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FitzHugh–Nagumo 方程式に現れる微細パターンについて

東京大学大学院数理科学研究科　大下 承民 (Yoshihito Oshita)
Graduate School of Mathematical Sciences,
University of Tokyo

1 Introduction

FitzHugh–Nagumo equation was introduced as a reduced equation of Hodgkin–Huxley model, which describes propagation of signals along a nerve axon. It has turned out to be related to the theory of the pattern formation in mathematical biology and wave propagation in excitable media. Refer to [2, 3, 5, 6, 7, 8]. FitzHugh–Nagumo equation is a system of reaction–diffusion equation consisting of two unknown functions u and v representing concentrations of activator and inhibitor respectively, and typically of the form

$$(E\text{-}1)_\varepsilon \qquad \begin{aligned} u_t &= \varepsilon^2\Delta u + f(u) - \kappa v, \\ \tau v_t &= D\Delta v + u - m - \gamma v, \end{aligned} \qquad \text{in } \Omega \times \mathbb{R}_+$$

with the homogeneous Neumann boundary condition on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain; $f(u) = -W'(u)$ ($W \in C^2(\mathbb{R})$ is a double–well potential which has global minima exactly at ± 1, and $W(\pm 1) = 0$) is a bistable nonlinearity; $m \in (-1, +1)$ is a constant; κ, τ, D and γ are positive constants and ε is a positive parameter. Throughout this survey we always impose the homogeneous Neumann boundary condition. We study the parameter scaling $\varepsilon \to 0$ in $(E\text{-}1)_\varepsilon$. We also study the following scaling.

$$(E\text{-}2)_\varepsilon \qquad \begin{aligned} u_t &= \varepsilon^2\Delta u + f(u) - \tfrac{\varepsilon}{\mu} v, \\ \tau v_t &= D\Delta v + u - m - \gamma v, \end{aligned} \qquad \text{in } \Omega \times \mathbb{R}_+$$

where μ, τ, D and γ are positive constants and $\varepsilon (\to 0)$ is a positive parameter. In addition, we study another scaling, that is,

$$(E\text{-}3)_{\varepsilon,D} \qquad \begin{aligned} u_t &= \varepsilon^2\Delta u + f(u) - \tfrac{\varepsilon}{\mu} v, \\ \tau v_t &= D\Delta v + u - m - \gamma v, \end{aligned} \qquad \text{in } \Omega \times \mathbb{R}_+$$

where μ, τ and γ are positive constants and $\varepsilon(\to 0)$ and $D(\to \infty)$ are positive parameters. Stationary solutions of $(E\text{-}1)_\varepsilon$ are functions u, v which satisfy the following system of elliptic equations

$$(1) \qquad \begin{array}{l} \varepsilon^2 \Delta u + f(u) - \kappa v = 0, \\ D\Delta v + u - m - \gamma v = 0, \end{array} \qquad \text{in } \Omega.$$

Similarly the stationary solutions of $(E\text{-}2)_\varepsilon$ and $(E\text{-}3)_{\varepsilon,D}$ solve

$$(2) \qquad \begin{array}{l} \varepsilon^2 \Delta u + f(u) - \frac{\varepsilon}{\mu} v = 0, \\ D\Delta v + u - m - \gamma v = 0, \end{array} \qquad \text{in } \Omega.$$

Note that these equations are independent of the constant τ. It is easy to see that if u, v solves (1), then u is a critical point of the functional I_ε defined by

$$I_\varepsilon[u] = \int_\Omega \frac{\varepsilon^2}{2}|\nabla u|^2 + W(u) + \frac{D\kappa}{2}|\nabla(T(u-m))|^2 + \frac{\kappa\gamma}{2}\{T(u-m)\}^2\, dx,$$

$$u \in H^1(\Omega),$$

where $T = (-D\Delta + \gamma)^{-1}$ is the Green operator of $-D\Delta + \gamma$ with the homogeneous Neumann boundary condition. We remark that if $\tau = 0$ were satisfied, the activator of $(E\text{-}1)_\varepsilon$, $u(\cdot, t)$ would be the gradient flow of I_ε. However since $\tau > 0$, the activator $u(\cdot, t)$ of $(E\text{-}1)_\varepsilon$ is different from a gradient flow of I_ε. In case of $(E\text{-}2)_\varepsilon$ and $(E\text{-}3)_{\varepsilon,D}$, we deal with the functionals J_ε and $J_{\varepsilon,D}$ respectively defined as follows:

$$J_{\varepsilon(,D)}[u] = \int_\Omega \frac{\varepsilon^2}{2}|\nabla u|^2 + W(u) + \frac{D\varepsilon}{2\mu}|\nabla(T(u-m))|^2 + \frac{\varepsilon\gamma}{2\mu}\{T(u-m)\}^2\, dx.$$

(Note that the operator T depends on D.) It is easy to see that the family of the functionals I_ε and $J_{\varepsilon(,D)}$ admit a global minimizer for each parameter. We are concerned with the asymptotic behavior of such minimizers for each parameter-scalings stated above. (For the stability, refer to [13].)

The homogenization problems with two length scales have been studied recently (refer to [1, 4, 9]). Also refer to [10, 12, 15] for the problem related to diblock copolymer.

We assume that f has polynomial growth at infinity and has three zeros: $-1, a, 1$ ($a \in (-1, 1)$) with $f'(\pm 1) < 0, f'(a) > 0$.

2 Statement of Main Results

To state the first result, we use the notion of Young measure, a useful tool for studying a sequence of functions which is oscillating and not convergent. We use the Young measure which is a map from Ω to the set of all probability measures on $\mathbb{R}$. A usual function $u(x)$ corresponds to the family of Dirac measures $\delta_{u(x)}$. The fundamental theorem for Young measure states the sufficient condition for relative compactness of a sequence of Young measures in an appropriate topology. We can get the limit Young measure instead of the limit function. (Refer to [14].)

In order to state the main result, define the constant

$$c_o = \frac{\sqrt{2}}{\int_{-1}^{1} \sqrt{W(s)}\, ds}$$

and the set of all admissible functions in the limiting problem which we will obtain later,

$$\mathcal{G}(\Omega) = \{u \in BV(\Omega)\,;\, |u(x)| = 1 \quad \text{for almost all } x \in \Omega\},$$

$$\mathcal{M}(\Omega) = \{u \in \mathcal{G}\,;\, \langle u \rangle_\Omega = m\}.$$

Here $\langle\cdot\rangle_\Omega$ denotes the average on Ω. We use the following notation: $P_\Omega(G)$ denotes the perimeter of $G \subset \Omega$ with respect to Ω.

Theorem 2.1. *The following statements hold:*

(i) *For any $\varepsilon > 0$, there exists a stable stationary solution $(u_\varepsilon, v_\varepsilon)$ of $(E\text{-}1)_\varepsilon$ such that for any sequence $\varepsilon_n \to 0$, u_{ε_n} is not convergent in $L^1(\Omega)$ and generates Young measure $\nu = (\nu_x)_{x\in\Omega}$ with $\nu_x = \frac{1-m}{2}\delta_{-1} + \frac{1+m}{2}\delta_1$ for almost all $x \in \Omega$.*

(ii) *For any sequence $\varepsilon_n \to 0$, there exists a subsequence $\varepsilon_k = \varepsilon_{n_k}$ and stable stationary solutions (u_k, v_k) of $(E\text{-}2)_{\varepsilon_k}$ such that u_k converges strongly in $L^1(\Omega)$ to a solution of*

$$(P)^\mu \qquad \min_{u\in\mathcal{G}} B^\mu(u), \quad B^\mu(u) = \frac{2}{c_0} P_\Omega(\{u=1\}) + \frac{1}{2\mu}\int_\Omega (u-m)T(u-m)\, dx.$$

(iii) *For any sequence $\varepsilon_n \to 0, D_n \to \infty$, there exist subsequences $\varepsilon_k = \varepsilon_{n_k}, D_k = D_{n_k}$ such that for each k, $(E\text{-}3)_{\varepsilon_k, D_k}$ has a stable stationary solution (u_k, v_k) which has the* *property that u_k converges strongly in $L^1(\Omega)$ to a solution of*

$$(\widetilde{P})^\mu \qquad \min_{u\in\mathcal{G}} \widetilde{B}(u), \quad \widetilde{B}(u) = \frac{2}{c_0} P_\Omega(\{u=1\}) + \frac{1}{2\mu\gamma}|\Omega|(\langle u\rangle - m)^2.$$

Note that the solutions in Theorem 2.1 (i) do not have a limit. In fact, from the result of [11], for $(E\text{-}1)_\varepsilon$, any stationary solutions which has a smooth surface as a limit must be unstable. In Theorem 2.1, we obtained the two limiting problems, $(P)^\mu$ and $(\widetilde{P})^\mu$, which are the geometric minimization problem with a parameter dependence, and determine the location of interior boundary layers. The next theorem concerns the asymptotic behavior of solutions of the two problems $(P)^\mu$ and $(\widetilde{P})^\mu$ as $\mu \to 0$.

Theorem 2.2. *The following statements hold:*

(i) *Let u^μ be a solution of $(P)^\mu$. Then for any sequence $\mu_k \to 0$, u^μ generates the same Young measure ν as in Theorem 2.1* (i).

(ii) *Let $\widetilde{u}^\mu$ be a solution of $(\widetilde{P})^\mu$. Then for any sequence $\mu_n \to 0$, there exists a subsequence $\mu_k = \mu_{n_k}$ such that $\widetilde{u}^{\mu_k}$ converges strongly in $L^1(\Omega)$ to a solution u^* of*

$$\min_{u\in\mathcal{M}} P_\Omega(\{u=1\}),$$

and generates the Young measure $\nu = (\nu_x)_{x\in\Omega}$ with $\nu_x = \delta_{u^*(x)}$ for almost all $x \in \Omega$.

Note that for the problem $(P)^\mu$, we obtained a similar result as Theorem 2.1 (i), which corresponds to the case $\varepsilon = \mu\kappa$. We see that we can construct a sequence of solutions for $(E\text{-}2)_\varepsilon$ which converges to a pattern with an arbitrary large perimeter if we choose sufficiently small μ.

In the next Theorem, we derive the geometric interface equation associated with the solutions of $(P)^\mu$ and $(\widetilde{P})^\mu$. We use the following notations: We take the sign of mean curvature such that principal curvature of the sphere is negative when the normal vector points to the center. ∂' denotes the relative boundary with respect to Ω.

Theorem 2.3. *The following statements hold:*

(i) *For fixed $\mu > 0$, let u be a solution of $(P)^\mu$ and $\Gamma = \partial'\{u=1\}$. Assume that Γ is smooth in a neighborhood U of a point $x_o \in \Gamma$. Then there holds*

$$\mu H = c_o T(u - m), \qquad \text{on } \Gamma \cap U,$$

where H denotes the mean curvature of Γ (when the normal vector points from $\{u = -1\}$ to $\{u=1\}$).

(ii) *For fixed $\mu > 0$, let $\widetilde{u}$ be a solution of $(\widetilde{P})^{\mu}$ and $\widetilde{\Gamma} = \partial'\{\widetilde{u} = 1\}$. Assume that $\widetilde{\Gamma}$ is smooth in a neighborhood $\widetilde{U}$ of a point $\widetilde{x}_o \in \widetilde{\Gamma}$. Then there holds*

$$\mu H = \frac{c_o}{\gamma}(\langle\widetilde{u}\rangle - m), \quad \text{on } \widetilde{\Gamma} \cap \widetilde{U},$$

where H denotes the mean curvature of $\widetilde{\Gamma}$ (when the normal vector points from $\{\widetilde{u} = -1\}$ to $\{\widetilde{u} = 1\}$).

Theorem 2.3 (ii) implies that solutions of $(\widetilde{P})^{\mu}$ typically involve a partition of Ω into regions separated by surfaces of a constant mean curvature. In [3], they obtained a limiting free boundary problem from an Allen–Cahn equation with a nonlocal term, which arises as a limit of a reaction–diffusion system. Then we see that any surface which corresponds to stationary solutions of the motion law obtained in [3] has also a constant mean curvature.

3 Remarks on Two Dimensional Problems

$u \in \mathcal{G}(\Omega)$ is called planar if $u = u(x_1, \ldots, x_N)$, $(x_1, \ldots, x_N) \in \Omega$ depends only on x_1.

Proposition 3.1. *Let $N = 2$ and $\Omega = (0,1)^2$. Then there exists a constant $m \in (-1,1)$, sufficiently close to -1, and a sequence $\mu_k \to 0$ such that every solution u^{μ_k} of $(P)^{\mu_k}$ is not planar.*

We think typical interfaces for solutions of $(\widetilde{P})^{\mu}$ should be lines or circles when $N = 2$. We believe that, for sufficiently close to 1, and μ small, an interface approximated by a circle of a small radius, centered near the points on the boundary, which have the maximum mean curvature, should arise as in Cahn–Hilliard theory.

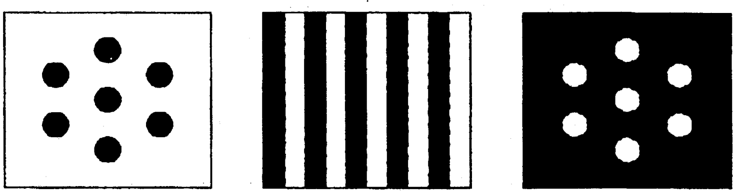

Figure 1 Typical Patterns; the black part is the region $u \sim 1$ and the white part is the region $u \sim -1$. (i) the left picture is the case $m < 0$; (ii) the central picture is the case $m \sim 0$; and the right picture is the case $m > 0$

We cannot expect that the minimizers of I_ε are precisely periodic in two dimensional arbitrary domain unlike the one dimensional case. However the Young measure generated by the global minimizers is constant in $x \in \Omega$. (See, Theorem 2.1 (i).) This suggests that the energy of global minimizers distribute somewhat uniformly. Then if the minimizers are not planar, what do they look like? In fact, non–planar minimizers which have hexagonal structures are observed (see Figure 1). We would like to give a mathematical account of this hexagonal pattern selection drawn in Figure 2.

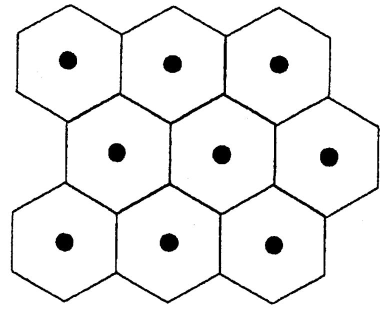

Figure 2 hexagon structure

Since the formal discussion suggests that we should study the pattern of the order $\varepsilon^{1/3}$, we use the following scaling and transformed functions

$$\hat{\epsilon} = \varepsilon^{2/3}, y = \frac{x}{\varepsilon^{1/3}},$$
$$u(x) = U(y), v(x) = \varepsilon^{2/3} V(y),$$
$$-D\Delta V + \gamma\hat{\epsilon} V = U - m.$$

Now let U, V be extended to the whole $\mathbb{R}^N$ in a symmetric and periodic way with a periodic unit domain Y. Then if $\{y\,;\,\hat{\epsilon} y \in \Omega\}$ is packed with a finite number of translated Y, we have

$$\varepsilon^{-2/3}|\Omega|^{-1} I_\varepsilon[u] =$$
$$\frac{1}{|Y|}\int_Y \frac{\hat{\epsilon}}{2}|\nabla U|^2 + \frac{W(U)}{\hat{\epsilon}} + \frac{(\langle U\rangle_Y - m)^2}{2\gamma\hat{\epsilon}} + \frac{D\kappa}{2}|\nabla V|^2 + \frac{\kappa\gamma\hat{\epsilon}}{2}(V - \langle V\rangle_Y)^2\, dy.$$

By using this rescaling argument and the Modica–Mortola theorem, we are led to the following reduced energy density:

$$\mathcal{E}[U] = \frac{1}{|Y|}\left[\frac{2}{c_o} P_Y(\{U = 1\}) + \frac{D\kappa}{2}\int_Y |\nabla V|^2\, dy\right]$$

if U, V are Y–periodic functions such that $W(U) = 0, \langle U \rangle_Y = m$ and $-D\Delta V = U - m$. Then we get

$$I_\varepsilon[u] \sim |\Omega| \mathcal{E}[U] \varepsilon^{2/3}.$$

Note that the isoperimetric constant, the minimum of the perimeter with a volume constraint, is achieved if and only if the interface is the sphere. Now consider the dimension $N = 2$ and define the periodic circular patterns as follows. Let α, β be two complex numbers with $\mathrm{Im}(\beta/\alpha) > 0$, and $\Sigma = \mathbb{Z}\alpha + \mathbb{Z}\beta$ be a lattice in the complex plane. Then let $U_\Sigma : \mathbb{R}^2 \to \{\pm 1\}$ be a function satisfying $U_\Sigma(x_1, x_2) = 1$ if $\mathrm{dist}(x_1 + ix_2, \Sigma) \leq r$ and $U_\Sigma(x_1, x_2) = -1$ if $\mathrm{dist}(x_1 + ix_2, \Sigma) > r$, with a constant $r > 0$ being determined by $\langle U \rangle_Y = m$. We assume that $r < \min\{|\alpha|, |\beta|\}$, which can be satisfied for a certain Σ if and only if $m \in (-1, \sqrt{3}\pi/3 - 1)$. Let $Y_\Sigma = \{(x_1, x_2)\,;$ there exist $s, t \in (0, 1)$ such that $x_1 + ix_2 = s\alpha + t\beta\}$ be a unit of parallelogram. See Figure 3.

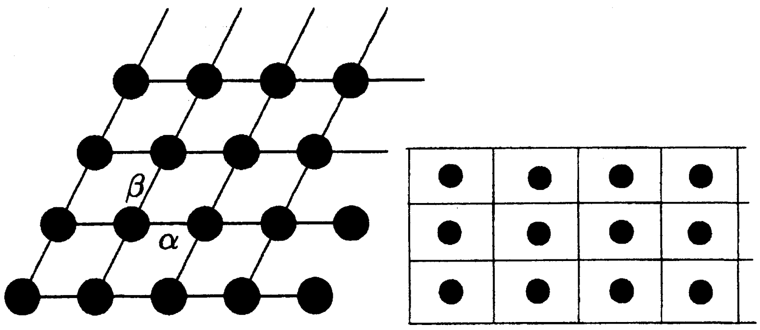

Figure 3 Periodic Circular Patterns U_Σ

One can show that the energy density for the triangle pattern (Figure 4) is larger than the hexagonal pattern (Figure 2).

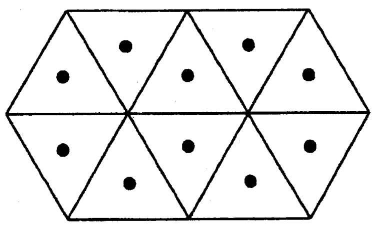

Figure 4 triangle structure

We will show that the energy density defined above achieves the minimum when Σ is a hexagonal structure. Then we will obtain the upper bound for the min I_ε for arbitrary domains by hexagonal structures, making a close study of the error by $\partial\Omega$.

Proposition 3.2 (X. Chen & Y. Oshita). *The following statements hold:*

(1) *For* $\Sigma = \mathbb{Z}\alpha + \mathbb{Z}\beta$, $\zeta = \beta/\alpha$, $\mathrm{Im}(\zeta) > 0$,

$$\mathcal{E}[U_\Sigma] = \frac{2}{c_o}\sqrt{\frac{2\pi(1+m)}{|Y_\Sigma|}} + \frac{\kappa(1+m)^2[R(\zeta)+c_1(m)]|Y_\Sigma|}{2D},$$

where

$$R(\zeta) = -\frac{1}{2\pi}\log\left|\sqrt{\mathrm{Im}(\zeta)}\, q^{1/12}\prod_{n=1}^{\infty}(1-q^n)^2\right|, \quad q = e^{2\pi i\zeta}$$

and

$$c_1(m) = \frac{1}{4\pi}\left(1 + \frac{m}{2} - \log(2\pi(1+m))\right).$$

(2) *The minimum of* $\mathcal{E}[U_\Sigma]$ *among all possible periodic circular patterns is*

$$\mathcal{E}^* = 3(1+m)(c_o)^{-2/3}D^{-1/3}\left[\pi\kappa\,(c_1(m) + R(\zeta^*))\right]^{1/3}, \quad \zeta^* = e^{i\pi/3},$$

which is attained when Σ *is equal to the lattice* $\mathbb{Z}\alpha^* + \mathbb{Z}\beta^*$,

$$|\alpha^*| = |\beta^*| = 2\pi^{1/6}3^{-1/4}(1+m)^{-1/2}D^{1/3}\left[c_o\kappa(c_1(m)+R(\zeta^*))\right]^{-1/3}, \qquad \frac{\beta^*}{\alpha^*} = \zeta^*.$$

(3) *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with the smooth boundary* $\partial\Omega$. *Then*

$$\min_{u\in H^1(\Omega)} I_\varepsilon[u] \le |\Omega|\varepsilon^{2/3}[\mathcal{E}^* + O(\varepsilon^{1/3}|\log\varepsilon|)],$$

as $\varepsilon \to 0$.

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