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On an anisotropic area-preserving crystalline motion and motion of nonadmissible polygons by crystalline curvature*

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Abstract

We split the present paper into two parts.
Part I: The asymptotic behavior of solutions to an anisotropic area-preserving crystalline motion is investigated. In this equation, the area enclosed by the solution polygon is preserved, while its total interfacial energy keeps on decreasing. By the concept of mixed area and the Brünn and Minkowski's inequality, the anisoperimetric inequality is established. From this and the theory of dynamical systems, we show that the asymptotic shape of a solution polygon is the boundary of the Wulff shape.
Part II: Behavior of solution polygons to a general crystalline motion is investigated. Polygon is called admissible if its normal angle's set is the same set of the Wulff shape. Main result says that if initial polygon is nonadmissible polygon, then edge disappearing occurs at most finitely many epochs and eventually a solution polygon becomes an admissible polygon.

Part I
On an anisotropic area-preserving crystalline motion

1 Introduction and a main result

In recent over ten years, several authors have investigated motion of polygonal curves by crystalline curvature in the plane. We refer the reader to the pioneer works for

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such motions by Taylor [11, 12] and Angenent and Gurtin [2], and also Taylor, Cahn and Handwerker [14] for a geometric and physical background. Motion by crystalline curvature, or crystalline motion, has been studied under various kinds of evolution law by several authors (see, e.g., Giga [4], and references therein). In Yazaki [16], we discussed the gradient flow of the total length functional of convex polygon keeping the area enclosed by the polygon constant, and showed that any polygon which evolves by this gradient flow converges to the circumscribed polygon of a circle (see Proposition 1.1 below). This result is corresponding to a semidiscrete version of Gage [3]. In the present paper, we consider the anisotropic case of [16].

Let $\mathcal{P}$ be closed, convex and n-sided polygon in $\boldsymbol{R}^2$. We define θ_j by normal angle of the j-th side and define its set by

$$\Theta_n = \{\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_0 + 2\pi\}.$$

Put $\vartheta_j = \theta_j - \theta_{j-1}$, then from the convexity of $\mathcal{P}$, $\vartheta_j < \pi$ holds and the angle between the j-th side and the $(j-1)$-th side is $\pi - \vartheta_j$. Here and hereafter, suffix j means an integer modulo n: $(\cdot)_n = (\cdot)_0$ and $(\cdot)_{-1} = (\cdot)_{n-1}$. The inward normal vector and the tangent vector of the j-th side are given as $\boldsymbol{n}_j = -{}^t(\cos\theta_j, \sin\theta_j)$ and $\boldsymbol{t}_j = {}^t(-\sin\theta_j, \cos\theta_j)$, respectively. Let $\boldsymbol{x}_j$ be the point of intersection between the line containing the j-th side of $\mathcal{P}$ and the line spanned by $\boldsymbol{n}_j$ which through the origin. Define $h_j = \langle \boldsymbol{x}_j, -\boldsymbol{n}_j \rangle$. Then h_j is the j-th support function of $\mathcal{P}$. Here $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\boldsymbol{R}^2$. The enclosed region of $\mathcal{P}$, say Ω, is given as $\Omega = \{\boldsymbol{x} \in \boldsymbol{R}^2 \,|\, \langle \boldsymbol{x}, -\boldsymbol{n}_j \rangle \le h_j,\ \theta_j \in \Theta_n\}$. Therefore, the shape of $\mathcal{P}$ is uniquely determined by $h = (h_0, h_1, \ldots, h_{n-1})$ and Θ_n.

By geometry, the length of the j-th side, say d_j, is given by $d_j = \gamma_j(\Delta_\theta h + h)_j$. Here

$$\gamma_j = \tan\frac{\vartheta_{j+1}}{2} + \tan\frac{\vartheta_j}{2} = \frac{\sin\vartheta_j + \sin\vartheta_{j+1} - \sin(\vartheta_j + \vartheta_{j+1})}{\sin\vartheta_j \sin\vartheta_{j+1}},$$

and

$$(\Delta_\theta h)_j = \frac{(\mathrm{D}_+ h)_j - (\mathrm{D}_+ h)_{j-1}}{\gamma_j}, \quad (\mathrm{D}_+ h)_j = \frac{h_{j+1} - h_j}{\sin\vartheta_{j+1}}.$$

Therefore, the total length of $\mathcal{P}$, say $\mathcal{L}$, is

$$\mathcal{L} = \mathcal{L}[h] = \sum_j d_j = \sum_j \gamma_j(\Delta_\theta h + h)_j = \sum_j \gamma_j h_j.$$

Here we have used the summation by parts:

$$\sum_j \gamma_j \varphi_j (\Delta_\theta \psi)_j = -\sum_j (\mathrm{D}_+\varphi)_j (\mathrm{D}_+\psi)_j \sin\vartheta_{j+1} = \sum_j \gamma_j (\Delta_\theta \varphi)_j \psi_j.$$

Here and hereafter we use the notation $\sum_j$ for $\sum_{0 \le j < n}$. The area of Ω, say $\mathcal{A}$, is

$$\mathcal{A} = \mathcal{A}[h] = \frac{1}{2}\sum_j h_j d_j = \frac{1}{2}\sum_j \gamma_j(\Delta_\theta h + h)_j h_j.$$

For any $\varphi = (\varphi_0, \varphi_1, \dots, \varphi_{n-1})$ and $\psi = (\psi_0, \psi_1, \dots, \psi_{n-1})$, we define the inner product on $\mathcal{P}$ by $(\varphi, \psi)_2 = \sum_j \varphi_j \psi_j d_j$. For a real valued function $\mathcal{E}[h]$, we defined the first variation of $\mathcal{E}$ with the metric $(\cdot, \cdot)_2$ by

$$\frac{d}{d\varepsilon}\mathcal{E}[h+\varepsilon\varphi]\bigg|_{\varepsilon=0} = \nabla\mathcal{E}[h] \bullet \varphi = \left(\frac{\delta\mathcal{E}[h]}{\delta h}, \varphi\right)_2.$$

The gradient flow of $\mathcal{E}[h]$ is $\dot{h}_j(t) = -(\delta\mathcal{E}[h]/\delta h)_j$. Here and hereafter, we use the notation $\dot{u}(t)$ for $du(t)/dt$. Then the j-th normal velocity of the gradient flow of $\mathcal{E}[h]$ is $v_j = \langle \dot{\boldsymbol{x}}_j, \boldsymbol{n}_j \rangle = -\dot{h}_j = (\delta\mathcal{E}[h]/\delta h)_j$.

The first variation of $\mathcal{L}$ is $(\delta\mathcal{L}[h]/\delta h)_j = \kappa_j$, since

$$\frac{d}{d\varepsilon}\mathcal{L}[h+\varepsilon\varphi]\bigg|_{\varepsilon=0} = \sum_j \gamma_j\varphi_j = \sum_j \kappa_j\varphi_j d_j = (\kappa, \varphi)_2.$$

Here

$$\kappa_j = \frac{\gamma_j}{d_j}.$$

This is a discrete analogue to the first variation of total length of a smooth curve being the curvature. In this sense, κ_j is a kind of curvature on the j-th side of $\mathcal{P}$. We call κ_j the *crystalline curvature* of the j-th side. On other characterizations of κ_j, see, e.g., Rybka [10] and Yazaki [15].

Let $f_j > 0$ be defined on the j-th side of $\mathcal{P}$. Assume $(\Delta_\theta f + f)_j > 0$ for all j. In the physical context, Ω is a crystal and f_j is the interfacial energy density defined on the j-th side of $\mathcal{P} = \partial\Omega$, and $\gamma_j(\Delta_\theta f + f)_j$ is the j-th length of the boundary of the Wulff polygon $\mathcal{W}_f$: $\mathcal{W}_f = \{\boldsymbol{x} \in \boldsymbol{R}^2 \mid \langle \boldsymbol{x}, -\boldsymbol{n}_j \rangle \le f_j,\ \theta_j \in \Theta_n\}$. The total interfacial energy of $\mathcal{P}$, say $\mathcal{F}$, is

$$\mathcal{F} = \mathcal{F}[h] = \sum_j f_j d_j = \sum_j \gamma_j f_j (\Delta_\theta h + h)_j = \sum_j \gamma_j (\Delta_\theta f + f)_j h_j,$$

and the first variation of $\mathcal{F}$ is $(\delta\mathcal{F}[h]/\delta h)_j = (\Delta_\theta f + f)_j \kappa_j$, since

$$\frac{d}{d\varepsilon}\mathcal{F}[h+\varepsilon\varphi]\bigg|_{\varepsilon=0} = \sum_j \gamma_j(\Delta_\theta f + f)_j\varphi_j = \sum_j (\Delta_\theta f + f)_j\kappa_j\varphi_j d_j = ((\Delta_\theta f + f)\kappa, \varphi)_2.$$

Put $w_j = (\Delta_\theta f + f)_j \kappa_j$. We often call f_j *crystalline energy* and also call w_j *crystalline curvature.*

Let φ be the direction of area-preserving:

$$\frac{d}{d\varepsilon}\mathcal{A}[h+\varepsilon\varphi]\bigg|_{\varepsilon=0} = (1, \varphi)_2 = 0.$$

Then we have the first variation of $\mathcal{F}$ in the direction φ:

$$\frac{d}{d\varepsilon}\mathcal{F}[h+\varepsilon\varphi]\bigg|_{\varepsilon=0} = (w - \overline{w}, \varphi)_2.$$

Here $\overline{w} = (1, w)_2/\mathcal{L} = \sum_k \gamma_k f_k/\mathcal{L}$ is the average of w_j. Thus we have the area-preserving gradient flow of $\mathcal{F}[h]$:

$$v_j = w_j - \overline{w} \quad (0 \le j < n). \tag{1.1}$$

From the relation $d_j = \gamma_j(\Delta_\theta h + h)_j$ and $v_j = -\dot{h}_j$, we have

$$\dot{d}_j = -\gamma_j(\Delta_\theta v + v)_j \quad (0 \le j < n). \tag{1.2}$$

The time derivative of the energy $\mathcal{F}$ is

$$\dot{\mathcal{F}} = -\sum_j \gamma_j(\Delta_\theta f + f)_j v_j = \frac{1}{\mathcal{L}}\left((\sum_j \gamma_j f_j)^2 - \sum_j \gamma_j(\Delta_\theta f + f)_j^2 \kappa_j \sum_j d_j\right).$$

From the relation $\sum_j \gamma_j f_j = \sum_j \gamma_j(\Delta_\theta f + f)_j$, by using the Schwarz inequality, we have $\dot{\mathcal{F}} \le 0$. Equality holds if and only if $(\Delta_\theta f + f)_j \kappa_j = const.$, namely $\mathcal{P} = const. \times \partial\mathcal{W}_f$.

Problem of the present paper is the following:

Problem 1 *For a given n-sided polygon $\mathcal{P}_0$ with its normal angle's set being Θ_n, find a family of polygons $\{\mathcal{P}(t)\}_{0\le t<T}$ satisfying* (1.1) *with $\mathcal{P}(0) = \mathcal{P}_0$ for some $T > 0$.*

We note that normal angle's set of $\mathcal{P}(t)$ is Θ_n as long as solution polygons exist.

A main result of the present paper is the following.

Theorem A *A solution polygon $\mathcal{P}(t)$ of Problem* 1 *exists globally in time keeping the area enclosed by the polygon constant $\mathcal{A}$ and, as t tends to infinity, $\mathcal{P}(t)$ converges to the shape of the boundary of the Wulff shape $\partial\mathcal{W}_{f^*}$ in the Hausdorff metric. Here*

$$f_j^* = \frac{f_j}{W}, \quad W = \sqrt{\frac{|\mathcal{W}_f|}{\mathcal{A}}} = \sqrt{\frac{\sum_k \gamma_k(\Delta_\theta f + f)_k f_k}{2\mathcal{A}}}.$$

Moreover, the constant W is asymptotically stable equilibrium point of evolution equations (see section 2*) equivalent to Problem* 1.

In [16], we treated the problem in the case where $f_j \equiv 1$ for all j and obtained the isoperimetric version of Theorem A:

Proposition 1.1 (Yazaki [16]) *Let $f_j \equiv 1$ $(0 \le j < n)$. Then a solution polygon $\mathcal{P}(t)$ of Problem* 1 *exists globally in time keeping the area enclosed by the polygon constant $\mathcal{A}$ and, as t tends to infinity, $\mathcal{P}(t)$ converges to the shape of the boundary of the Wulff shape $\partial\mathcal{W}_{h_*}$ in the Hausdorff metric. Here $h_* = \sqrt{2\mathcal{A}/\sum_k \gamma_k}$.*

2 Proof of Theorem A

2.1 Scenario of the proof

From (1.1), together with (1.2) and the length $\mathcal{L} = \sum_j \gamma_j(\Delta_\theta f + f)_j w_j^{-1}$, we can restate Problem 1 as follows. Throughout this paper, put $a_j = (\Delta_\theta f + f)_j$ for all $0 \le j < n$.

Problem 2 *Find a function $w(t) = (w_j(t))_{0\le j<n} \in (C[0,T) \cap C^1(0,T))^n$ satisfying*

$$\dot{w}(t) = a_j^{-1}w_j^2(\Delta_\theta w + w)_j - \frac{\sum_k \gamma_k a_k}{\sum_i \gamma_i a_i w_i^{-1}} a_j^{-1} w_j^2, \quad 0 \le j < n, \quad 0 < t < T, \tag{2.1}$$

$$w_j(0) = a_j \kappa_j^0, \quad 0 \le j < n, \tag{2.2}$$

$$w_{-1}(t) = w_{n-1}(t), \quad w_n(t) = w_0(t), \quad 0 < t < T, \tag{2.3}$$

where κ_j^0 is the j-th initial crystalline curvature of $\mathcal{P}_0$.

Problem 1 and Problem 2 are equivalent except the indefiniteness of position of a solution polygon. See Yazaki [16, Remark 2.1]. Since Problem 2 is the initial value problem of ordinary differential equations, there exists a unique time local solution. Moreover, by using a similar argument as in Taylor [13, Proposition 3.1], Ishii and Soner [7, Lemma 3.4], Yazaki [16, Lemma 3.1], we obtain the time global solvability.

Lemma 2.1 (time global existence) *A solution w of Problem 2 and a solution polygon of Problem 1 exist globally in time, i.e., a solution polygon does not develop singularities in a finite time.*

In the following, we will show three lemmas, which play an important roll in a scenario of the proof of Theorem A.

Let the anisoperimetric ratio be

$$\mathcal{J} = \frac{\mathcal{F}^2}{4|\mathcal{W}_f|\mathcal{A}}$$

for a polygon $\mathcal{P}$ with the normal angle's set being Θ_n. Here $|\mathcal{W}_f|$ is the area of the Wulff polygon $\mathcal{W}_f$. The first key lemma is the anisotropic version of the isoperimetric inequality.

Lemma 2.2 (anisoperimetric inequality) *For a polygon $\mathcal{P}$ with the normal angle's set being Θ_n, the anisoperimetric inequality*

$$\mathcal{J} \ge 1$$

holds. The equality $\mathcal{J}(t) \equiv 1$ holds if and only if $w_j = (\Delta_\theta f + f)_j \kappa_j \equiv$ const. for all $0 \le j < n$ i.e., Ω (the enclosed region of $\mathcal{P}$) satisfies $\Omega = k\mathcal{W}_f$ for some constant $k > 0$.

We will prove this lemma in the next section by using the mixed area and the Brünn and Minkowski's inequality.

From this lemma and Lemma 2.1, for a solution polygon $\mathcal{P}(t)$,

$$\mathcal{J}(t) = \frac{\mathcal{F}(t)^2}{4|\mathcal{W}_f|\mathcal{A}} \geq 1$$

holds for $t \geq 0$. Moreover, we have the following second key lemma.

Lemma 2.3 $\lim_{t\to\infty} \mathcal{F}(t) = 2\sqrt{|\mathcal{W}_f|\mathcal{A}}$ *and* $\lim_{t\to\infty} \mathcal{J}(t) = 1$ *hold.*

Proof of this lemma closely follows [16, Lemma 5.6].

From this lemma, if a solution polygon $\mathcal{P}(t)$ is n-sided polygon at the time infinity, then $\Omega(t)$ approaches to $k\mathcal{W}_f$ $(k > 0)$ as t tends to infinity. Therefore, for the assertion of Theorem A, it is required that the estimate $\inf_{0<t<\infty} \min_{0\leq j<n} d_j(t) > 0$ holds. As a matter of fact, the following the third key lemma holds. This lemma is a strong assertion compare with the above estimate.

Lemma 2.4 *Let the equilibrium point of* (2.1) *in Problem* 2 *be* W. *Then* $W = \sqrt{|\mathcal{W}_f|/\mathcal{A}}$. *Moreover, The equilibrium point* W *is asymptotically stable and*

$$\lim_{t\to\infty} w_j(t) = W$$

holds for $0 \leq j < n$.

One can prove this lemma by the general theory of dynamical systems or the Lyapunov theorem (see [16, Lemma 5.8]).

Proof of Theorem A. From Lemma 2.4, we have

$$\lim_{t\to\infty} (\Delta_\theta h(t) + h(t))_j = \frac{a_j}{W}, \quad 0 \leq j < n.$$

From this limit and the theory of the generalized eigenvalue space, there exists a vector $\boldsymbol{c}(t) = {}^t(c_1(t), c_2(t)) \in \boldsymbol{R}^2$ such that $h_j(t) - \langle \boldsymbol{c}(t), -\boldsymbol{n}_j\rangle$ converges to $f_j^* = f_j/W$ for all $0 \leq j < n$ as t tends to infinity. Hence for any $\varepsilon > 0$ there exists $t' > 0$ such that $\mathcal{P}(t) \subset \mathcal{W}_{(1+\varepsilon)f^*} \backslash \mathcal{W}_{(1-\varepsilon)f^*}$ holds for $t \geq t'$. Then the assertion holds. ∎

2.2 Proof of Lemma 2.2

The result of Lemma 2.2 follows from a classical convex geometry by using a concept of mixed area and the Brünn and Minkowski's inequality.

Let $\mathcal{Q}_0$ and $\mathcal{Q}_1$ be polygons with the normal angle's set being Θ_n. For $i = 0, 1$, denote the j-th support function and length of the j-th side by $h_j^{(i)}$ and $d_j^{(i)}$, respectively. Note that $d_j^{(i)} = \gamma_j(\Delta_\theta h^{(i)} + h^{(i)})_j$ holds for $0 \leq j < n$. Let the enclosed region of $\mathcal{P}_i$ be $\Omega_i = \{\boldsymbol{x} \in \boldsymbol{R}^2 \mid \langle \boldsymbol{x}, -\boldsymbol{n}_j\rangle \leq h_j^{(i)},\ \theta_j \in \Theta_n\}$ for $i = 0, 1$. Define the linear interpolant of Ω_0 and Ω_1 by $\Omega_s = (1-s)\Omega_0 + s\Omega_1 = \{(1-s)\boldsymbol{x} + s\boldsymbol{y} \,|\, \boldsymbol{x} \in \Omega_0, \boldsymbol{y} \in \Omega_1\}$, and $\mathcal{Q}_s = (1-s)\mathcal{Q}_0 + s\mathcal{Q}_1$ for $0 \le s \le 1$. Then the j-th support function and length of the j-th side of $\mathcal{Q}_s$ are given as $h_j^{(s)} = (1-s)h_j^{(0)} + sh_j^{(1)}$ and $d_j^{(s)} = (1-s)d_j^{(0)} + sd_j^{(1)}$, respectively. The area of Ω_s, say $\mathcal{A}(\Omega_s)$, is

$$\mathcal{A}(\Omega_s) = \frac{1}{2}\sum_j d_j^{(s)} h_j^{(s)} = (1-s)^2\mathcal{A}(\Omega_0) + s^2\mathcal{A}(\Omega_1) + \frac{s(1-s)}{2}\sum_j \left(d_j^{(1)}h_j^{(0)} + d_j^{(0)}h_j^{(1)}\right).$$

By summation by parts:

$$\sum_j d_j^{(1)}h_j^{(0)} = \sum_j \gamma_j(\Delta_\theta h^{(1)} + h^{(1)})_j h_j^{(0)} = \sum_j \gamma_j(\Delta_\theta h^{(0)} + h^{(0)})_j h_j^{(1)} = \sum_j d_j^{(0)}h_j^{(1)}, \quad (2.4)$$

we have

$$\mathcal{A}(\Omega_s) = \frac{1}{2}\sum_j d_j^{(s)} h_j^{(s)} = (1-s)^2\mathcal{A}(\Omega_0) + s^2\mathcal{A}(\Omega_1) + 2s(1-s)\frac{1}{2}\sum_j d_j^{(0)}h_j^{(1)}.$$

The coefficient of $2s(1-s)$ is called *mixed area* of Ω_0 and Ω_1, and denoted by

$$\mathcal{A}(\Omega_0, \Omega_1) = \frac{1}{2}\sum_j d_j^{(0)}h_j^{(1)}.$$

Note that $\mathcal{A}(\Omega_0, \Omega_1) = \mathcal{A}(\Omega_1, \Omega_0)$ holds by (2.4).

H. Brünn and H. Minkowski proved the following inequality:

$$\sqrt{\mathcal{A}(\Omega_s)} \ge (1-s)\sqrt{\mathcal{A}(\Omega_0)} + s\sqrt{\mathcal{A}(\Omega_1)}, \quad 0 \le s \le 1. \quad (2.5)$$

Equality holds if and only if $\Omega_0 = k\Omega_1$ ($k > 0$). The Brünn and Minkowski's inequality (2.5) is equivalent to the following inequality:

$$\mathcal{A}(\Omega_0, \Omega_1) \ge \sqrt{\mathcal{A}(\Omega_0)\mathcal{A}(\Omega_1)}. \quad (2.6)$$

From this inequality, we obtain the anisoperimetric inequality as follows. Let Ω_0 be the enclosed region, say Ω, of a polygon $\mathcal{P}$, and the enclosed area $\mathcal{A}(\Omega) = \mathcal{A}$. For the interfacial energy $f_j > 0$ with $(\Delta_\theta f + f)_j > 0$, let Ω_1 be the Wulff region $\mathcal{W}_f$. Then the area of $\mathcal{W}_f$ is $\mathcal{A}(\mathcal{W}_f) = |\mathcal{W}_f| = \sum_k \gamma_k(\Delta_\theta f + f)_k f_k/2$. Then the mixed area is $\mathcal{A}(\Omega, \mathcal{W}_f) = \sum_j f_j d_j/2 = \mathcal{F}/2$, which is a half of the total interfacial energy on $\mathcal{P}$. Hence by (2.6), $\mathcal{F}/2 \ge \sqrt{\mathcal{A}|\mathcal{W}_f|}$, namely,

$$\mathcal{J} = \frac{\mathcal{F}^2}{4|\mathcal{W}_f|\mathcal{A}} \ge 1. \quad (2.7)$$

The equality $\mathcal{J}(t) \equiv 1$ holds if and only if $\Omega = k\mathcal{W}_f$ for some constant $k > 0$, i.e., $(\Delta_\theta f + f)_j \kappa_j \equiv$ *const.* for all $0 \le j < n$.

In particular, if $f_j \equiv 1$, then $\mathcal{F} = \mathcal{L}$ and $|\mathcal{W}_f| = \sum_j \gamma_j/2$, so we have

$$\mathcal{I} = \frac{\mathcal{L}^2}{2\sum_j \gamma_j \mathcal{A}} \ge 1, \quad (2.8)$$

which is the isoperimetric inequality of polygons. The equality $\mathcal{I}(t) \equiv 1$ holds if and only if $\kappa_j \equiv const.$ for all $0 \le j < n$. See, e.g., Yazaki [16] for another proof by using the crystalline motion.

The isoperimetric inequality (2.8) represents the variational problem: what is the shape which has the least total length of a polygon for the fixed enclosed area? The answer (the case where $\mathcal{I} = 1$) is the boundary of the Wulff shape $\partial\mathcal{W}_{h_*}$, which is the circumscribed polygon of the circle with radius $h_* = \sqrt{2\mathcal{A}/\sum_i \gamma_i}$ (cf. Proposition 1.1). Similarly, the anisoperimetric inequality (2.7) represents the variational problem: what is the shape which has the least total interfacial energy of a polygon for the fixed enclosed area? The answer (the case where $\mathcal{J} = 1$) is the boundary of the Wulff shape $\partial\mathcal{W}_{f^*}$ (cf. Theorem A).

Part II

On motion of nonadmissible polygons by crystalline curvature

3 Introduction and a main result

We consider an evolution equation of a closed convex polygon $\mathcal{P}(t)$ in the plane $\boldsymbol{R}^2$:

$$v_j = g\left(\theta_j, \frac{l_f(\theta_j)}{d_j}\right) \tag{3.1}$$

at time t with the normal angle of the j-th side being θ_j. Here v_j denotes the normal velocity of the j-th side of $\mathcal{P}(t)$ in the direction of the inward unit normal $\boldsymbol{n}_j = -{}^t(\cos\theta_j, \sin\theta_j)$, and d_j is the length of the j-th side. On the j-th side of $\mathcal{P}(t)$, the interfacial energy (density) $f_j > 0$ is defined, and f_j is specified by the Wulff shape:

$$\mathcal{W}_f = \left\{ (x, y) \in \boldsymbol{R}^2 \,\middle|\, x\cos\theta_j + y\sin\theta_j \le f_j \quad \text{for all } \ \theta_j \in \Theta \right\}$$

with Θ, a normal angle's set of $\partial\mathcal{W}_f$:

$$\Theta = \{\theta_0 < \theta_1 < \cdots < \theta_{n-1} < \theta_0 + 2\pi\}.$$

The boundary of the Wulff shape $\partial\mathcal{W}_f$ is n-sided polygon. In (3.1) $l_f(\theta_j)$ is the (positive) length of the j-th side of $\partial\mathcal{W}_f$ if $\theta_j \in \Theta$ and $l_f(\theta_j) = 0$ if $\theta_j \notin \Theta$. The function $g(\theta_j, \lambda)$ is a given positive function for $\lambda > 0$. We assume that $g(\theta_j, \lambda)$ is monotone nondecreasing in λ, and $\lim_{\lambda\to\infty} g(\theta_j, \lambda) = \infty$ and $g(\theta_j, 0) \equiv 0$ hold for all θ_j. Under these assumptions, if the j-th normal angle θ_j of $\mathcal{P}(t)$ belongs to Θ, then $v_j > 0$, and if $\theta_j \notin \Theta$, then $v_j = 0$.

A polygon is called *admissible polygon* if its normal angle's set equals Θ. Note that the original concept of the admissibility is defined for piecewise linear curves (not necessarily convex), see, e.g., Gurtin [6]. In the physical context, enclosed region of the polygon $\mathcal{P}$ is crystal, and the interfacial energy f_j is called *crystalline energy* and $l_f(\theta_j)/d_j$ is called *crystalline curvature.* Motion of admissible polygons by the evolution equation (3.1) is called *crystalline motion* or motion by crystalline curvature. See the pioneer works by Angenent and Gurtin [2], Taylor [12, 13] and Taylor, Cahn and Handwerker [14] for the background story of this motion.

Let Θ_0 be a normal angle's set of initial polygon $\mathcal{P}(0) = \mathcal{P}_0$. Our aim in the present paper is to show the behavior of a solution polygon in the case where $\Theta_0 \supset \Theta$, i.e., $\mathcal{P}_0$ is not admissible polygon. A main result is as follows.

Theorem B Let Θ_t be a normal angle's set of a solution polygon $\mathcal{P}(t)$ of (3.1) with initial polygon $\mathcal{P}(0) = \mathcal{P}_0$. Assume that $\lambda \mapsto g(\theta_j, \lambda)$ $(\theta_j \in \Theta_0)$ is locally Lipschitz continuous on $\boldsymbol{R}_+$. If $\Theta_0 \supset \Theta$, then there exists a finite time sequence $0 = t_0 < t_1 < t_2 < \cdots < t_m$ such that

$$\Theta_0 \supset \Theta_{t_1} \supset \Theta_{t_2} \supset \cdots \supset \Theta_{t_m} = \Theta$$

holds. On each interval $t \in [t_k, t_{k+1})$, there exists a unique solution polygon $\mathcal{P}(t)$ with initial polygon $\mathcal{P}(t_k)$ for $k = 0, 1, \ldots, m-1$.

The result says that no edges of a solution polygon $\mathcal{P}(t)$ disappear for $t \in [t_k, t_{k+1})$ starting with nonadmissible polygon $\mathcal{P}(t_k)$, some edge, say the j-th side, disappears as $t \to t_{k+1}$ if $\theta_j \in \Theta_{t_k} \backslash \Theta$, and eventually a solution polygon becomes an admissible polygon at finite time t_m.

After the time t_m, a solution polygon $\mathcal{P}(t)$ with initial admissible polygon $\mathcal{P}(t_m)$ evolves, while its admissibility is preserved, and finally it shrinks to a single point or collapses to a lines segment with positive length in a finite time, say $T > t_m$, depending on the growth condition of $g(\theta_j, \lambda)$ with respect to λ. No edges of $\mathcal{P}(t)$ disappear for $t \in [t_m, T)$. This result was proved by M.-H. Giga and Y. Giga [5]. The case where a solution polygon collapses to a line segment is called degenerate pinching. Andrews [1] showed a condition of initial polygon $\mathcal{P}_0$ in degenerate pinching case. Moreover, Ishiwata and Yazaki [8, 9] showed that, in the case where $g(\theta_j, \lambda) = a_j \lambda^\alpha$ $(a_j > 0, 0 < \alpha < 1)$, the blow-up order of v is $(T-t)^{-\alpha}$ in degenerate pinching phenomenon under a monotonicity assumption.

In the next section 2, we will present two examples of this motion. The main theorem will be proved in the last section 3.

4 Examples

We present two examples. Throughout this paper we use the notation $\dot{u}(t)$ for $du(t)/dt$.

Example 4.1 Let us consider the case where $g(\theta_j, \lambda) = \lambda^\alpha$ $(\alpha > 0)$,

$$\Theta = \left\{ \theta_j = \frac{\pi}{2} j \,\middle|\, j = 0, 1, 2, 3 \right\},$$

and $f_j \equiv 1/2$, i.e., $\partial \mathcal{W}_f$ is a square with the length of side being $l_f(\theta_j) \equiv 1$. See Figure 1 (far left).

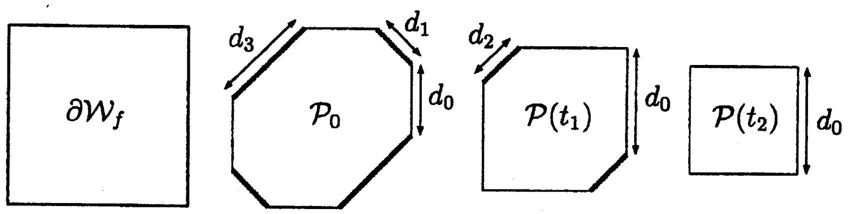

Figure 1: Isotropic motion of nonadmissible polygon by $g(\theta_j, \lambda) = \lambda^\alpha$ $(\alpha > 0)$

Initial data and stage 1. Let $\mathcal{P}_0$ be symmetric octagon with

$$\Theta_0 = \left\{ \theta_j = \frac{\pi}{4} j \,\middle|\, j = 0, 1, \ldots, 7 \right\},$$

and $d_0(0) = d_i(0)$ $(i = 2, 4, 6)$, $d_1(0) = d_5(0)$ and $d_3(0) = d_7(0)$. See Figure 1 (second from left). Assume $d_3(0) > d_1(0)$. Evolution equations are $\dot{d}_0 = 2v_0$, $\dot{d}_1 = \dot{d}_3 = -2\sqrt{2}v_0$, and $v_0 = d_0^{-\alpha}, v_1 = v_3 = 0$. Then we have exact solutions

$$d_0(t) = \left(d_0(0)^{\alpha+1} + 2(\alpha+1)t \right)^{1/(\alpha+1)}, \quad d_i(t) = d_i(0) + \sqrt{2}(d_0(0) - d_0(t))$$

for $i = 1, 3$ and for $0 \le t < t_1$. Here

$$t_1 = \frac{\left(\dfrac{d_1(0)}{\sqrt{2}} \right)^{\alpha+1} - d_0(0)^{\alpha+1}}{2(\alpha+1)}$$

Therefore it holds that $d_3(t) > d_1(t)$ for $0 \le t < t_1$, $\lim_{t \to t_1} d_1(t) = 0$, $\inf_{0<t<t_1} d_i(t) > 0$ $(i = 0, 3)$ and $\Theta_t \equiv \Theta_0$ for $0 \le t < t_1$.

Stage 2. Initial polygon $\mathcal{P}(t_1)$ is symmetric hexagon with

$$\Theta_{t_1} = \left\{ \theta_0 = 0 < \frac{\pi}{2} < \frac{3\pi}{4} < \pi < \frac{3\pi}{2} < \frac{7\pi}{4} \right\},$$

and $d_0(t_1) = d_i(t_1)$ $(i = 1, 3, 4)$, $d_2(t_1) = d_5(t_1)$. See Figure 1 (third from left). Evolution equations are $\dot{d}_0 = 0$, $\dot{d}_2 = -2\sqrt{2}v_0$, and $v_0 = d_0^{-\alpha}, v_2 = 0$. Then we have exact solutions

$$d_0(t) \equiv d_0(t_1), \quad d_2(t) = \frac{2\sqrt{2}}{d_0(t_1)^\alpha}(t_2 - t) \quad (t_1 \le t < t_2).$$

Here

$$t_2 = \frac{d_0(t_1)^\alpha d_2(t_1)}{2\sqrt{2}} + t_1.$$

Hence it holds that $\lim_{t \to t_2} d_2(t) = 0$, $\inf_{t_1 < t < t_2} d_0(t) > 0$ and $\Theta_t \equiv \Theta_{t_1}$ for $t_1 \le t < t_2$.
Final stage. Initial polygon $\mathcal{P}(t_2)$ is admissible square with $\Theta_{t_2} = \Theta$ and $d_0(t_2) = d_i(t_2)$ $(i = 1, 2, 3)$. See Figure 1 (far right). Evolution equation is $\dot{d}_0 = -2v_0$, and $v_0 = d_0^{-\alpha}$. Then we have an exact solution

$$d_0(t) = (2(\alpha+1)(T-t))^{1/(\alpha+1)} \quad (t_2 \le t < T).$$

Here

$$T = \frac{d_0(t_2)^{\alpha+1}}{2(\alpha+1)} + t_2.$$

A solution polygon shrinks to a single point as $t \to T$ and $\Theta_t \equiv \Theta$ holds for $t_2 \le t < T$.

Example 4.2 Let us consider the case where $g(\theta_j, \lambda) = a_j \lambda$ $(a_j > 0)$,

$$\Theta = \left\{ \theta_j = \frac{\pi}{2} j \,\middle|\, j = 0, 1, 2, 3 \right\},$$

and $f_j \equiv 1/2$, i.e., $\partial \mathcal{W}_f$ is a square with the length of side being $l_f(\theta_j) \equiv 1$. See Figure 2 (left).

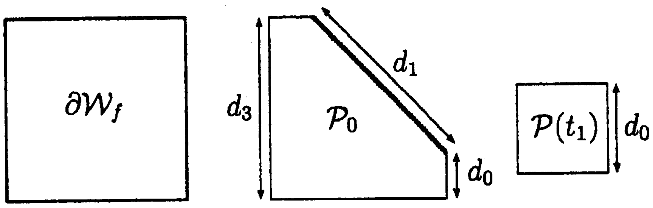

Figure 2: Anisotropic motion of nonadmissible polygon by $g(\theta_j, \lambda) = a_j \lambda$ $(a_j > 0)$

Initial data and stage 1. Let $\mathcal{P}_0$ be symmetric pentagon with

$$\Theta_0 = \left\{ \theta_0 = 0 < \frac{\pi}{4} < \frac{\pi}{2} < \pi < \frac{3\pi}{2} \right\},$$

and $d_0(0) = d_2(0)$, $d_3(0) = d_4(0)$. See Figure 2 (middle). Assume $a_0 = a_2$ and $a_3 = a_4$. Evolution equations are $\dot{d}_0 = v_0 - v_3$, $\dot{d}_1 = -2\sqrt{2} v_0$ and $\dot{d}_3 = -v_0 - v_3$, and $v_0 = a_0/d_0, v_1 = 0, v_3 = a_3/d_3$. Put $C(t) = d_3(t)^2 + 2d_0(t)^2 d_3(t)^2 - d_0(t)^2$ and $C_1 = 4(a_0 + a_3) > 0$. Then $\dot{C}(t) = -C_1$ holds and we have exact solutions

$$d_1(t) = \sqrt{2}(d_3(t) - d_0(t)), \quad d_3(t) = -d_0(t) + \sqrt{2d_0(t)^2 + C(t)}.$$

Here $C(t) = C(0) - C_1 t$ and $C(0) > 2d_0(0)^2 > 0$. Hence there exists $t_1 > 0$ satisfying

$$t_1 = \frac{C(0) - C(t_1)}{C_1}, \quad C(t_1) = 2d_0(t_1)^2,$$

and it holds that $\lim_{t\to t_1} d_1(t) = 0$, $\lim_{t\to t_1} d_0(t) = \lim_{t\to t_1} d_3(t) > 0$ and $\Theta_t \equiv \Theta_0$ for $0 \le t < t_1$.

Final stage is the same situation as in Example 4.1: initial polygon $\mathcal{P}(t_1)$ is admissible square with $\Theta_{t_1} = \Theta$ and $d_0(t_1) = d_i(t_1)$ $(i = 1, 2, 3)$. See Figure 2 (right).

5 Proof of Theorem B

A simple calculation shows that $d_j(t)$'s satisfy a system of ordinary differential equations:

$$\dot{d}_j(t) = (\cot\vartheta_{j+1} + \cot\vartheta_j)\, v_j - \frac{v_{j+1}}{\sin\vartheta_{j+1}} - \frac{v_{j-1}}{\sin\vartheta_j}. \tag{5.1}$$

Here $\vartheta_j = \theta_j - \theta_{j-1}$ and $\theta_j \in \Theta_t$.

Combining (3.1) and (5.1), we obtain the local existence theorem from a general theory.

Lemma 5.1 Assume that $\lambda \mapsto g(\theta_j, \lambda)$ $(\theta_j \in \Theta_0)$ is locally Lipschitz continuous on $\boldsymbol{R}_+$. Then there is a constant $t_* > 0$ and unique solution polygon $\mathcal{P}(t)$ of (3.1) with initial polygon $\mathcal{P}_0$ and a normal angle's set $\Theta_t \equiv \Theta_0$ for $t \in [0, t_*)$.

We will see that some sides disappear in a finite time. Let $\mathcal{L}(t)$ be a total length of $\mathcal{P}(t)$:

$$\mathcal{L}(t) = \sum_{\theta_j \in \Theta_0} d_j(t) \quad (t \in [0, t_*)).$$

From (5.1), we have

$$\dot{\mathcal{L}}(t) = -\sum_{\theta_j \in \Theta_0} \gamma_j v_j = -\sum_{\theta_j \in \Theta} \gamma_j v_j \quad (t \in [0, t_*)),$$

since $v_j = 0$ for $\theta_j \in \Theta_0 \backslash \Theta$. Here

$$\gamma_j = \tan\frac{\vartheta_{j+1}}{2} + \tan\frac{\vartheta_j}{2}.$$

Then $\mathcal{L}(t) \le \mathcal{L}(0)$ holds. By geometry, $d_j(t) \le \mathcal{L}(t)$ holds for all j. Since $g(\theta_j, \lambda)$ is monotone nondecreasing in λ, if $\theta_j \in \Theta$, then g is bounded from blow by a positive constant, say C_0:

$$g\left(\theta_j, \frac{l_f(\theta_j)}{d_j}\right) \ge \min_{\theta_k \in \Theta} g\left(\theta_k, \frac{l_f(\theta_k)}{d_k}\right) \ge \min_{\theta_k \in \Theta} g\left(\theta_k, \frac{l_f(\theta_k)}{\mathcal{L}(0)}\right) = C_0 > 0 \quad (\theta_j \in \Theta).$$

Therefore we have a positive constant C_1 satisfying

$$\dot{\mathcal{L}}(t) \le -nC_0 \min_{\theta_j \in \Theta} \gamma_j = -C_1 < 0 \quad (t \in [0, t_*)),$$

and then it holds that

$$\min_{\theta_j \in \Theta_0} d_j(t) \le \mathcal{L}(t) \le \mathcal{L}(0) - C_1 t \quad (t \in [0, t_*)).$$

Hence we have the following lemma.

Lemma 5.2 Let t_* be the same as in Lemma 5.1. There exist $\theta_k \in \Theta_0$ and $t_1 \ge t_*$ such that

$$\lim_{t \to t_1} d_k(t) = 0 \quad \text{and} \quad d_j(t) > 0 \quad (\theta_j \in \Theta_0, t \in [0, t_1)).$$

Note that the limit $\lim_{t\to t_1} d_k(t) = 0$ follows from a weaker condition $\liminf_{t\to t_1} d_k(t) = 0$, see, e.g., Ishii and Soner [7] and Yazaki [16].

Theorem B follows from the following lemma.

Lemma 5.3 Let t_1 be the same as in Lemma 5.2. Put

$$\mathcal{J} = \left\{ \theta_j \in \Theta_0 \,\middle|\, \lim_{t \to t_1} d_j(t) = 0 \right\}.$$

Then $\mathcal{J} \subset \Theta_0 \backslash \Theta$ holds.

Proof. If $\mathcal{J} \subset \Theta_0 \backslash \Theta$ does not hold, then $\mathcal{J} \cap \Theta \ne \phi$ holds. One can divide into $\mathcal{J} = \bigoplus_k \mathcal{J}_k$, where $\mathcal{J}_k$'s are maximal subsets having m_k elements of the form

$$\mathcal{J}_k = \{\theta_j \in \mathcal{J} \mid j = j_k, j_{k+1}, \dots, j_k + m_k - 1\},$$

with the boundary of $\mathcal{J}_k$:

$$\partial \mathcal{J}_k = \{\theta_j \mid j = j_k - 1, j_k + m_k\}.$$

Note that $m_k \ge 1$ and $\partial \mathcal{J}_k \subset \Theta_0 \backslash \mathcal{J}$, i.e., $\inf_{0<t<t_1} d_j(t) > 0$ holds for $\theta_j \in \partial \mathcal{J}_k$ and for all k.

Let $p = j_k - 1$ and $q = j_k + m_k$ for simplicity. By geometry, we have

$$|\theta_q - \theta_p| \le \pi.$$

The following argument closely follows Taylor [13, PROPOSITION 3.1], Ishii and Soner [7, LEMMA 3.4] and Yazaki [16, Lemma 3.1].

Let $L_j(t)$ be the line extending the j-th side of $\mathcal{P}(t)$ for all $\theta_j \in \Theta_0$ and let $\boldsymbol{B}_j(t)$ be the intersection point of $L_j(t)$ and $L_{j-1}(t)$, that is $\boldsymbol{B}_j(t)$ the j-th vertex of $\mathcal{P}(t)$. By the definition of $\mathcal{J}_k$, vertices $\boldsymbol{B}_{p+1}, \dots, \boldsymbol{B}_q$ converge to the same point $\boldsymbol{B}_*$:

$$\boldsymbol{B}_* \in \bigcap_{0 \le t < t_1} \bigcap_{p \le j \le q} \left\{ \boldsymbol{x} \in \boldsymbol{R}^2 \,\middle|\, \langle \boldsymbol{x} - \boldsymbol{B}_j(t), \boldsymbol{n}_j \rangle \ge 0 \right\}.$$

Here $\langle\cdot,\cdot\rangle$ is the usual Euclidean inner product. Note that intersection takes over $p \leq j \leq q$ since the sign of v_j is nonnegative for all $p \leq j \leq q$.

We assume $\mathcal{J}_k \cap \Theta \neq \phi$, without loss of generality, since $\mathcal{J} \cap \Theta \neq \phi$. Then there exists $\theta_r \in \mathcal{J}_k \cap \Theta$. Note that $p < r < q$ holds, and $\inf_{0<t<t_1} v_r(t) > 0$ and $\lim_{t\to t_1} v_r(t) = \lim_{t\to t_1} g(\theta_r, l_f(\theta_r)/d_r(t)) = \infty$ hold.

Let $\boldsymbol{y}(t)$ be the intersection point of $L_p(t)$ and $L_q(t)$. We define

$$a(t) = \langle \boldsymbol{B}_* - \boldsymbol{y}(t), \boldsymbol{n}_r\rangle, \quad b(t) = \langle \boldsymbol{B}_* - \boldsymbol{B}_r(t), \boldsymbol{n}_r\rangle.$$

Then $a(t) \geq b(t)$ holds for $t \in [0, t_1)$ and $\lim_{t\to t_1} a(t) = \lim_{t\to t_1} b(t) = 0$ holds.

If $\theta_p, \theta_q \in \Theta_0\backslash\Theta$, then $\dot{\boldsymbol{y}} = 0$, which contradicts to convergence of $\boldsymbol{B}_j$'s to $\boldsymbol{B}_*$ for $p < j \leq q$. If either $\theta_p \in \Theta$ or $\theta_q \in \Theta$ hold, then there exists a positive constant, say C_*, such that $\sup_{0<t<t_1} |\dot{\boldsymbol{y}}(t)| \leq C_*$ holds, since $\theta_p, \theta_q \notin \mathcal{J}$.

Therefore by $\dot{a} = -\langle \dot{\boldsymbol{y}}, \boldsymbol{n}_r\rangle$ and $\dot{b} = -v_r$, there exists $\eta \in (t, t_1)$ such that

$$\int_t^{t_1} v_r(\tau)\, d\tau = -\int_t^{t_1} \dot{b}(\tau)\, d\tau = b(t) \leq a(t) = -\dot{a}(\eta)(t_1 - t) \leq C_*(t_1 - t).$$

This contradicts the fact $v_r \to \infty$ as $t \to t_1$. Hence $\mathcal{J}_k \cap \Theta = \phi$ for all k, i.e., $\mathcal{J} \subset \Theta_0\backslash\Theta$ holds. ∎

Proof of Theorem B. If $\Theta_0\backslash\Theta \neq \phi$, then by Lemma 5.1, 5.2 and 5.3, there exist $t_1 > 0$ and $\theta_k \in \Theta_0\backslash\Theta$ such that

$$\lim_{t\to t_1} d_k(t) = 0, \quad \inf_{0<t<t_1} d_j(t) > 0 \quad (\theta_j \in \Theta) \quad \text{and} \quad d_j(t) > 0 \quad (\theta_j \in \Theta_0, t \in [0, t_1))$$

hold. Therefore $\Theta_0 \supset \Theta_{t_1} \supset \Theta$ holds. If $\Theta_{t_1}\backslash\Theta \neq \phi$, then we can repeat the same argument as above and obtain $t_2 > t_1$ (if not, $\Theta_0 \supset \Theta_{t_1} = \Theta$ holds). Since the number of edges is finite, edge disappearing occurs at most finitely many epochs $0 < t_1 < t_2 < \cdots < t_m$ and eventually $\Theta_{t_m} = \Theta$ holds. ∎

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