Free logarithmic Sobolev inequalities and free transportation cost inequalities

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Introduction

Since its first systematic study done by L. Gross in 1975, the logarithmic Sobolev inequality (LSI) has been discussed by many authors in various contexts, in particular, in close connection with the notions of hypercontractivity and spectral gap. An LSI can be understood to compare the relative Fisher information with the relative entropy. Among other things, we here refer to the LSI due to D. Bakry and M. Emery [1] in the general Riemannian manifold setting. Another interesting inequality was presented by M. Talagrand [10] in 1996, called the transportation cost inequality (TCI). A TCI compares the (quadratic) Wasserstein distance $W(\mu,\nu)$ between probability measures μ,ν (for the definition see §2 below) with $\sqrt{S(\mu,\nu)}$, the square root of the relative entropy. Indeed, in [10] Talagrand proved the inequality $W(\mu,\nu)\leq\sqrt{S(\mu,\nu)}$ when ν is the standard Gaussian measure on $\mathbb{R}^n$, and an exposition in the case of more general ν can be found in [8] for example. On the other hand, in [9] F. Otto and C. Villani succeeded in discovering links between the LSI and the TCI in the Riemannian manifold setting. This, combined with [1], implies the TCI in the same situation as Bakry and Emery's LSI. See [7, 8, 11] for more about these classical LSI and TCI as well as related topics.

Voiculescu's inequality in [14, Proposition 7.9] is the first free probabilistic analog of the LSI. Extending its single variable case (see (1.5) in §1), Biane obtained in [2] the free LSI (Theorem 1.2) for measures on $\mathbb{R}$. To prove this, Biane applied the classical LSI on the Euclidean space to the related selfadjoint random matrices and used the weak convergence of their mean eigenvalue distributions. In Theorem 1.3 we show the variant of Biane's free LSI for measures on $\mathbb{T}$. The proof is based on random matrix approximation; we can apply Bakry and Emery's classical LSI on the special unitary group $\mathrm{SU}(n)$, a Riemannian manifold, to the related $n\times n$ special unitary random matrices and pass to the scaling limit as n goes to ∞.

In [5] Biane and Voiculescu obtained the free analog of Talagrand's TCI for compactly supported $\mu\in\mathcal{M}(\mathbb{R})$ (Theorem 2.3). Their proof involves the free process and the complex Burgers' equation, and it is a realization of free probability parallel of not only the result itself but also the proof in [9]. The proof itself justifies the above inequality to be the right free analog of Talagrand's TCI. But, we can reprove Biane and Voiculescu's free TCI in a slightly more general setting (Theorem 2.4) by making use of random matrix approximation. Certain $n \times n$ selfadjoint random matrices, i.e., probability measures on the space M_n^{sa} ($\cong \mathbb{R}^{n^2}$) of $n \times n$ selfadjoint matrices, are considered, and the classical TCI for these measures asymptotically approaches, as n goes to ∞, to the free TCI. Furthermore, a similar method using special unitary random matrices can work to prove the free TCI for measures on $\mathbb{T}$ (Theorem 2.5).

The detailed version of this notes is [6].

1 Free LSI for measures on $\mathbb{R}$ and $\mathbb{T}$

Let M be a smooth complete Riemannian manifold of dimension m with the volume measure dx, and let $\mathrm{Ric}(M)$ denote the *Ricci curvature tensor* of M. For a real-valued C^2 function Ψ on M, the *Hessian* of Ψ is denoted by $\mathrm{Hess}(\Psi)$. The set of all Borel probability measures on M is denoted by $\mathcal{M}(M)$. For $\mu, \nu \in \mathcal{M}(M)$, the *relative entropy* of μ with respect to ν is denoted by $S(\mu, \nu)$, which is defined by

$$S(\mu,\nu) := \int_M \log \frac{d\mu}{d\nu}\, d\mu$$

when μ is absolutely continuous with respect to ν; otherwise $S(\mu,\nu) := +\infty$.

Among huge contributions to the LSI topic, Bakry and Emery [1] gave a simple "local" criterion, the so-called Bakry and Emery criterion, for a given measure on M to satisfy an LSI.

Theorem 1.1 (Bakry and Emery [1]) *Let $\Psi \in C^2(M)$ and set $d\nu(x) := \frac{1}{Z} e^{-\Psi(x)} dx$ with a normalization constant Z. Assume that the Bakry and Emery criterion*

$$\mathrm{Ric}(M) + \mathrm{Hess}(\Psi) \ge \rho I_m$$

holds with a constant $\rho > 0$. Then, for every $f \in C^\infty(M)$,

$$\int_M f^2 \log f^2 \, d\nu - \left(\int_M f^2 \, d\nu\right) \log\left(\int_M f^2 \, d\nu\right) \le \frac{2}{\rho} \int_M \|\nabla f(x)\|^2 \, d\nu(x).$$

Equivalently, for every $\mu \in \mathcal{M}(M)$ absolutely continuous with respect to ν one has

$$S(\mu,\nu) \le \frac{1}{2\rho} \int_M \left\| \nabla \log \frac{d\mu}{d\nu} \right\|^2 d\mu,$$

whenever the density $d\mu/d\nu$ is smooth on M.

In case $M = \mathbb{R}$, we notice

$$S(\mu,\nu) = -S(\mu) + \int_{\mathbb{R}} \Psi(x)\, d\mu(x) + \log Z, \tag{1.1}$$

$$\int_{\mathbb{R}} \left| \frac{d}{dx} \log \frac{d\mu}{d\nu}(x) \right|^2 d\mu(x) = \int_{\mathbb{R}} \left(\frac{p'(x)}{p(x)} + \Psi'(x) \right)^2 d\mu(x) \tag{1.2}$$

where $p := d\mu/dx$.

For each $\mu \in \mathcal{M}(\mathbb{R})$, Voiculescu [12] introduced the *free entropy* of μ

$$\Sigma(\mu) := \iint_{\mathbb{R}^2} \log|x-y|\, d\mu(x)\, d\mu(y),$$

which is the "main component" of the free entropy $\chi(\mu)$ of μ introduced in [13]:

$$\chi(\mu) = \Sigma(\mu) + \frac{3}{4} + \frac{1}{2}\log 2\pi.$$

Assume that $\mu \in \mathcal{M}(\mathbb{R})$ has the density $p = d\mu/dx$ (with respect to the Lebesgue measure dx) belonging to the L^3-space $L^3(\mathbb{R}) := L^3(\mathbb{R}, dx)$. In [12] Voiculescu also introduced the *free Fisher information* of μ

$$\Phi(\mu) := \frac{4\pi^2}{3} \int_{\mathbb{R}} p(x)^3\, dx = 4 \int_{\mathbb{R}} ((Hp)(x))^2 d\mu(x),$$

where Hp is the *Hilbert transform* of p

$$(Hp)(x) := \lim_{\varepsilon \searrow 0} \int_{|x-t|>\varepsilon} \frac{p(t)}{x-t}\, dt.$$

Let Q be a real-valued C^1 function on $\mathbb{R}$. For each $\mu \in \mathcal{M}(\mathbb{R})$, Biane and Speicher [4] introduced the *relative free Fisher information* $\Phi_Q(\mu)$ of μ relative to Q, and it is defined to be

$$\Phi_Q(\mu) := 4 \int_{\mathbb{R}} \left((Hp)(x) - \frac{1}{2} Q'(x) \right)^2 d\mu(x) \tag{1.3}$$

when μ has the density $p = d\mu/dx$ belonging to $L^3(\mathbb{R})$; otherwise to be $+\infty$. When Q is a real-valued continuous function on $\mathbb{R}$ such that

$$\lim_{|x|\to+\infty} |x| \exp(-\varepsilon Q(x)) = 0 \quad \text{for every } \varepsilon > 0,$$

the *weighted energy integral* associated with Q is defined by

$$E_Q(\mu) := -\Sigma(\mu) + \int_{\mathbb{R}} Q(x)\, d\mu(x) \quad \text{for } \mu \in \mathcal{M}(\mathbb{R}).$$

According to a fundamental result in the theory of weighted potentials, there exists a unique $\mu_Q \in \mathcal{M}(\mathbb{R})$ such that

$$E_Q(\mu_Q) = \inf\{E_Q(\mu) : \mu \in \mathcal{M}(\mathbb{R})\},$$

and $E_Q(\mu_Q)$ is finite (hence so is $\Sigma(\mu_Q)$). Moreover, μ_Q is known to be compactly supported. The minimizer μ_Q is sometimes called the *equilibrium measure* associated with Q. Set $B(Q) := -E_Q(\mu_Q)$ so that the function

$$\widetilde{\Sigma}_Q(\mu) := -\Sigma(\mu) + \int_{\mathbb{R}} Q(x)\, d\mu(x) + B(Q) \quad \text{for } \mu \in \mathcal{M}(\mathbb{R}) \tag{1.4}$$

is nonnegative and is zero only when $\mu = \mu_Q$. Following Biane and Speicher [4] and Biane [2], we call the function $\widetilde{\Sigma}(\mu)$ the *relative free entropy* (or *modified free entropy*) of μ relative to Q. We note that the formula (1.4) resembles (1.1) while the formula (1.3) is similar to (1.2). An important point is that $\widetilde{\Sigma}(\mu)$ for $\mu \in \mathcal{M}(\mathbb{R})$ is the good rate function of the large deviation principle in the scale $1/n^2$ for the empirical eigenvalue distribution of a certain $n \times n$ selfadjoint random matrix associated with Q.

The following free analog of LSI was shown by Biane.

Theorem 1.2 (Biane [2]) *Assume that Q is a real-valued C^1 function on $\mathbb{R}$ such that $Q(x) - \frac{\rho}{2}x^2$ is convex on $\mathbb{R}$ with a constant $\rho > 0$. Then, for every $\mu \in \mathcal{M}(\mathbb{R})$ one has*

$$\widetilde{\Sigma}_Q(\mu) \le \frac{1}{2\rho}\Phi_Q(\mu).$$

In particular, when $Q(x) = \frac{\rho}{2}x^2$ with $\rho > 0$,

$$\widetilde{\Sigma}_Q(\mu) = -\Sigma(\mu) + \frac{\rho}{2}\int_{\mathbb{R}} x^2\, d\mu(x) - \frac{1}{2}\log\rho - \frac{3}{4},$$

whose minimizer is the $(0, 1/\rho)$-semicircular distribution, and

$$\Phi_Q(\mu) = \Phi(\mu) - 2\rho + \rho^2 \int_{\mathbb{R}} x^2\, d\mu(x).$$

Hence, the free LSI of Theorem 1.2 becomes

$$\Sigma(\mu) \ge -\frac{1}{2\rho}\Phi(\mu) - \frac{1}{2}\log\rho + \frac{1}{4}.$$

Maximizing the right-hand side over $\rho > 0$ gives Voiculescu's inequality

$$\chi(\mu) \ge \frac{1}{2}\log\frac{2\pi e}{\Phi(\mu)}. \tag{1.5}$$

The *free entropy* of $\mu \in \mathcal{M}(\mathbb{T})$ is similarly defined as

$$\Sigma(\mu) := \iint_{\mathbb{T}^2} \log|\zeta - \eta| \, d\mu(\zeta)\, d\mu(\eta).$$

When μ has the density $p = d\mu/d\zeta$ with respect to $d\zeta = d\theta/2\pi$ ($\zeta = e^{\sqrt{-1}\theta}$) belonging to the L^3-space $L^3(\mathbb{T}) := L^3(\mathbb{T}, d\zeta)$, the *free Fisher information* of μ was introduced in [15] by

$$F(\mu) := \int_{\mathbb{T}} ((Hp)(\zeta))^2 \, d\mu(\zeta),$$

where Hp is the Hilbert transform of p

$$(Hp)\left(e^{\sqrt{-1}\theta}\right) := \lim_{\varepsilon \searrow 0} \int_{\varepsilon \le |t| < \pi} \frac{p\left(e^{\sqrt{-1}(\theta - t)}\right)}{\tan\left(\frac{t}{2}\right)} \frac{dt}{2\pi}.$$

Note [15] that $F(\mu)$ is also written as

$$F(\mu) = \frac{1}{3}\left(-1 + \int_{\mathbb{T}} p(\zeta)^3 \, d\zeta\right).$$

When μ has no such density as above, $F(\mu)$ is defined to be $+\infty$.

Let Q be a real-valued C^1 function on $\mathbb{T}$. As in the case of measures on $\mathbb{R}$, for each $\mu \in \mathcal{M}(\mathbb{T})$ we define the *relative free Fisher information* $F_Q(\mu)$ by

$$F_Q(\mu) := \int_{\mathbb{T}} ((Hp)(\zeta) - Q'(\zeta))^2 \, d\mu(\zeta) - \left(\int_{\mathbb{T}} Q'(\zeta)\, d\mu(\zeta)\right)^2$$

when μ has the density $p = d\mu/d\zeta$ belonging to $L^3(\mathbb{T})$; otherwise $F_Q(\mu) := +\infty$. Here, Q' means the derivative of $Q(e^{\sqrt{-1}\theta})$ in θ, i.e., $Q'(e^{\sqrt{-1}\theta}) = \frac{d}{d\theta} Q(e^{\sqrt{-1}\theta})$. The weighted energy integral

$$-\Sigma(\mu) + \int_{\mathbb{T}} Q(\zeta)\, d\mu(\zeta) \quad \text{for } \mu \in \mathcal{M}(\mathbb{T})$$

admits a unique minimizer $\mu_Q \in \mathcal{M}(\mathbb{T})$, the equilibrium measure associated with Q. Set $B(Q) := \Sigma(\mu_Q) - \int_{\mathbb{T}} Q(\zeta)\, d\mu_Q(\zeta)$ and introduce the *relative free entropy* of $\mu \in \mathcal{M}(\mathbb{T})$ relative to Q by

$$\widetilde{\Sigma}_Q(\mu) := -\Sigma(\mu) + \int_{\mathbb{T}} Q(\zeta)\, d\mu(\zeta) + B(Q).$$

It is known that $\widetilde{\Sigma}_Q(\mu)$ for $\mu \in \mathcal{M}(\mathbb{T})$ is the rate function of the large deviation for the empirical eigenvalue distribution of a certain $n \times n$ unitary (or special unitary) random matrix.

Our free analog of LSI for measures on $\mathbb{T}$ is

Theorem 1.3 ([6]) *Let Q be a real-valued C^1 function on $\mathbb{T}$ such that $Q\left(e^{\sqrt{-1}t}\right) - \frac{\rho}{2}t^2$ is convex on $\mathbb{R}$ with a constant $\rho > -1/2$. Then, for every $\mu \in \mathcal{M}(\mathbb{T})$ one has*

$$\widetilde{\Sigma}_Q(\mu) \le \frac{1}{1 + 2\rho} F_Q(\mu).$$

The proof is based on the procedure of random matrix approximation. Namely, the free analog arises as the scaling limit in the scale $1/n^2$ of the classical one (Theorem 1.1) on the special unitary group $\mathrm{SU}(n)$. We need the convergence of the empirical eigenvalue distribution of the random matrix not only in the mean but also in the almost sure sense that is a consequence of the corresponding large deviation principle. We also need the exact computation of the Ricci curvature tensor of $\mathrm{SU}(n)$ (with respect to the Riemannian structure associated with the usual trace on $M_n(\mathbb{C})$) to check the so-called Bakry and Emery criterion in Theorem 1.1.

2 Free TCI for measures on $\mathbb{R}$ and $\mathbb{T}$

Let $\mathcal{X}$ be a Polish space with a metric d. The (quadratic) *Wasserstein distance* between $\mu, \nu \in \mathcal{M}(\mathcal{X})$ is defined by

$$W(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \sqrt{\iint_{\mathcal{X}\times\mathcal{X}} \frac{1}{2} d(x,y)^2 \, d\pi(x,y)},$$

where $\Pi(\mu,\nu)$ denotes the set of all probability measures on $\mathcal{X} \times \mathcal{X}$ with marginals μ and ν, i.e., $\pi(\cdot \times \mathcal{X}) = \mu$ and $\pi(\mathcal{X} \times \cdot) = \nu$. The Wasserstein distance is sometimes defined with the integral of $d(x,y)^2$ instead of $\frac{1}{2}d(x,y)^2$.

In the typical case where $\mathcal{X} = \mathbb{R}^n$ and $d(x,y) = \|x-y\|$, the usual Euclidean metric, the celebrated TCI of Talagrand [10] is

$$W(\mu, g_n) \le \sqrt{S(\mu, g_n)}, \qquad \mu \in \mathcal{M}(\mathbb{R}^n),$$

where g_n is the standard Gaussian measure, i.e., $dg_n(x) := (2\pi)^{-n/2} e^{-\|x\|^2/2} \, dx$ (dx means the Lebesgue measure on $\mathbb{R}^n$). This inequality is a bit extended as follows (see [8]).

Theorem 2.1 *Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ and assume that $\Psi(x) - \frac{\rho}{2}\|x\|^2$ is convex on $\mathbb{R}^n$ with a constant $\rho > 0$. If $d\nu(x) := \frac{1}{Z} e^{-\Psi(x)} \, dx \in \mathcal{M}(\mathbb{R}^n)$ with a normalization constant Z, then*

$$W(\mu,\nu) \le \sqrt{\frac{1}{\rho} S(\mu,\nu)}, \qquad \mu \in \mathcal{M}(\mathbb{R}^n).$$

In [9] Otto and Villani established the interrelation between LSI and TCI by a technique using partial differential equations. Their result, combined with Bakry and Emery's LSI [1] (or Theorem 1.1), implies the following TCI in a setup on Riemannian manifolds, where M is an m-dimensional smooth complete Riemannian manifold equipped with the geodesic distance $d(x,y)$ and the volume measure dx.

Theorem 2.2 (Bakry and Emery [1] and Otto and Villani [9]) *Let Ψ be a real-valued C^2 function on M and set $d\nu(x) := \frac{1}{Z} e^{-\Psi(x)} \, dx \in \mathcal{M}(M)$ with a normalization constant*

Z. If the Bakry and Emery criterion Ric(M) + Hess$(\Psi) \geq \rho I_m$ *holds with a constant* $\rho > 0$, *then*

$$W(\mu,\nu) \leq \sqrt{\frac{1}{\rho}S(\mu,\nu)}, \qquad \mu \in \mathcal{M}(M).$$

On the other hand, the following free analog of Talagrand's TCI is shown by Biane and Voiculescu [5].

Theorem 2.3 (Biane and Voiculescu [5]) *For every compactly supported* $\mu \in \mathcal{M}(\mathbb{R})$,

$$W(\mu,\gamma_{0,2}) \leq \sqrt{-\Sigma(\mu) + \int \frac{x^2}{2}\,d\mu(x) - \frac{3}{4}}, \tag{2.1}$$

where $\gamma_{0,2}$ *is the standard semicircular measure, i.e.,* $d\gamma_{0,2}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\chi_{[-2,2]}(x)\,dx$.

In [6] we present a new proof of the above free TCI in a more general situation by using a random matrix technique. In fact, the classical TCI on the matrix space M_n^{sa} asymptotically approaches to the free analog when the matrix size goes to ∞. The following is our free TCI for probability measures on $\mathbb{R}$, where the relative entropy in the classical TCI is replaced by the relative free entropy.

Theorem 2.4 ([6]) *Let* Q *be a real-valued function on* $\mathbb{R}$. *If* $Q(x) - \frac{\rho}{2}x^2$ *is convex on* $\mathbb{R}$ *with a constant* $\rho > 0$, *then*

$$W(\mu,\mu_Q) \leq \sqrt{\frac{1}{\rho}\widetilde{\Sigma}_Q(\mu)}$$

for every compactly supported $\mu \in \mathcal{M}(\mathbb{R})$.

In particular, when $Q(x) = x^2/2$ and so $\rho = 1$, the relative free entropy $\widetilde{\Sigma}_Q(\mu)$ is the inside of the square root in (2.1) and its minimizer is $\gamma_{0,2}$; hence Theorem 2.4 is a generalization of Theorem 2.3.

Next, we consider two kinds of Wasserstein distances between probability measures $\mu,\nu \in \mathcal{M}(\mathbb{T})$. The one is the Wasserstein distance with respect to the usual metric $|\zeta-\eta|$, $\zeta,\eta \in \mathbb{T}$, and the other is with respect to the geodesic distance (i.e., the angular distance) on $\mathbb{T}$. We write $W_{|\cdot|}(\mu,\nu)$ for the former and $W(\mu,\nu)$ for the latter. Of course, one has

$$W_{|\cdot|}(\mu,\nu) \leq W(\mu,\nu), \qquad \mu,\nu \in \mathcal{M}(\mathbb{T}).$$

The next theorem is our free TCI for measures on $\mathbb{T}$ comparing the Wasserstein distance with the relative free entropy.

Theorem 2.5 ([6]) *Let* Q *be a real-valued function on* $\mathbb{T}$. *If there exists a constant* $\rho > -\frac{1}{2}$ *such that* $Q(e^{\sqrt{-1}t}) - \frac{\rho}{2}t^2$ *is convex on* $\mathbb{R}$, *then*

$$W_{|\cdot|}(\mu,\mu_Q) \leq W(\mu,\mu_Q) \leq \sqrt{\frac{2}{1+2\rho}\widetilde{\Sigma}_Q(\mu)}$$

for every $\mu \in \mathcal{M}(\mathbb{T})$.

The special case where $Q \equiv 0$ and $\rho = 0$ is

$$W_{|\cdot|}\left(\mu, \frac{d\theta}{2\pi}\right) \le W\left(\mu, \frac{d\theta}{2\pi}\right) \le \sqrt{-2\Sigma(\mu)}, \qquad \mu \in \mathcal{M}(\mathbb{T}).$$

3 Some remarks

1. The Ricci curvature tensor of $\mathrm{U}(n)$ is known to be degenerate, while that of $\mathrm{SU}(n)$ to be of positive constant and a straightforward computation shows that the Ricci curvature tensor of $\mathrm{SU}(n)$ with respect to the Riemannian structure associated with Tr_n is

$$\mathrm{Ric}(\mathrm{SU}(n)) = \frac{n}{2} I_{n^2-1}.$$

This is the reason why we need the large deviation for the eigenvalue distribution of special unitary random matrices instead of unitary ones.

2. A special orthogonal random matrix model can be used as well to obtain the free LSI in Theorem 1.3 and the free TCI in Theorem 2.5. Here, note that the Ricci curvature tensor of $\mathrm{SO}(n)$ is

$$\mathrm{Ric}(\mathrm{SO}(n)) = \frac{n-2}{4} I_{n(n-1)/2}.$$

Similarly, the free TCI in Theorem 2.4 can be shown by using a real symmetric random matrix model as well.

3. We do not know whether the bounds $1/2\rho$ in the free LSI of Theorem 1.2 as well as $1/(1+2\rho)$ in Theorem 1.3 are best possible or not. However, a simple computation says that the bound $1/2\rho$ in Theorem 1.2 cannot be smaller than $1/4\rho$.

4. In the case of the uniform probability measure $d\theta/2\pi$ on $\mathbb{T}$, our free TCI is

$$W\left(\mu, \frac{d\theta}{2\pi}\right) \le \sqrt{-2\Sigma(\mu)}, \qquad \mu \in \mathcal{M}(\mathbb{T}),$$

while to the authors' best knowledge the sharpest classical TCI is

$$W\left(\mu, \frac{d\theta}{2\pi}\right) \le \sqrt{S\left(\mu, \frac{d\theta}{2\pi}\right)}, \qquad \mu \in \mathcal{M}(\mathbb{T}).$$

Thus, it seems interesting to compare these two TCI's. But, some concrete examples show that these are not comparable; in fact, the ratio

$$\frac{-\Sigma\left(\mu_k(n)\right)}{S\left(\mu_k(n), \frac{d\theta}{2\pi}\right)}$$

can be arbitrarily small and also arbitrarily large.

5. The free LSI of Theorem 1.2 is applicable in particular for measures supported in $\mathbb{R}^+ = [0, \infty)$, but we can also show a different inequality which might be a proper free LSI in the case where the whole space is $\mathbb{R}^+$ instead of $\mathbb{R}$. To do so, we use the symmetrization technique transforming measures on $\mathbb{R}^+$ to symmetric ones on $\mathbb{R}$.

6. For an N-tuple $(X_1,\dots,X_N)$ of noncommutative selfadjoint random variables in a tracial W^*-probability space $(\mathcal{M},\tau)$, Voiculescu [13] introduced the free entropy $\chi(X_1,\dots,X_N)$ in the microstates approach. Furthermore, in [14] he introduced the free Fisher information $\Phi^*(X_1,\dots,X_N)$ and the free entropy $\chi^*(X_1,\dots,X_N)$ in the microstates-free approach. His inequality in [14]

$$\chi^*(X_1,\dots,X_N) \geq \frac{N}{2}\log\frac{2\pi Ne}{\Phi^*(X_1,\dots,X_N)}$$

is a kind of multivariable free LSI. On the other hand, Biane and Voiculescu [5] extended the Wasserstein distance to the multivariable case: $W((X_1,\dots,X_N),(Y_1,\dots,Y_N))$ for two N-tuples of noncommutative random variables. In this situation, challenging problems are to show free LSI and free TCI for noncommutative multivariables. In this connection, the inequality

$$\chi^*(X_1,\dots,X_N) \geq \chi(X_1,\dots,X_N)$$

obtained by Biane, Capitane and Guionnet [3] is remarkable.

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