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Noncausal Cauchy problem for the noncausal SDEs

Shigeyoshi OGAWA

Dept. of Mathematical Sciences, Ritsumeikan University,
Kusatsu, Shiga, 525-8577 Japan
ogawa-s@se.ritsumei.ac.jp

1 Causality in stochastic calculus

The SDE, short for the stochastic differential equation, may be viewed as a mathematical description for such stochastic dynamical systems driven or perturbed by some random force. In many physical situations the random force can be represented by the Gaussian white noise, that is the derivative $\frac{d}{dt}W_t$ of the Brownian motion W_t, or more generally by the derivative $\frac{d}{dt}M_t$ of some martingales M_t having appropriate properties.

The traditional theory of the SDE introduced by K.Itô is constructed on such implicit but essential hypothesis that all random functions should be adapted to the so-called natural filtration $\{\mathcal{F}_t, t \geq 0\}$, $\mathcal{F}_t = \sigma\{M_s; s \leq t\}$ generated by the driving process M_t. As long as the principle of causality regimes the phenomena in all levels of scale, microscopic or macroscopic, this hypothesis of causality could not arise as veritable constraint for the stochastic theory. However in the analysis of many phenomena of statistical nature where we can observe only the statistical average of those underlying microscopic quantities, such as the pressure in statistical physics or the temperature in thermodynamics, we might not need that the principle of causality holds in all scale levels of phenomena. In other words, for the validity of the causality in macroscopic scale it would suffice that the causality regimes the phenomena only in that level and we need not care whether it still remains working in microscopic level.

The stochastic analysis of some phenomena in social sciences would be the case. One of such could be found in the analysis of the *insider trading* problem at the stochastic theory of finances (cf.[3]). In a very primitive understanding of the terminology "insider" the corresponding SDE model would be of such noncausal type that the parameters are no longer supposed to be adapted to the filtration. Since this violates the fundamental hypothesis of *causality* at the Itô's theory, we need to prepare a new stochastic calculus that can be developed without that hypothesis. The *noncausal stochastic calculus* introduced by the author in 1979 [11] would serve for this purpose. Already in 1985, this noncausal theory was applied by the author to the study of the Cauchy problem for the SDE with noncausal initial data and some basic results about the existence and uniqueness properties of the solution were established.

The aim of the present article is to develop the study for a more general case, namely for the Cauchy problem of an SDE with noncausal initial data and coefficients. For the simplicity of the discussion, we would be contented to treat the case of real SDE that is driven by the Brownian motion. However the results, except the Corollary 5 given in the last paragraph, might be extended to the case of multi-dimensional SDEs.

2 Noncausal SDE and the Cauchy problem

Let $W_t(\omega)$ $(t \geq 0,\ \omega \in \Omega)$ be the real Brownian motion defined one a probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t\}$ be the natural filtration of the σ-fields generated by the W_t that is, $\mathcal{F}_t = \sigma\{W_s; s \leq t\}$. By the random functions we understand those real valued functions $f(t, x, \omega)$ which are measurable in (t, x, ω) with respect to the field $B_{[0,T]} \times B_R \times \mathcal{F}$ and satisfy the condition,

$$P\{\int_0^T f^2(t, x, \omega)dt < \infty\} = 1, \quad \text{for each } x \in R$$

A random function $f(t, x, \omega)$ is called *causal* (or anticipative) if, for each $(t, x) \in [0, T] \times R^1$ it is adapted to the filtration $\mathcal{F}_t$.

Given real valued functions $a(t, x, y), b(t, x, y)$ $((t, x, y) \in [0, T] \times R^2)$ and arbitrary real random variables $\xi(\omega), \eta(\omega)$, we are to study the SDE of noncausal type as follows,

$$\begin{cases} dX_t = a(t, X_t, \eta(\omega))dt + b(t, X_t, \eta(\omega))dW_t, \ t \in (0, T], \\ X_0(\omega) = \xi(\omega) \end{cases} \tag{1}$$

Notice that the random variables $\xi(\omega), \eta(\omega)$ are not supposed to be independent of the Brownian motion $W_t, t \geq 0$. Thus the SDE should be treated in the frame work of the noncausal stochastic calculus, in other words the stochastic integral in the equation should be understood in that sense. In stead of the usual notation $\int dW_t$ the symbol like $\int d_\varphi W_t$ will be used for this purpose, where the $\{\varphi_n\}$ is a c.o.n.s. in $L^2(0, T)$. We will give in the next paragraph a short review of the noncausal calculus based on that integral.

As we have noticed, the stochastic calculus of noncausal type, introduced by the author ([11], [7],[10],[9] and [5], etc.), will play a principal role throughout the discussion. We are going to give in this paragraph a very rapid review of some fundamental results in the theory of noncausal stochastic, mainly following the recent article [4].

In what follows, we will fix the probability space once for all $(\Omega, \mathcal{F}, \mathcal{P})$ on which is defined the real or R^d-valued Brownian motion. We will denote by $\mathbf{H}$ the totality of all random functions $f(t,\omega)$, measurable in (t,ω) with respect to the field $\mathcal{B}_{\mathcal{R}_+} \times \mathcal{F}$, such that $P\{\int_0^T |f(t,\omega)|^2 dt < \infty\} = 1$, and by $\mathbf{M}$ the subset of all *causal* random functions that satisfy the following conditions;

(M.1) measurable in (t,ω) with respect to the field $\mathcal{B}_{\mathcal{R}_+} \times \mathcal{F}$, and especially

(M.2) adapted to the family of σ-fields $\{F_t\}$, where $F_t = \sigma\{W_s; 0 \le s \le t\}$,

(M.3) belong to the class L^2 in t, $P\{\int_0^T |f(t,\omega)|^2 dt < \infty\} = 1$.

2.1 Causal functions and B-differentiability

An $\mathbf{H}$-class random function $g(t,\omega)$ is said to be differentiable with respect to the Brownian motion W_t (or B-differentiable) provided that there exists an $\mathbf{M}$-class random function say $\hat{g}(t,\omega)$ such that, for small enough $h > 0$,

$$\sup_{t,s,|t-s|<h} E|g(t,\omega) - g(s,\omega) - \int_s^t \hat{g}(r,\omega) d^0 W_r|^2 = o(h)$$

where the integral $\int d^0 W$ stands for the Itô's stochastic integral. The function $\hat{g}$ is called the B-derivative of the g. It is not difficult to see that if the function $g(t,\omega)$ is B-differentiable then its B-derivative is uniquely determined (see [12]). The B-differentiability of the random function with respect to the multi-dimensional Brownian motion is defined in a similar way.

Remark 1 *Let $g(t,\omega)$ be a functional of the multi-dimensional Brownian motion, $\mathbf{W}_t = (W_t^1, W_t^2, \cdots, W_t^n)$ where the W^i, $(1 \le i \le n)$ are independent copies of the 1-dim. Brownian motion W_t. Then the B-derivative of such function, say $\nabla_w g$, can be defined in the following way: the $\nabla_w g = (\frac{\partial}{\partial W_t^1} g, \frac{\partial}{\partial W_t^2} g, \cdots, \frac{\partial}{\partial W_t^n} g)^t$ is a causal random vector such that,*

$$\sup_{t,s|t-s|<h} E|g(t,\omega) - g(s,\omega) - \sum_{k=1}^n \int_s^t \frac{\partial}{\partial W_r^k} g(r,\omega) d^0 W_r^k|^2 = o(h)$$

Here we notice that the Itô integral is defined for the causal random functions $f(t,\omega) \in$ $\mathbf{M}$ and roughly speaking the symmetric integrals (i.e. $\mathcal{I}_{1/2}$ of Ogawa [12] and Stratonovich-Fisk integral) are defined for the causal and B-differentiable functions.

2.2 Noncausal stochastic integral

Given a random function $f(t,\omega) \in \mathbf{H}$ and an arbitrary complete orthonormal system $\{\varphi_n\}$ in $L^2([0,1])$, we consider the formal random series

$$\sum_n^{\infty} \int_0^1 f(t,\omega)\varphi_n(t)dt \int_0^1 \varphi_n(t)dW_t.$$

The stochastic integral of noncausal type was introduced by the author in 1979 ([11]), in the following way,

Definition 1 : *A random function $f(t,\omega) \in \mathbf{H}$ is said to be integrable with respect to the basis $\{\varphi_n\}$ (or φ-integrable) when the random series above converges in probability and the sum, denoted by $\int_0^1 f(t,\omega)d_\varphi W_t$, is called the stochastic integral of noncausal type with respect to the basis $\{\varphi_n\}$.*

In general case, the way of convergence of the random series being conditional, the integrability and the sum may depend on the basis. If the function is integrable with respect to any basis $\{\varphi_n\}$ and the sum does not depend on the choice of the basis, we will say that the function is *universally* integrable (or shortly u-integrable).

Here are some equivalent expressions and a possible variation of the above definition, which are worth to be remarked so that we may have better understanding of the nature of our noncausal integral.

(a) As a limit of the sequence of random Stieltjes integrals;
$\int_0^1 f d_\varphi W_t := \lim \int_0^1 f dW_n^\varphi(t)$ (limit in probability),
where $W_n^\varphi(t) = \sum_{k=1}^{n} \int_0^t \varphi_k(s)ds \int_0^1 \varphi_k(s)dW_s$ is a pathwise smooth approximation of the Brownian motion $W(t,\omega)$.

(b) Riemannian definition: As a special case of the above expression, let us take the Haar functions $\{H_{n,i}(t),\ 0 \le i \le 2^n - 1,\ 0 \le n\}$ as basis $\{\varphi_n\}$. Then we easily see that,

$$\int_0^1 f d_H W_t = \lim_{n\to\infty} \sum_{i=0}^{2^n-1} 2^n \int_{2^{-n}i}^{2^{-n}(i+1)} f(s)ds \cdot \{W(2^{-n}(i+1)) - W(2^{-n}i)\}.$$

This type of definition can be found in recent publications of some authors. However as we notice here, this is a special case of our integral.

(c) Let $D_n(t,s)$ be the kernel given by, $D_n(t,s) = \sum_{k=1}^{n} \varphi_k(t)\varphi_k(s),\ (t,s \in [0,1])$.
Then we have the following representation for the noncausal integral,

$$\int_0^1 f d_\varphi W(t) = \lim_{n\to\infty} \int_0^1 dt \int_0^1 f(t,\omega) D_n(t,s) dW_s \quad \text{(limit in probability).}$$

For the case of trigonometric functions, the kernel $D_n(t,s)$ is the Dirichlet kernel appearing in the theory of Fourier series.

(d) A generalization of the above view: Replace the kernels $\{D_n(t,s)\}$ in the above interpretation by any δ- sequence say $\{K_n(t,s)\}$, then we will get a generalized formula for the noncausal integral;

$$\int_0^1 f d_K W := \lim_{n\to\infty} \int_0^1 dt \int_0^1 f(t,\omega) K_n(t,s) dW_s$$

2.3 Condition for the integrability

Let $\mathbf{H}_0$ be the totality of all random functions $f(t,\omega) \in \mathbf{H}$ such that, $E\int_0^1 |f(t,\omega)|^2 dt < \infty$. By Wiener-Itô's theory of Homogeneous Chaos, we know that such function $f \in \mathbf{H}_0$ can be decomposed into the sum of multiple Wiener integrals, that is:

There exists a set of kernels, say $\{k_n^f(t;t_1,\cdots,t_n)\}_{n=0}^{\infty}$, such that $k_n^f \in L^2([0,1]^{n+1})$ with $\sum_n n!||k_n^f||_{n+1}^2 < \infty$, symmetric in n-parameters $(t_1,\cdot,t_n) \in [0,1]^n$ and that,

$$f(t,\omega) = \sum_{n=0}^{\infty} I_n(k_n^f(t;\cdot)), \quad I_n(k_n^f(t;\cdot)) = \int\int\cdots\int k_n^f(t;t_1,\cdots,t_n) dW_{t_1} dW_{t_2}\cdots dW_{t_n}$$

where $||\cdot||_n$ stands for the norm in $L^2([0,1]^n)$-space.

We will denote by $\mathbf{H}_1$ the totality of all $\mathbf{H}_0$- functions $f(t,\omega)$ such that, $\sum_{n=1}^{\infty} nn!||k_n^f||_{n+1}^2 < \infty$. Given a function $f \in \mathbf{H}_1$ we introduce its stochastic derivative $\mathcal{D}f$ by the following formula,

$$\mathcal{D}f(t,s) = \sum_{n=1}^{\infty} nI_{n-1}(k_n^f(t;s,\cdot)).$$

Since $E\int_0^1\int_0^1 (\mathcal{D}f(t,s))^2 dtds = \sum nn!||k_n^f||_{n+1}^2$, we notice that the stochastic derivative $\mathcal{D}f(t,s)$ is well defined for the $f \in \mathbf{H}_1$. Now we can state the condition for the φ-integrability of the $\mathbf{H}_1$-class functions in the following theorem that was established by the author in 1984.

Theorem 2.1 ([10]) *Let* $f \in \mathbf{H}_1$ *and let* $\{\varphi_n\}$ *be an arbitrary orthonormal basis. Then the necessary and sufficient condition for the random function* f *to be* φ*-integrable is that the* $\lim_{n\to\infty} \int_0^1 \int_0^1 \mathcal{D}f(t,s)D_n(t,s)dtds$ *exists in probability.*

2.4 Relation between symmetric and noncausal integrals

We call a random function $f(t,\omega)$ semi martingale when it admits the decomposition, $f(t,\omega) = a(t,\omega) + \int^t \hat{f}d^0Wt$ where $\hat{f} \in \mathbf{M}$ and $a(t)$ is such that almost every sample path is of bounded variation in t over $[0,1]$. Notice that if $\sup_{t,s|t-s|<h} E|a(t)-a(s)|^2 = o(h)$ then f is B-differentiable.

The followings are the basic results concerning the relation between the symmetric integrals with the noncausal integral.

Theorem 2.2 ([7]) *Every causal B- differentiable function is integrable in noncausal sense with respect to the system of Haar functions and the sum coincides with that of the symmetric integrals:*

$$\int_0^1 f d_H W = \int_0^1 f d^0 W + \frac{1}{2}\int_0^1 \hat{f} dt$$

We say that a c.o.n.s basis $\{\varphi_n\}$ is regular provided that it satisfies the next condition:

$$\sup_n ||u_n||_2 < \infty, \quad u_n(t) = \sum_{k\leq n} \varphi_k(t) \int_0^t \varphi_k(s)ds \tag{2}$$

Remark 2 *Notice that this condition 2 is equivalent to the fact that,*

$$w - \lim_{n\to\infty} u_n = \frac{1}{2} \quad (in\ L^2)$$

namely to the fact that, for any $f(t) \in L^2(0,1)$ *it holds the following,*

$$\lim_{n\to\infty} \int_0^1 u_n(t)f(t)dt = \frac{1}{2}\int_0^1 f(t)dt$$

Theorem 2.3 ([7]) *Every semi martingale (causal or not) becomes* φ*-integrable, iff the basis* $\{\varphi_n\}$ *is regular. In this case the noncausal integral coincides with the symmetric integrals.*

Related to this result is a natural and interesting question asking whether there can or can not be a basis $\{\varphi_n\}$ which is not regular. This question is affirmatively answered by P.Mejer and M.Mancino [1]. We can go on further. The next result shows that *a smoothness in* W_t of the integrand ensures the integrability with respect to any orhtonormal basis.

Theorem 2.4 ([7]) *Every semi martingale that is twice B-differentiable, namely the B-derivative* $\hat{f}$ *is again a semi martingale, is u-integrable.*

3 Cauchy problem for the noncausal SDEs – Known results

First notice that the SDE in (1) becomes meaningful in the frame work of the noncausal stochastic calculus, that is;

$$\begin{cases} dX_t = a(t, X_t, \eta(\omega))dt + b(t, X_t, \eta(\omega))d_\varphi W_t, \ t \in (0,T], \\ X_0(\omega) = \xi(\omega) \end{cases} \tag{3}$$

here the $\{\varphi_n\}$ is a regular basis in $L^2(0,1)$, which we will fix throughout the discussion.

We notice at this stage that when the parameters $\xi(\omega), \eta(\omega)$ are not random and the solution X_t can be supposed to be causal, then by virtue of the Theorem 2.3 the SDE in (3) is reduced to the usual SDE with symmetric integration,

$$\begin{cases} dX_t = a(t, X_t, \eta)dt + b(t, X_t, \eta)dW_t, \ t \in (0,T], \\ X_0(\omega) = \xi \end{cases} \tag{4}$$

The Cauchy problem for the noncausal SDE was first studied by the author [8] for such simple case where the parameter η is not random or does not appear in $a(t,x)$, $b(t,x)$ and only the initial data $\xi(\omega)$ arises as a noncausal factor.

$$\begin{cases} dX_t = a(t, X_t)dt + b(t, X_t)d_\varphi W_t, \ t \in (0,T], \\ X_0(\omega) = \xi(\omega) \end{cases} \tag{5}$$

For this case, the existence and a kind of uniqueness property of the solution are proved under a milder assumption on the regularity of the coefficients $a(\cdot), b(\cdot)$ as follows;

Assumption 1 *The coefficients $a(t,x)$, $b(t,x)$ are sufficiently regular in such sense that,*

1. *$a(t,x), \frac{\partial^2}{\partial x^2}b(t,x)$ are of C^1-class,*
2. *$a(t,x), b(t,x)$ are sufficiently regular in the sense that the causal Cauchy problem 4 admits the unique strong solution $X(t,\omega;\xi)$ and that the $X(t,\omega;\xi)$ is continuous in (t,ξ) with probability one.*

derivatives are bounded on $[0,1] \times R^1$. Notice that under such conditions the composite

$$\tilde{X}(t,\omega) = X(t,\omega;\xi(\omega))$$

of the strong solution $X(t,\xi,\omega)$ of the (4) and the random variable $\xi(\omega)$ is well defined, which we expect to be a solution of the noncausal Cauchy problem (5). In fact we have the following,

Theorem 3.1 (1985 [8]) *The composite $\tilde{X}(t,\omega)$ is a solution of the noncausal Cauchy problem (5).*

We have also found that this solution $\tilde{X}(t,\omega)$ verifies the Itô formula of noncausal type, that is;

Proposition 3.1 (1985, [6]) *For any function $F(x) \in C^4$ it holds the equality,*

$$dF(\tilde{X}_t) = F'(\tilde{X}_t)\{a(t,\tilde{X}_t)dt + b(t,\tilde{X}_t)d_\varphi W_t\} \quad 0 \le t \le 1$$

As an application of this we can show the following result that concerns the uniqueness of the solution for the noncausal problem (5),

Corollary 3.1 ([6]) *When the $b(x) \neq 0$, the composed function $\tilde{X}(t,\omega) = X(t,\omega;\xi(\omega))$ is the unique solution among all random functions verifying the Itô formula 3.1 of noncausal type.*

The proof of this together with that of the previously presented Proposition 3.1 will be given in the next paragraph for a more general case.

4 Discussion for the more general case

We are going to give in this paragraph the results on the Cauchy problem for the more general case (3).

Assumption 2 *We suppose that the coefficients $a(t,x;\eta), b(t,x;\eta)$ are sufficiently regular in such sense that, for an arbitrary couple of parameters (ξ,η) the causal Cauchy problem 4 admits the unique strong solution $X(t,\omega;\xi,\eta)$ and that the $X(t,\omega;\xi,\eta)$ is continuous in (t,ξ,η) with probability one.*

Remark 3 *The assumption is satisfied when, for example, the $a(t,x;\eta), b(t,x;\eta)$ are of the C^4-class in x, of C^1-class in η and all derivatives are bounded on $[0,1] \times R^1$.*

We also notice that under the Assumption 2 the composite

$$\tilde{X}(t,\omega) = X(t,\omega;\xi(\omega),\eta(\omega))$$

of the strong solution $X(t,\omega;\xi,\eta)$ of the (4) and the random variables $\xi(\omega),\eta(\omega)$ is well defined, and as in the previous case we expect this composite $\tilde{X}(t,\omega)$ to be a solution of the noncausal Cauchy problem (3). In fact we have the following,

Theorem 4.1 *The $\tilde{X}$ gives a noncausal solution of the noncausal Cauchy problem 3.*

For the verification of this, we need some preparations.

Proposition 4.1 *Let $f(t,\omega;\xi,\eta)$ $(\xi,\eta\in[-A,A])$ be a semi-martingale such that for each fixed (ξ,η),*

$$df(t,\omega;\xi,\eta)=g(t,\omega;\xi,\eta)dt+h(t,\omega;\xi,\eta)d^0W_t \tag{6}$$

where $g(\cdot),h(\cdot)$ are causal random functions satisfying the following condition,

$$P[\int_{-A}^{A}d\xi\int_{-A}^{A}d\eta\int_0^1\{g^2(t,\omega;\xi,\eta)+h^2(t,\omega;\xi,\eta)\}dt<\infty]=1$$

(i) Then for any regular basis $\{\varphi_n\}$ in $L^2(0,1)$, it holds the following equality,

$$\lim_{n\to\infty}\int_{-A}^{A}d\xi\int_{-A}^{A}d\eta[\int_0^1 f(t,\omega;\xi,\eta)\{d_\varphi W(t)-dW_\varphi^n(t)\}]^2=0 \quad \textit{(in probability)} \tag{7}$$

(ii) Moreover if the coefficient $h(t,\omega;\xi,\eta)$ in the decomposition (6) again becomes a semi-martingale satisfying the same condition as the $f(\cdot)$, then the equality (7) still holds true for any basis $\{\varphi_n\}$.

(Proof) Put $f=f_1+f_2$ where,

$$f_1(t,\omega;\xi,\eta)=f(0,\omega;\xi,\eta)+\int_0^t g(s,\omega;\xi,\eta)ds$$

and

$$f_2(t,\omega;\xi,\eta)=\int_0^t h(s,\omega;\xi,\eta)d^0W_s$$

. Then we have for f_1, the equality;

$$\int_0^1 f_1(s,\omega;\xi,\eta)\{d_\varphi W(s)-dW_\varphi^n(s)\}$$
$$=f_1(1,\omega;\xi,\eta)\{W(1)-W_\varphi^n(1)\}-\int_0^1\{W(s)-W_\varphi^n(s)\}g(s,\omega;\xi,\eta)ds$$

Hence with the help of the Theorem of Nishio-Itô we confirm that,

$$\lim_{n\to\infty}\int_{-A}^{A}d\xi\int_{-A}^{A}d\eta[\int_0^1 f_1(t,\omega;\xi,\eta)\{d_\varphi W(t)-dW_\varphi^n(t)\}]^2=0 \quad \text{(in probability)}$$

For the term f_2 we have the decomposition,

$$\int_0^1 f_2(t,\omega;\xi,\eta)\{d_\varphi W(t)-dW_\varphi^n(t)\}=\sum_{i=1}^4 I_{i,n}(\xi,\eta)$$

where,

$$I_{1,n} = \sum_{k=n+1}^{\infty} f_2(1,\omega;\xi,\eta)\tilde{\varphi}_k(1)Z_k, \ (\tilde{\varphi}_k(t) = \int_0^t \varphi_k(s)ds)$$

$$I_{2,n} = \sum_{k=n+1}^{\infty} \int_0^1 \tilde{\varphi}_k(t)\varphi_k(t)h(t,\omega;\xi,\eta)dt$$

$$I_{3,n} = \sum_{k=n+1}^{\infty} \int_0^1 \varphi_k(t)d^0W(t)\int_0^t \tilde{\varphi}_k(s)h(s,\omega;\xi,\eta)d^0W(s)$$

$$I_{4,n} = \sum_{k=n+1}^{\infty} \int_0^1 \tilde{\varphi}_k(t)h(t,\omega;\xi,\eta)d^0W(t)\int_0^t \varphi_k(s)d^0W(s)$$

and here, $Z_n = \displaystyle\int_0^1 \varphi_n(t)dW(t)$.

We are to show that; $\displaystyle\lim_{n\to\infty}\int_{-A}^{A} d\xi \int_{-A}^{A} I_{i,n}^2(\xi,\eta) = 0$ (in probability), $(1 \le i \le 4)$.
Since for the quantities $I_{i,n}$ $(i = 1,3,4)$ this could be easily done by a usual routine, it would suffice to show the result only for the term $I_{2,n}$.
By taking the Remark2 into account, we see that for each fixed (ξ,η) we have,

$$\lim_{n\to\infty} I_{2,n}(\xi,\eta) = \lim_{n\to\infty}\int_0^1 h(t,\omega;\xi,\eta)\{\frac{1}{2} - u_n(t)\}dt = 0$$

On the other hand we have,

$$I_{2,n}^2 \le (\frac{1}{2} + 2U^2)\int_0^1 h^2(t,\omega;\xi,\eta)dt \quad \text{where, } U = \sup_n ||u_n||_{L^2} < \infty.$$

Hence we confirm the result, $\displaystyle\lim_{n\to\infty}\int_{-A}^{A}\int_{-A}^{A} I_{2,n}^2(\xi,\eta)d\xi d\eta = 0$ □

Now given the unique solution $X(t,\omega;\xi,\eta)$ of the causal problem 4, we introduce the sequence of random functions in the following way,

$$X_\varphi^n(t,\omega;\xi,\eta) = \xi + \int_0^t a(s,X(s,\omega;\xi,\eta);\eta)ds + \int_0^t b(s,X(s,\omega;\xi,\eta);\eta)dW_\varphi^n(s) \quad (8)$$

where W_φ^n is the approximate process of the Brownian motion introduced in the previous paragraph.

We easily see by the Theorem 2.3 that for each fixed t, (ξ,η), we have $\displaystyle\lim_{n\to\infty} X_\varphi^n(t,\omega;\xi,\eta) = X(t,\omega;\xi,\eta)$, (in probablity). Moreover we can see that this convergence is uniform in (ξ,η) on every finite set $C_A = [-A,A]\times[-A,A]$.

Proposition 4.2 *For an arbitrarily large $A > 0$ it holds the following relation at each fixed $t \in [0,1]$,*

$$\lim_{n\to\infty}\sup_{(\xi,\eta)\in C_A} |X_\varphi^n(t,\omega;\xi,\eta) - X(t,\omega;\xi,\eta)| = 0 \quad (\textit{in probability})$$

(Proof) Put

$$\Delta_n(t,\omega;\xi,\eta) = X^n_\varphi(t,\omega;\xi,\eta) - X(t,\omega;\xi,\eta)$$

From equations (4),(8) we obtain the following;

$$\Delta_n(t,\omega;\xi,\eta) = \int_0^t b(X(s;\xi,\eta);\eta)\{dW^n_\varphi(s) - d_\varphi W(s)\} \tag{9}$$

On the other hand we have the following expression,

$$\Delta_n(t,\omega;\xi,\eta) = \int_0^\xi d\xi_1 \int_0^\eta d\eta_1 \frac{\partial^2}{\partial\xi\partial\eta}\Delta_n(t,\omega;\xi_1,\eta_1)$$
$$+\int_0^\xi \frac{\partial}{\partial\eta}\Delta_n(t,\omega;\xi_1,0)d\xi_1 + \int_0^\eta \frac{\partial}{\partial\xi}\Delta_n(t,\omega;0,\eta_1)d\eta_1$$

which implies that,

$$\sup_{(\xi,\eta)\in C_A} |\Delta_n(t,\omega;\xi,\eta)| \le J_1(n) + J_2(n) + J_3(n)$$

where

$$J_1(n) = 4A^2\int_{-A}^A d\xi_1 \int_{-A}^A d\eta_1 |\frac{\partial^2}{\partial\xi\partial\eta}\Delta_n(t,\omega;\xi_1,\eta_1)|^2$$
$$J_2(n) = 2A\int_{-A}^A |\frac{\partial}{\partial\eta}\Delta_n(t,\omega;\xi_1,0)|^2 d\xi_1, \quad J_3(n) = 2A\int_{-A}^A |\frac{\partial}{\partial\xi}\Delta_n(t,\omega;0,\eta_1,0)|^2 d\eta_1$$

We are to show that for each fixed t these $J_1(n), J_2(n), J_3(n)$ tend to zero in probability as $n \longrightarrow \infty$. Since at this stage the parameters ξ,η remain as deterministic constants, we notice that the $X(t,\omega;\xi,\eta)$ is causal and derivable in ξ,η. In fact under the assumption (3) on the regularity of the coefficients $a(\cdot), b(\cdot)$ it is easy to verify that the derivatives,

$$X_1(t) = \frac{\partial}{\partial\xi}X(t,\omega;\xi,\eta), \quad X_2(t) = \frac{\partial}{\partial\eta}X(t,\omega;\xi,\eta), \quad X_3(t) = \frac{\partial^2}{\partial\xi\partial\eta}X(t,\omega;\xi,\eta),$$

are given as the solutions of the following symmetric type SDEs, which can be solved explicitly;

$$dX_1(t) = a_x(t,X,\eta)X_1(t)dt + b_x(t,X,\eta)X_1(t)dW_t,$$
$$X_1(0) = 1$$

$$\begin{cases} dX_2(t) & = \{a_\eta(t,X,\eta)dt + b_\eta(t,X,\eta)dW_t\} \\ & +\{a_x(t,X,\eta)X_2(t)dt + b_x(t,X,\eta)X_2(t)dW_t\}, \\ X_2(0) & = 0, \end{cases}$$

$$\begin{cases} dX_3(t) & = \{a_{xx}(t,X,\eta)X_2(t) + a_{x,\eta}(t,X,\eta)\}X_1(t)dt \\ & +\{b_{xx}(t,X,\eta)X_2(t) + b_{x,\eta}(t,X,\eta)\}X_1(t)dW_t \\ & +a_x(t,X,\eta)X_3(t)dt + b_x(t,X,\eta)X_3(t)dW_t, \\ X_3(0) & = 0 \end{cases}$$

This combined with the expression (9) would imply that the quantity $\Delta_n(t,\omega;\xi,\eta)$ is derivable in ξ,η and that the order of the derivation in ξ,η and the integration is exchangeable. For example,

$$\frac{\partial^2}{\partial\xi\partial\eta}\Delta_n = \int_0^t \frac{\partial^2}{\partial\xi\partial\eta} b(s, X(s;\xi,\eta)\{dW_\varphi^n(s) - dW_\varphi(s)\}$$

Hence by virtue of the Proposition 4.1 we only need to show that the following quantities,

$$\frac{\partial^2}{\partial\xi\partial\eta} b(X(t;\xi,\eta);\eta),\ \frac{\partial}{\partial\xi} b(X(t;\xi,0);0),\ \frac{\partial}{\partial\eta} b(X(t;0,\eta);\eta)$$

are semi-martingales satisfying the condition in that Proposition. Since this can be verified by a simple routine work, we see that we are done. □

Now we are going to give the proof for our Theorem,

(Proof) Fix a positive A in an arbitrary way and put,

$$\xi_A(\omega) = \xi(\omega)\mathbf{1}_{C_A}(\xi(\omega)) - A\mathbf{1}_{(-\infty,-A]}(\xi(\omega)) + A\mathbf{1}_{[A,\infty)}(\xi(\omega))$$

$$\eta_A(\omega) = \eta(\omega)\mathbf{1}_{C_A}(\eta(\omega)) - A\mathbf{1}_{(-\infty,-A]}(\eta(\omega)) + A\mathbf{1}_{[A,\infty)}(\eta(\omega))$$

For an arbitrary positive ϵ we have,

$$P\{|X_\varphi^n(t,\omega;\xi(\omega),\eta(\omega)) - X(t,\omega;\xi(\omega),\eta(\omega))| > \epsilon\}$$

$$\leq P\{|X_\varphi^n(t,\omega;\xi_A(\omega),\eta_A(\omega)) - X(t,\omega;\xi_A(\omega),\eta_A(\omega,ega))| > \epsilon\}$$

$$+P(|\xi(\omega)| > A) + P(|\eta(\omega)| > A)$$

Since $|\xi_A(\omega)|, |\eta_A(\omega)| \leq A$, we confirm that

$$\lim_{n\to\infty} P\{|X_\varphi^n(t,\omega;\xi_A(\omega),\eta_A(\omega)) - X(t,\omega;\xi_A(\omega),\eta_A(\omega))| > \epsilon\} = 0$$

by virtue of the Proposition 4.2. The A being arbitrary this implies that,

$$\lim_{n\to\infty} P\{|X_\varphi^n(t,\omega;\xi(\omega),\eta(\omega)) - X(t,\omega;\xi(\omega),\eta(\omega))| > \epsilon\} = 0 \quad \square$$

5 On the question of uniqueness

The noncausal solution of the problem (3), $\tilde{X}(t,\omega) = X(t,\omega;\xi(\omega),\eta(\omega))$ constructed in the Theorem 4.1, has a remarkable property as stated in the next,

Theorem 5.1 (Noncausal Itô formula) *For any random variable $\zeta(\omega)$ and any function $F(x,y)$, which is differentiable in (x,y) and of C^4-class in x with bounded derivatives, it holds the following equality;*

$$dF(\tilde{X}_t,\zeta(\omega)) = (\partial_x F)(\tilde{X}_t,\zeta(\omega))\{a(\tilde{X}_t;\eta(\omega))dt + b(\tilde{X}_t;\eta(\omega))d_\varphi W_t\} \qquad (10)$$

(Proof) Let $X(t,\omega;\xi,\eta)$ be the unique solution of the causal SDE (4) with deterministic parameters (ξ,η). Then by the usual Itô formula for causal functions, we have for each fixed deterministic parameters $(\xi,\eta,\ \zeta)$, the following relation;

$$F(X(t;\xi,\eta),\zeta) = F(\xi,\zeta) + \int_0^t (\partial_x F)(X(s,\omega;\xi,\eta),\zeta)\{a(X_s;\eta)ds + b(X_s;\eta)dW_s\} \tag{11}$$

Here the stochastic integral $\int dW_t$ stands for the causal symmetric integral.
Given this we introduce the approximation sequence as follows,

$$F^n(t,\omega;\xi,\eta,\zeta) = F(\xi,\zeta) + \int_0^t (\partial_x F)(X(s,\omega;\xi,\eta),\zeta)\{a(X_s;\eta)ds + b(X_s;\eta)dW^n_\varphi(s)\} \tag{12}$$

Following the same argument as in the proof of Proposition 4.1, we would easily verify that for each fixed $t \in [0,1]$ the sequence $F^n(t,\omega;\xi,\eta,\zeta)$ converges to $F(X(t,\omega;\xi,\eta),\zeta)$ in probability as $n \to \infty$, uniformly in $(\xi,\eta,\zeta) \in C'_A$ on any finite set $C'_A = [-A,A]^3$. Hence we confirm, again following the same argument as in the proof of the Theorem 4.1, that for each fixed t the sequence $F^n(t,\omega;\xi(\omega),\eta(\omega),\zeta(\omega))$ converges in probability to the $F(X(t,\omega;\xi(\omega),\eta(\omega),\zeta(\omega)) = F(\tilde{X}(t,\omega),\zeta(\omega))$. Now from the equation (12) we see that the following limit,

$$\lim_{n\to\infty} \int_0^t (\partial_x F)(X(s,\omega;\xi(\omega),\eta(\omega)),\zeta(\omega))b(X(s,\omega;\xi(\omega),\eta(\omega));\eta(\omega))dW^n_\varphi(s)$$

should converge in probability to the limit,

$$\int_0^t (\partial_x F)(\tilde{X}(s,\omega),\zeta(\omega))b(\tilde{X}(s,\omega);\eta(\omega))dW_\varphi(s)$$

by definition of the $\tilde{X}(t,\omega)$ and by definition of the noncausal integral with respect to the basis $\{\varphi_n\}$.
Thus from this fact we get the desired equality (10), by letting $n \to \infty$ on both sides of the equality (12). □

As we have mentioned in the previous paragraph, this fact that the solution $\tilde{X}(t,\omega) = X(t,\omega;\xi(\omega),\eta(\omega))$ of the noncausal problem (3) satisfies the Itô formula of noncausal type (10) would give us a partial answer to the question of uniqueness of the solution of our noncausal problem. In fact we have the following result that is valid for the case of 1-dimensional SDE.

Corollary 5.1 *If the $b(t,x;\eta)$ does not depend on the t and $b(x;\eta) > 0$ (or < 0) for all (t,x,η), then the solution $\tilde{X}(t,\omega)$ is unique among the all random functions that verify the noncausal Itô formula (10).*

(Proof) Without loss of generality we suppose that $b(\cdot) > 0$. Put $Y(t) = F(\tilde{X}(t))$ where $F(x)$ is as follows,

$$F(x) = \int_0^x \frac{dy}{b(y,\eta)}$$

. Then we have, $\tilde{X}(t,\omega) = F^{-1}(Y(t,\omega))$. By applying the noncausal Itô formula to the function $Y(t)$ we get,

$$Y(t) = F(\xi(\omega)) + \int_0^t (\frac{a}{b})(F^{-1}(Y(s)); \eta(\omega))ds + W(t)$$

Since this is merely a family ordinary integral equations parametrized by the ω, we see the uniqueness of its solution $Y(t)$ for each fixed and hence the uniqueness of the $\tilde{X}(t,\omega)$. This completes the proof. □

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