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Translation-invariant quantum Markov states

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1 Introduction

The notion of quantum Markov states was first introduced by Accardi and Frigerio ([1], [3]), and was further discussed from a somewhat different viewpoint by Fannes, Nachtergaele and Werner ([5]). A Markov state by Accardi and Frigerio is defined on a UHF algebra and is determined by an initial state and a family of completely positive quasi-conditional expectations However, thanks to [2], [3] and also [6], conditional expectations can be used in place of quasi-conditional expectations. Although Accardi and Frigerio defined their Markov states without translation-invariance, we restrict ourselves to translation-invariant ones and clarify their fine structure.

In [4] it was implicitly stated that any translation-invariant Markov state in the sense of [3] is determined by a single conditional expectation (so that it is a C^*-finitely correlated state in [5]), and an explicit form of translation-invariant Markov states was given. In Section 2 we make the relation between two notions of quantum Markov states more precise and consider the question concerning the commutativity of local density matrices of a Markov state. In Section 3 we see explicit form of quantum Markov states due to [4].

2 Characterization of translation-invariant Markov states

Let $\mathfrak{A}_i = M_d = M_d(\mathbb{C})$, the $d \times d$ complex matrix algebra, for $i \in \mathbb{N}$ and $\mathfrak{A}$ be the infinite C^*-tensor product $\bigotimes_{i=1}^{\infty} \mathfrak{A}_i$. We denote $\mathfrak{A}_\Lambda = \bigotimes_{n \in \Lambda} \mathfrak{A}_n$ for arbitrary subset $\Lambda \subset \mathbb{N}$. The translation γ is the right shift on $\mathfrak{A}$. We write $\phi_{[1,n]}$ for the localization $\phi|\mathfrak{A}_{[1,n]}$, and in particular ϕ_1 for $n = 1$. The following definition is from [3] with a slight modification.

Definition 2.1 A state ϕ on $\mathfrak{A}$ is called a (quantum) Markov state if for each $n \in \mathbb{N}$ there exists a conditional expectation E_n from $\mathfrak{A}_{[1,n+1]}$ into $\mathfrak{A}_{[1,n]}$ such that $E_n(\mathfrak{A}_{[1,n+1]}) \supset \mathfrak{A}_{[1,n-1]}$ and $\phi_{[1,n+1]} = \phi_{[1,n]} \circ E_n$. A Markov state is said to be translation-invariant if $\phi \circ \gamma = \phi$.

Although the above definition is a bit different from the original one of Accardi and Frigerio in [3], it is known that both definition are equivalent ([2], [3] and also [6]).

We assume that ϕ is locally faithful, i.e. $\phi_{[1,n]}$ is faithful for all $n \in \mathbb{N}$. The next theorem was implicitly stated in [4]; we here give a proof.

Theorem 2.2 *Let ϕ be a state on $\mathfrak{A}$. Then the following are equivalent.*

(i) *ϕ is a translation-invariant Markov state,*

(ii) *There exists a conditional expectation E from $M_d \otimes M_d$ into M_d such that $\phi_1 \circ E(I \otimes A) = \phi_1(A)$ for all $A \in M_d$ and*

$$\phi(A_1 \otimes \cdots \otimes A_n) = \phi_1(E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_{n-1} \otimes A_n) \cdots)))$$

for all $A_1, \ldots, A_n \in M_d$.

Proof. (ii) $\Rightarrow$ (i). Assume (ii), and define conditional expectation $E_n : \mathfrak{A}_{[1,n+1]} \to \mathfrak{A}_{[1,n]}$, $n \in \mathbb{N}$, by

$$E_n(X \otimes A) = X \otimes E(A)$$

for $X \in \mathfrak{A}_{[1,n-1]}$ and $A \in \mathfrak{A}_{[n,n+1]}$. Then for $A_1, \ldots, A_n \in M_d$,

$$\begin{aligned} \phi(A_1 \otimes \cdots \otimes A_n) &= \phi_1 \circ E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_{n-1} \otimes A_n) \cdots) \\ &= \phi(A_1 \otimes \cdots \otimes A_{n-2} \otimes E(A_{n-1} \otimes A_n)) \\ &= \phi \circ E_{n-1}(A_1 \otimes \cdots \otimes A_n) \end{aligned}$$

and

$$\begin{aligned} \phi(I \otimes A_1 \otimes \cdots \otimes A_n) &= \phi_1 \circ E(I \otimes E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_{n-1} \otimes A_n) \cdots) \\ &= \phi_1 \circ E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_{n-1} \otimes A_n) \cdots) \\ &= \phi(A_1 \otimes \cdots \otimes A_n). \end{aligned}$$

So ϕ is a translation-invariant Markov state.

(i) $\Rightarrow$ (ii). We fix $n \in \mathbb{N}$, and define $F_{n-1} = \gamma^{-1} \circ E_n \circ \gamma$. This is well defined. Indeed, for any $A \in \mathfrak{A}_{[1,n-1]}$ and $B \in \mathfrak{A}_1$,

$$E_n(I \otimes A) \cdot B \otimes I^{\otimes n-1} = E_n(B \otimes A) = B \otimes I^{\otimes n-1} \cdot E_n(I \otimes A).$$

Hence, $E_n(I \otimes A) \in \mathfrak{A}_{[2,n]}$. Similarly, we define $F_i = \gamma^{-(n-i)} \circ E_n \circ \gamma^{n-i}$ $(1 \leq i \leq n-1)$. Then for $1 \leq i \leq n$ and $A_1, \ldots, A_{i+1} \in M_d$, we have

$$\begin{aligned} F_i(A_1 \otimes \cdots \otimes A_{i+1}) &= (A_1 \otimes \cdots \otimes A_{i-1} \otimes I^{\otimes 2}) \cdot F_i(I^{\otimes i-1} \otimes A_i \otimes A_{i+1}) \\ &= A_1 \otimes \cdots \otimes A_{i-1} \otimes F_1(A_i \otimes A_{i+1}). \end{aligned}$$

Now, let $\mathfrak{F}_n$ denote the set of all conditional expectations $F : M_d \otimes M_d \to M_d$ such that if we define $F_i(A_1 \otimes \cdots \otimes A_{i+1}) = A_1 \otimes \cdots \otimes A_{i-1} \otimes F(A_i \otimes A_{i+1})$, for $A_1, \ldots, A_{i+1} \in M_d$, then $\phi_{[1,i]} \circ F_i = \phi_{[1,i+1]}$ for each $1 \leq i \leq n$. Then the above argument guarantees the non-emptyness of $\mathfrak{F}_n$. Since $\mathfrak{F}_n$'s are compact and $\mathfrak{F}_1 \supset \mathfrak{F}_2 \cdots$, it follows that $\bigcap_{n\in\mathbb{N}} \mathfrak{F}_n$ is not empty. Choose $E \in \bigcap_{n\in\mathbb{N}} \mathfrak{F}_n$ and define

$$E_n(A_1 \otimes \cdots \otimes A_{n+1}) = A_1 \otimes \cdots \otimes A_{n-1} \otimes E(A_n \otimes A_{n+1})$$

for $A_1, \dots, A_{n+1} \in M_d$. Then

$$\begin{aligned}\phi(A_1 \otimes \cdots \otimes A_n) &= \phi_1 \circ E_1 \circ \cdots \circ E_n(A_1 \otimes \cdots \otimes A_n) \\ &= \phi_1 \circ E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_{n-1} \otimes A_n) \cdots)),\end{aligned}$$

and

$$\phi_1 \circ E(I \otimes A) = \phi(I \otimes A) = \phi_1(A)$$

for $A \in M_d$. □

The following definition is from [5].

Definition 2.3 A state ϕ on $\mathfrak{A}$ is called a C^*-finitely correlated state if there exist a finite dimensional C^*-algebra $\mathfrak{B}$, a completely positive map $E : M_d \otimes \mathfrak{B} \to \mathfrak{B}$ and a state ρ on $\mathfrak{B}$ such that

$$\rho(E(I_d \otimes B)) = \rho(B)$$

for all $B \in \mathfrak{B}$ and

$$\phi(A_1 \otimes \cdots \otimes A_n) = \rho(E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_n \otimes I_{\mathfrak{B}}) \cdots)))$$

for all $A_1, \dots, A_n \in M_d$.

Let ϕ be a translation-invariant Markov state, and E be as in (ii) of Theorem 2.2. We set $\mathfrak{B} = E(M_d \otimes M_d)$ and $\hat{E} = E|M_d \otimes \mathfrak{B}$. Then ϕ is a C^*-finitely correlated state with a triple $(\mathfrak{B}, \hat{E}, \phi|\mathfrak{B})$. Hence any translation-invariant Markov state becomes a C^*-finitely correlated state.

Now, let $q_1, \dots, q_k$ be minimal central projections of $\mathfrak{B}$, so that $\mathfrak{B}q_i \cong M_{d_i}$ for some $d_i \in \mathbb{N}$. Let m_i be the multiplicity of M_{d_i} in M_d. Then

$$\mathfrak{B} = \bigoplus_{i=1}^{k} \mathfrak{B}q_i = \bigoplus_{i=1}^{k} M_{d_i}.$$

Moreover, we set

$$\mathfrak{C} = \bigoplus_{i=1}^{k} M_{d_i} \otimes M_{m_i}$$

and let $E_{\mathfrak{C}} : M_d \to \mathfrak{C}$ be the pinching $A \mapsto \sum_{i=1}^{k} q_i A q_i$.

The next proposition is included in [4].

Proposition 2.4 *There exist positive linear functionals ρ_{ij} on $M_{m_i} \otimes M_{d_j}$ $(1 \leq i, j \leq k)$ such that*

$$\hat{E} = (\bigoplus_{i,j=1}^{k} \mathrm{id}_{M_{d_i}} \otimes \rho_{ij})(E_{\mathfrak{C}} \otimes \mathrm{id}_{\mathfrak{B}}).$$

We remark that the unitality of $\hat{E}$ is equivalent to the condition that $\bigoplus_{j=1}^{k} \rho_{ij}$ is a state on $M_{m_i} \otimes \mathfrak{B}$ for each $1 \leq i \leq k$. Furthermore, the condition that $\phi_1 \circ \hat{E}(I \otimes B) = \phi_1(B)$ for all $B \in \mathfrak{B}$ is equivalent to the condition that for $B_j \in M_{d_j}$ $(1 \leq j \leq k)$,

$$\sum_{i=1}^{k} \psi_i(q_i)\rho_{ij}(I_{m_i} \otimes B_j) = \psi_j(B_j), \tag{1}$$

where $\phi|\mathfrak{B} = \bigoplus_{i=1}^{k} \psi_i$. Set $\pi_{ij} = \rho_{ij}(I_{m_i} \otimes q_j)$ and $\alpha_i = \psi_i(q_i)$; then the equation (1) says

$$\begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} \begin{bmatrix} \pi_{11} & \cdots & \pi_{1k} \\ \vdots & \ddots & \vdots \\ \pi_{k1} & \cdots & \pi_{kk} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix}.$$

The unitality of $\hat{E}$ means that the matrix $[\pi_{ij}]$ is a stochastic matrix. The faithfullness of $\hat{E}$ guarantees $\pi_{ij} > 0$ for all $1 \leq i, j \leq k$. Hence, $\{\alpha_i\}$ is uniquely determined by $\{\pi_{ij}\}$ from the Perron-Frobenius theorem. So, by (1), $\{\psi_i\}$ is also uniquely determined by $\{\rho_{ij}\}$.

By Proposition 2.4, we get the next corollary.

Corollary 2.5 *Let S_i and T_{ij} be the density matrices of ψ_i and ρ_{ij}, respectively. Then the density matrix $\hat{D}_n$ of $\phi|\mathfrak{A}_{[1,n-1]} \otimes \mathfrak{B}$ is*

$$\bigoplus_{i_1,\ldots,i_n} S_{i_1} \otimes T_{i_1 i_2} \otimes \cdots \otimes T_{i_{n-1} i_n}.$$

In the above the density matrix $\hat{D}_n$ is taken with respect to the usual trace on $\mathfrak{A}_{[1,n-1]} \otimes \mathfrak{B}$, i.e. the trace having the value 1 for each rank one projection. If all summands of $\mathfrak{B}$ are of multiplicity one, then the density in the above corollary is actually the density matrix D_n of $\phi_{[1,n]}$. Hence, the densities D_n are all commuting in this case (see [6]).

Consider the case $d = 2$. Subalgebras $\mathfrak{B}$ of M_2 are M_2 or $\mathbb{C} \oplus \mathbb{C}$ or $\mathbb{C}$. If $\mathfrak{B}$ is M_2 or $\mathbb{C}$, any translation-invariant Markov state relative to $\mathfrak{B}$ is a product state. If $\mathfrak{B}$ is $\mathbb{C} \oplus \mathbb{C}$, all summands of $\mathfrak{B}$ are of multiplicity one. Hence, the density matrices D_n's are commuting in this case (see [3]).

The following is the simplest example where the densities $\{D_n\}$ are non-commuting.

Example 2.6 We set $d = 3$ and $\mathfrak{B} = \mathbb{C} \oplus \mathbb{C} = \mathbb{C}(e_{11} + e_{22}) + \mathbb{C}e_{33}$, where e_{ij} $(1 \leq i, j \leq 3)$ are the matrix unit of M_3. Assume that the density matrix of $\bigoplus_{i,j=1}^{2} \rho_{ij}$ is $T_{11} \oplus T_{12} \oplus T_{21} \oplus T_{22} = A_1 \oplus A_2 \oplus c_1 \oplus c_2 \in M_2 \oplus M_2 \oplus \mathbb{C} \oplus \mathbb{C}$, where $A_1, A_2 \in M_2$, $A_1, A_2 \geq 0$, $\mathrm{Tr}(A_1 + A_2) = 1$, and $c_1, c_2 \in \mathbb{R}^+$, $c_1 + c_2 = 1$. We define

$$\psi_1(e_{11} + e_{22}) = \frac{c_1}{c_1 + \mathrm{Tr}(A_2)}, \quad \psi_2(e_{33}) = \frac{\mathrm{Tr}(A_2)}{c_1 + \mathrm{Tr}(A_2)},$$

then it is easily seen that the condition (1) is satisfied. In this case, the density matrix D_n of $\mathfrak{A}_{[1,n]}$ is

$$\bigoplus_{i_1,\dots,i_{n-1}} S_{i_1} \otimes T_{i_1 i_2} \otimes \cdots \otimes T_{i_{n-2} i_{n-1}} \otimes (T_{i_{n-1}1} \otimes (A_1 + A_2) \oplus T_{i_{n-1}2} \otimes (c_1 + c_2)).$$

So, D_n's are non-commuting if so are A_1 and A_2.

3 Disintegration of quantum Markov states

In this section, we survey the explicit form of Markov states due to [4]. Let ϕ be a translation-invariant Markov state as in Section 2. We put $\Omega_n = \{1, \dots k\}$,

$$\Omega = \prod_{n \in \mathrm{N}} \Omega_n$$

and

$$(x_1, x_2, \dots, x_n) = \{(y_1, y_2, \dots) \in \Omega \mid y_i = x_i, 1 \leq i \leq n\}.$$

We define the mesure ν on Ω by

$$\begin{aligned} \nu((x_1, x_2, \dots, x_n)) &= \phi(q_{x_1} \otimes q_{x_2} \otimes \cdots q_{x_n}) \\ &= \alpha_{x_1} \cdot \prod_{i=1}^{n-1} \pi_{x_i x_{i+1}}. \end{aligned}$$

Then ν is a probability mesure on Ω. For an arbitraly element $\omega = (\omega_1, \omega_2, \dots) \in \Omega$, we set

$$\mathfrak{B}_\omega = M_{d_{\omega_1}} \otimes M_{m_{\omega_1}} \otimes M_{d_{\omega_2}} \cdots$$

and the state ψ_ω on $\mathfrak{B}_\omega$ by

$$\psi_\omega = \tilde{\psi}_{\omega_1} \otimes \bigotimes_{i=1}^{\infty} \tilde{\rho}_{\omega_i \omega_{i+1}},$$

where $\tilde{\psi}_i = \psi_i / \alpha_i$ and $\tilde{\rho}_{ij} = \rho_{ij} / \pi_{ij}$. Let $E_\omega : \mathfrak{A} \to \mathfrak{B}_\omega$ be a completely positive map defined by

$$E_\omega(A_1 \otimes A_2 \otimes \cdots \otimes A_n) = q_{\omega_1} A_1 q_{\omega_1} \otimes \cdots \otimes q_{\omega_n} A_n q_{\omega_n}$$

for any $A_1, \dots, A_n \in M_d$, and $\phi_\omega = \psi_\omega \circ E_\omega$.

Theorem 3.1 *Define Ω, ν and ϕ_ω as above. Then*

$$\phi = \int_\Omega \phi_\omega \nu(d\omega).$$

Proof. If $\omega, \omega' \in (x_1, \dots, x_n)$, then

$$\begin{aligned}
& \phi_\omega(A_1 \otimes \cdots \otimes A_{n-1}) \\
= \; & \psi_\omega(q_{x_1} A_1 q_{x_1} \otimes \cdots \otimes q_{x_n} A_{n-1} q_{x_n}) \\
= \; & \frac{1}{\nu((x_1, \dots, x_n))} \cdot (\psi_{x_1} \otimes \bigotimes_{i=1}^{n-1} \rho_{x_i x_{i+1}})(q_{x_1} A_1 q_{x_1} \otimes \cdots \otimes q_{x_{n-1}} A_{n-1} q_{x_{n-1}}) \\
= \; & \phi_{\omega'}(A_1 \otimes \cdots \otimes A_{n-1})
\end{aligned}$$

for any $A_1, \dots, A_{n-1} \in M_d$. Therefore,

$$\begin{aligned}
& \phi(A_1 \otimes \cdots \otimes A_{n-1}) = \phi \circ E_{\mathfrak{C}}(A_1 \otimes \cdots \otimes A_{n-1}) \\
= \; & \sum_{x_1, \dots, x_n} (\psi_{x_1} \otimes \bigotimes_{i=1}^{n-1} \rho_{x_i x_{i+1}})(q_{x_1} A_1 q_{x_1} \otimes \cdots \otimes q_{x_{n-1}} A_{n-1} q_{x_{n-1}}) \\
= \; & \sum_{x_1, \dots, x_n} \nu((x_1, \dots, x_n)) \cdot \phi_{\omega_{(x_1, \dots, x_n)}}(A_1 \otimes \cdots \otimes A_{n-1}) \\
= \; & \int_\Omega \phi_\omega(A_1 \otimes \cdots \otimes A_{n-1}) \nu(d\omega),
\end{aligned}$$

where $\omega_{(x_1, \dots, x_n)}$ is an arbitraly element in $(x_1, \dots, x_n)$. □

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