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Bell's results on, and representations of finitely connected planar domains

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1 Ahlfors maps and Bergman kernels

Let D be a domain in $\mathbb{C}$. Consider the subspace $A^2(D)$ of the Hilbert space $L^2(D)$ (of all square integrable functions on D with respect to the Lebesque meaure on $\mathbb{C}$) consisting of all elements in $L^2(D)$ holomorphic on D. Then there is the natural projection

$$P : L^2(D) \to A^2(D),$$

which is called the *Bergman projection.* The coresponding kernel $K(z,w)$ is called the *Bergman kernel.*

When D is the unit disc,

$$K(z,w) = \frac{1}{\pi(1-z\overline{w})^2}.$$

Hence the Bergman kernel function $K(z,w)$ associated to a simply connected domain D can be written by using the Riemann map $f_a(z)$ (determined uniquely by the conditions $f_a(a) = 0$ and $f_a'(a) > 0$) and its derivative:

$$K(z,w) = \frac{f_a'(z)\overline{f_a'(w)}}{\pi(1-f_a(z)\overline{f_a(w)})^2}.$$

Let D be a non-degenerate multiply connected planar domain with smooth boundary. Fix a point a in D, and let f_a be the *Ahlfors map* associated with the pair (D, a). Among all holomorphic functions h which map D into the unit disc and satisfy $h(a) = 0$, the Ahlfors map f_a is the unique function which maximizes $h'(a)$ under the condition $h'(a) > 0$. Such proper holomorphic maps can recover the Bergman projections and kernels in general.

Theorem 1 *Let $f : D_1 \to D_2$ be a proper holomorphic map between planar (proper) domains. Let P_j be the Bergman projection for D_j. Then*

$$P_1(f' \cdot (\phi \circ f)) = f' \cdot ((P_2 \phi) \circ f)$$

for all $\phi \in L^2(D_2)$.

But the translation formula for the Bergman kernels is not so simple in general. For instance, it is hard to write down the following formula explicitly.

Proposition 2 *Let $f : D_1 \to D_2$ be a proper holomorphic map between planar (proper) domains. Then the Bergman kernels $K_j(z, w)$ associated to D_j transform according to*

$$f'(z) K_2(f(z), w) = \sum_{k=1}^{m} K_1(z, F_k(w)) \overline{F_k'(w)}$$

for $z \in D_1$ and $w \in D_2 - V$ where the multiplicity of the map f is m and the functions F_k, $k = 1, \cdots, m$, denote the local inverses to f and V is the set of critical values.

S. Bell obtained several kinds of simpler representations of Bergman kernel functions.

Theorem 3 **([1])** *For a non-degenarate multiply connected planar domain D, we can find two points a, b in D such that*

$$K(z, w) = f_a'(z) \overline{f_b'(w)} R(z, w)$$

with a rational combination $R(z, w)$ of f_a and f_b.

Here we say that a function $R(z,w)$ is a *rational combination of* f_a *and* f_b if it is a rational function of

$$f_a(z), f_b(z), \overline{f_a(w)}, \overline{f_b(w)}.$$

Such representation as above has the following variant.

Theorem 4 ([5]) *For a non-degenarate multiply connected planar domain D, we can find two points a, b in D such that*

$$K(z,w) = \frac{f_a'(z)\overline{f_a'(w)}}{(1 - f_a(z)\overline{f_a(w)})^2}\left(\sum_{j,k} H_j(z)\overline{K_k(w)}\right)$$

where f_a, f_b *are the Ahlfors functions, H and K are rational functions of them, and the sum is a finite sum.*

Actually, we can use any proper holomorphic maps.

Theorem 5 ([2]) *Let D be a non-degenarate multiply connected planar domain, and f a proper holomorphic map of D onto the unit disk U. Then* $K(z,w)$ *is an algebraic function of*

$$f(z), f'(z), \overline{f(w)}, \overline{f'(w)}.$$

Moreover, we have the following

Theorem 6 ([2]) *Let D be a non-degenerate multiply connected planar domain. The following conditions are equivalent.*

(1) The Bergman kernel $K(z,w)$ *associated to D is algebraic, i.e. an algebraic function of z and* $\overline{w}$.

(2) The Ahlfors map $f_a(z)$ *is an algebraic function of z.*

(3) There is a proper holomorphic mapping $f : D \to U$ *which is an algebraic function.*

(4) Every proper holomorphic mapping from D onto the unit disc U is an algebraic function.

Also we have

Theorem 7 ([4]) *Let D be a non-degenerate multiply connected planar domain. There are two holomorphic functions F_1 and F_2 on D such that the Bergman kernel on D is a rational combination of F_1 and F_2 if and only if there is a proper holomorphic map f of D onto U such that f and f' are algebraically dependent: i.e. there is a polynomial Q such that $Q(f, f') = 0$.*

Then, for every proper holomorphic map f of D to U, f and f' are algebraically dependent.

Proposition 8 ([4]) *Let D be a simply connected planar (proper) domain. The Bergman kernel on D is a rational combination of a function of a complex variable if and only if the Riemann map f of D and f' are algebraically dependent.*

Finally, we note the following facts.

Proposition 9 ([2]) *If $K(z, w)$ is algebraic, and f be a proper holomorphic map to U. Then $K(z, w)$ is an algebraic function of $f(z)$ and $\overline{f(w)}$.*

Corollary 1 ([2]) *Let D_1 and D_2 have algebraic Bergman kernels, then every biholomorphic map of D_1 onto D_2 is algebraic.*

2 Bell representations

Now the issue is to find a family of canonical domains which admit a *simple* proper holomorphic map to U. Bell proposed such a family, and actually, they are enough.

Theorem 10 ([6]) *Every non-degenerate n-connected planar domain with $n > 1$ is mapped biholomorphically onto a domain $W_{\mathbf{a},\mathbf{b}}$ defined by*

$$W_{\mathbf{a},\mathbf{b}} = \left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

with suitable complex vectors $\mathbf{a} = (a_1, a_2, \cdots, a_{n-1})$ and $\mathbf{b} = (b_1, b_2, \cdots, b_{n-1})$.

The above theorem is considered as a natural generalization of the classical Riemann mapping theorem for simply connected planar domains. The function $f_{\mathbf{a},\mathbf{b}}$ defined by

$$f_{\mathbf{a},\mathbf{b}}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}$$

is a proper holomorphic mapping from $W_{\mathbf{a},\mathbf{b}}$ to the unit disc which is rational. Actually, it is a very classical fact that, for such an $f = f_{\mathbf{a},\mathbf{b}}$ as above, f and f' are algebraically dependent. Hence the above proposition implies the following corollary.

Corollary 2 *Every non-degenerate n-connected planar domain D with $n > 1$ is biholomorphic to a domain with the algebraic Bergman kernel.*

Corollary 3 *There are two holomorphic functions F_1 and F_2 such that the Bergman kernel on $W_{\mathbf{a},\mathbf{b}}$ is a rational combination of F_1 and F_2.*

Definition The locus $\mathbf{B}_n$ in $\mathbb{C}^{2n-2}$ consisting of $(\mathbf{a}, \mathbf{b})$ such that the corresponding domain $W_{\mathbf{a},\mathbf{b}}$ is a non-degenerate n-connected planar domain.

We call this locus $\mathbf{B}_n$ the *coefficient body* for non-degenerate n-connected canonical domains.

It is obvious that $\mathbf{B}_n$ is contained in the product space

$$(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C},$$

which has the same homotopy type as that of

$$X = (S^1)^{n-1} \times F_{0,n-1}\mathbb{C},$$

where

$$F_{0,n-1}\mathbb{C} = \{(z_1, \cdots, z_{n-1} \in \mathbb{C}^{n-1} \mid z_j \neq z_k \quad \text{if} \quad j \neq k\}$$

is called a *configuration space.*

To clearify the topological structure of the coefficent body, it is more convenient to use the following modified representation space.

Definition We set

$$\mathbf{B}_n^* = \{(a_1, \cdots, a_{n-1}, \mathbf{b}) \in (\mathbb{C})^{2n-2} \mid (a_1^2, \cdots, a_{n-1}^2, \mathbf{b}) \in \mathbf{B}_n\},$$

and call it the *modified coefficient body.*

Theorem 11 $\mathbf{B}_n^*$ *is a circular domain, and has the same homotopy type as that of the product space* X.

Corollary 4 *The homotopy type of* $\mathbf{B}_n$ *is the same as that of* X.

Remark The fundamental group of $F_{0,n-1}\mathbb{C}$ is called the *pure braid group,* and its structure is well-known.

Problem

1. Determine the *Ahlfors locus* of $\mathbf{B}_n$ which consists of all $(\mathbf{a}, \mathbf{b})$ such that $f_{\mathbf{a},\mathbf{b}}$ gives an Ahlfors map (, or more precisely, $e^{i\theta} f_{\mathbf{a},\mathbf{b}}$ with a suitable $\theta \in \mathbb{R}$ is an Ahlfors map).
2. Fix a point $(\mathbf{a}, \mathbf{b})$ in $\mathbf{B}_n$, and let $W = W_{\mathbf{a},\mathbf{b}}$ be the corresponding n-conenncted canonical domain. Determine the *leaf* $E(W)$ of $\mathbf{B}_n$ for W, consisting of all points which correspond to n-connected canonical domains biholomorphically equivalent to W.
3. Determine the *collision locus* C of $\mathbf{B}_n$ which consists of all $(\mathbf{a}, \mathbf{b})$ such that the correcponding map $f_{\mathbf{a},\mathbf{b}}$ has a pair of critical points (counted with multiplicities) whose image is the same. (Note that $\mathbf{B}_n - C$ is a finite-sheeted holomorphic smooth cover of the intersection of $F_{0,2n-2}\mathbb{C}$ and the unit polydisc.)

Example 1

$$\mathbf{B}_2^* = \{(a, b) \in \mathbb{C}^2 : a \neq 0, |b + 2a| < 1, |b - 2a| < 1\},$$

which is biholomorphic to the polydisc deleted the diagonal.

Next, the set

$$\left\{ (a, b) \in \mathbf{B}_2^* : \left| \frac{4a^2}{1 - \overline{(b + 2a)}(b - 2a)} \right| = \frac{4r}{4 + r^2} \right\}$$

corresponds to a leaf of $\mathbf{B}_2$ *for every given* $r > 2$, *and the collision locus of* $\mathbf{B}_2$ *is empty.*

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