\mathrm{I}_1 \cup \mathrm{I}_2)$. Take any $x = x^* \in \mathcal{A}(\mathrm{I}_1)_-$ and $y = y^* \in \mathcal{A}(\mathrm{I}_2)_-$ satisfying $\|x\|\|y\| \leq \lambda$. Let $\varphi_1(A_1) \equiv \mathbf{Tr}(\varrho A_1)$ for $A_1 \in \mathcal{A}(\mathrm{I}_1)$ and $\varphi_2(A_2) \equiv \mathbf{Tr}(\varrho A_2)$ for $A_2 \in \mathcal{A}(\mathrm{I}_2)$. Then

$$\varphi'_\varrho(A) \equiv \mathbf{Tr}(\varrho' A), \quad \varrho' \equiv \varrho + ixy.$$

for $A \in \mathcal{A}(\mathrm{I}_1 \cup \mathrm{I}_2)$ is a state of $\mathcal{A}(\mathrm{I}_1 \cup \mathrm{I}_2)$ and has φ_1 and φ_2 as its restrictions to $\mathcal{A}(\mathrm{I}_1)$ and $\mathcal{A}(\mathrm{I}_2)$, irrespective of the choice of x and y satisfying the above conditions.

Example 2

Let I_1 and I_2 be mutually disjoint subsets of $\mathbb{N}$. Let φ and ψ be states of $\mathcal{A}(\mathrm{I}_1)$ and $\mathcal{A}(\mathrm{I}_2)$ such that

$$\varphi = \sum_i \lambda_i \varphi_i, \quad \psi = \sum_i \lambda_i \psi_i, \ (0 < \lambda_i, \ \sum_i \lambda_i = 1),$$

where φ_i and ψ_i are states of $\mathcal{A}(\mathrm{I}_1)$ and $\mathcal{A}(\mathrm{I}_2)$ which have a joint extension χ_i for each i.

$$\chi = \sum_i \lambda_i \chi_i$$

is a joint extension of φ and ψ.

This simple example yields next more elaborate ones.

Example 3

Let φ and ψ be states of $\mathcal{A}(\mathrm{I}_1)$ and $\mathcal{A}(\mathrm{I}_2)$ for disjoint I_1 and I_2 with (non-trivial) decompositions

$$\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2, \quad \psi = \mu\psi_1 + (1-\mu)\psi_2, \quad (0 < \lambda, \mu < 1)$$

where φ_1 and φ_2 are even. Product state extensions $\varphi_i\psi_j$ of φ_i and ψ_j yield

$$\begin{aligned} \chi \;\equiv\; & (\lambda\mu + \kappa)\varphi_1\psi_1 + (\lambda(1-\mu) - \kappa)\varphi_1\psi_2 \\ & ((1-\lambda)\mu - \kappa)\varphi_2\psi_1 + ((1-\lambda)(1-\mu) + \kappa)\varphi_2\psi_2, \end{aligned}$$

which is a joint extension of φ and ψ for all $\kappa \in \mathbb{R}$ satisfying

$$-\min(\lambda\mu,\ (1-\lambda)(1-\mu)) \leq \kappa \leq \min((1-\lambda)\mu,\ \lambda(1-\mu)).$$

Example 4

Let φ_k, $k = 1, \cdots, m$ and ψ_l, $l = 1, \cdots, n$ be states of $\mathcal{A}(\mathrm{I}_1)$ and $\mathcal{A}(\mathrm{I}_2)$ for disjoint I_1 and I_2. Let

$$\varphi = \sum_{k=1}^{m} \lambda_k \varphi_k, \quad \psi = \sum_{l=1}^{n} \mu_l \psi_l$$

with λ_k, $\mu_l > 0$, $\sum \lambda_k = \sum \mu_l = 1$. Assume that there exists a joint extension χ_{kl} of φ_k and ψ_l for each k and l. Then

$$\chi = \sum_{kl} (\lambda_k \mu_l + \kappa_{kl}) \chi_{kl} \tag{4}$$

is a joint extension if

$$(\lambda_k \mu_l + \kappa_{kl}) \geq 0, \quad \sum_l \kappa_{kl} = \sum_k \kappa_{kl} = 0.$$

Since the constraint for mn parameters $\{\kappa_{kl}\}$ are effectively $m+n-1$ linear relations (because $\sum_{kl} \kappa_{kl} = 0$ is common for $\sum_l \kappa_{kl} = 0$ and $\sum_k \kappa_{kl} = 0$), we have $mn - (m+n-1) = (m-1)(n-1)$ parameters for the joint extension (4).

The above is an excerpt from the paper "State Extension from Subsystems to the Joint System" submitted to Commun.Math.Phys.

References

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