数理解析研究所講究録 1356 巻 2004 年 100-107

Stability of Stationary Interfaces of Binary-Tree Type: Overview

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1 Intoroduction

A curvature-driven motion with a triple junction has been introduced by Mullins [10] as a model of grain boundary motion in two dimensions. Later, the motion was derived formally by Bronsard and Reitich [1] as the singular limit of a vector-valued Allen-Cahn equation. Bronsard and Reitich [1] also showed short-time existence of the motion. Let $\Gamma_i(t)$ $(i=1,2,3)$ represent curves at time $t>0$ contained in a two-dimensional bounded region Ω with smooth boundary $\partial\Omega$. Suppose $\Gamma_i(t)$ $(i=1,2,3)$ meet at one point $m(t)$. The evolving interface that we consider is subject to the following laws:

(M1) The normal velocity of the interface is given by its curvature.

(M2) At the triple junction $m(t)$, the contact angle θ_k between $\Gamma_i(t)$ and $\Gamma_j(t)$ is given by Young's law, where $(i,j,k)=(1,2,3),(2,3,1),(3,1,2)$. That is, for positive constants σ_1, σ_2, σ_3,

$$\frac{\sin\theta_1}{\sigma_1}=\frac{\sin\theta_2}{\sigma_2}=\frac{\sin\theta_3}{\sigma_3},$$

where $0<\theta_k<\pi$ and $\theta_1+\theta_2+\theta_3=2\pi$.

(M3) At the other end of each curve, $\Gamma_i(t)$ touches $\partial\Omega$ at the right angle.

The interfaces have Energy $E(t)$, which decreases as time goes:

$$E(t)=\sigma_1|\Gamma_1(t)|+\sigma_2|\Gamma_2(t)|+\sigma_3|\Gamma_3(t)|,$$

where $|\Gamma_i(t)|$ $(i=1,2,3)$ mean the lengths of curves $\Gamma_i(t)$. Stationary interfaces of the motion can be viewed as critical points of the energy. In this connection Sternberg and

Ziemer [12] have proved the existence of local minimizers of the energy in clover-like regions. Here we remark that stationary interfaces consist of straight line segments.

On the other hand, Ikota and Yanagida [8] have studied stabilities of stationary interfaces of the motion (M1)–(M3) by linearizing corresponding equations around the stationary interfaces. They linearized the equations formally and analyzed the resulting elliptic operator rigorously to obtain a stability criterion which determines the unstable dimension in terms of the parameters. They also checked by numerical experiments that the linearized stability criterion suggests a nonlinear stability.

Recently Ikota and Yanagida have extended their results to stationary interfaces of binary-tree type with more than one triple junctions. In this article we give an overview of their results.

2 Formulation of the Problem

Consider a network of curves with triple junctions in Ω. We assume the network $\Gamma = \Gamma(t)$ consists of n curves denoted by $\gamma_i = \gamma_i(t)$, $i = 1, 2, \ldots, n$, and contacts with $\partial\Omega$ at endpoints (see Figure 1). We regard Γ as the set of the curves $\{\gamma_i\}$, and denote

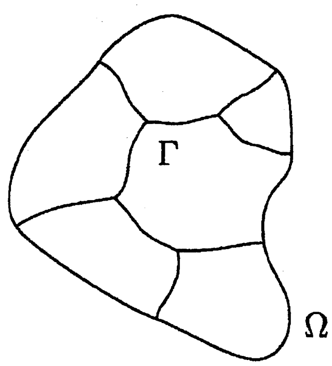

Figure 1: An interface Γ with triple junctions.

by B the subset of Γ that consists of curves touching $\partial\Omega$. Let σ_i be positive constants representing surface energy of γ_i per unit length. We denote by $L(t)$ the lengths of γ_i and by $V = \{x_l\}$ the set of triple junctions.

Every curve γ_i is driven to the center of curvature at the normal speed V_i that is equal to the curvature of γ_i at each point. At each triple junction x_l, three curves, say

γ_i, γ_j, γ_k, meet with prescribed angles and satisfy Young's law:

$$\frac{\sin\theta_{l,i}}{\sigma_i} = \frac{\sin\theta_{l,j}}{\sigma_j} = \frac{\sin\theta_{l,k}}{\sigma_k},$$

where $\theta_{l,i}$ is the angle between γ_j and γ_k at x_l and $\theta_{l,j}$, $\theta_{l,k}$ are alike. Each $\gamma_i \in B$ contacts with $\partial\Omega$ at the right angle.

We consider perturbations that can be represented as graphs of functions on Γ, and describe the motion of nearby interfaces by using some nonlinear partial differential equations with moving boundaries.

Suppose that γ_i, γ_j, γ_k meet at a triple junction $x_l \in V$. We take x_l as the origin of ξ-η coordinate system. For γ_i, the ξ-axis is taken along γ_i, and the η-axis is taken by rotating the ξ-axis by $\pi/2$ radian counter-clockwise. In this coordinate system we consider a perturbation which can be represented as a graph of $\eta = w_i(\xi)$. Approximating the time evolution of w_i, we obtain a linear operator $\mathcal{L}$ at γ_i. We take coordinate systems for γ_j and γ_k in the same way, and describe perturbations by using some functions $w_j(t,\xi)$ and $w_k(t,\xi)$. For details of this procedure, we refer to our previous paper [8].

There are two ways of introducing a coordinate system on γ_i because both end points can be the origin. We will choose one of these coordinate systems according to situations in order to make the presentation simple.

Now let us describe the linear operator $\mathcal{L}$ more precisely. Put $\mathbf{u} = (u_1, u_2, \dots, u_n)$, where u_i is defined on γ_i. Let L_i be the length of γ_i. Then $\mathcal{L}$ is written as

$$\mathcal{L}[\mathbf{u}] = \frac{\partial^2 \mathbf{u}}{\partial \xi^2}. \tag{1}$$

The associated boundary conditions are given as follows.

1. For $\gamma_i \in B$,

$$\frac{\partial u_i}{\partial \xi}(L_i) + h_i u_i(L_i) = 0. \tag{2}$$

2. If γ_i, γ_j, γ_k meet at $x_l \in V$,

$$\sigma_i u_i(0) + \sigma_j u_j(0) + \sigma_k u_k(0) = 0, \tag{3}$$

$$\frac{\partial u_i}{\partial \xi}(0) = \frac{\partial u_j}{\partial \xi}(0) = \frac{\partial u_k}{\partial \xi}(0). \tag{4}$$

We set

$$\boldsymbol{H} := \bigoplus_{\gamma_i \in \Gamma} L^2(0, L_i),$$

and treat $\mathcal{L}$ as an operator from $\boldsymbol{H}$ to $\boldsymbol{H}$ with a domain of definition

$$\mathcal{D}(\mathcal{L}) = \left\{ \mathbf{u} \in \bigoplus_{\gamma_i \in \Gamma} H^2(0, L_i) \ \mid \ \mathbf{u} \text{ satisfies conditions (3) and (4) } \right\}.$$

The inner product $(\cdot, \cdot)_{\boldsymbol{H}}$ of $\boldsymbol{H}$ is given by

$$(\mathbf{u}, \mathbf{v})_{\mathbf{H}} := \sum_{\gamma_i \in \Gamma} \left\{ \sigma_i \int_0^{L_i} u_i v_i d\xi \right\}.$$

3 Results

Our results are stated as follows.

Theorem 3.1. *Let $\Gamma = \{\gamma_i\}$ be a stationary interface that is homeomorphic to a binary tree. Define a characteristic index D by*

$$D = \sum_{\gamma_i \in \Gamma} \sigma_i L_i \times \prod_{\gamma_i \in B} h_i + \sum_{\gamma_i \in B} \left\{ \sigma_i \prod_{\gamma_j \in B \setminus \{\gamma_i\}} h_j \right\},$$

where h_i denotes the curvature of $\partial\Omega$ at the point of contact with $\gamma_i \in B$. (Note that h_i is taken to be nonpositive if Ω is convex.)

(i) *The unstable dimension N_{U} is given by*

$$N_{\mathrm{U}} = \begin{cases} m - 1 & \text{for} \quad (-1)^m D \leq 0, \\ m & \text{for} \quad (-1)^m D > 0, \end{cases}$$

where $m = \#\{h_i < 0\}$.

(ii) *The stationary interface is degenerate (i.e., there exists a zero eigenvalue) if and only if $D = 0$.*

We remark that the index D is independent of the topology of Γ.

4 Variational Methods

The operator $\mathcal{L}$ naturally leads to a bilinear form.

Definition 4.1. *A bilinear form $J : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ is defined by*

$$J(\mathbf{u}, \mathbf{v}) := \sum_{\gamma_i \in B} h_i u_i(L_i) v_i(L_i) + \sum_{\gamma_i \in \Gamma} \sigma_i \int_0^{L_i} \partial_\xi u_i(\xi) \partial_\xi v_i(\xi) d\xi,$$

where

$$\boldsymbol{V} := \{\mathbf{u} \in \bigoplus_{\gamma_i \in \Gamma} H^1(0, L_i) \mid \mathbf{u} \text{ satisfies the condition } (3)\}.$$

The inner product $(\cdot,\cdot)_{\mathbf{V}}$ is given by

$$(\mathbf{u}, \mathbf{v})_{\mathbf{V}} := \sum_{\gamma_i \in \Gamma} \left\{ \sigma_i \int_0^{L_i} (u_i v_i + \partial_\xi u_i \, \partial_\xi v_i) d\xi \right\}.$$

In addition we introduce a functional $I : \boldsymbol{V} \setminus \{0\} \to \mathbb{R}$ defined by

$$I(\mathbf{u}) := \frac{J(\mathbf{u}, \mathbf{u})}{(\mathbf{u}, \mathbf{u})_{\mathbf{H}}}.$$

We can characterize the eigenvalues of $\mathcal{L}$ in terms of I. Discussions similar to [8] yield the following result:

Proposition 4.1. *There exist positive numbers c and d such that*

$$\|\mathbf{u}\|_{\mathbf{V}}^2 \le c(\mathbf{u}, \mathbf{u})_{\mathbf{H}} + dJ(\mathbf{u}, \mathbf{u}) \qquad \text{for all } \mathbf{u} \in \boldsymbol{V}.$$

From this we deduce that the operator $\mathcal{L}$ is self-adjoint.

Let $\mathfrak{H}$ be the family of all finite dimensional subspaces of $\boldsymbol{H}$. Denote by λ_j the jth eigenvalue of $\mathcal{L}$. Then we have $\lambda_j \ge \lambda_{j+1}$. The eigenvalues λ_j are characterized by the sup-inf principle:

$$-\lambda_j = \sup_{\substack{K \in \mathfrak{H} \\ \dim K \le j-1}} \inf_{v \in K^\perp \setminus \{0\}} I(\mathbf{v}), \tag{5}$$

where

$$K^\perp := \{\mathbf{u} \in \boldsymbol{V} \mid (\mathbf{u}, \mathbf{v})_{\mathbf{H}} = 0 \text{ for all } \mathbf{v} \in K\}.$$

For the proof, see Section 1, Chapter 13 of [11].

If we take $\{h_i\}$ as parameters, each eigenvalue is a continuous and monotone decreasing function of h_i. See Theorems 6 and 9 in Chapter 6 of [2].

In view of the above observations we can prove the proposition below.

Proposition 4.2. *Put $m = \#\{h_i < 0 \mid \gamma_i \in B\}$. Then $N_{\mathrm{U}} \ge m - 1$*

5 Characteristic Functions

As in [8] we make characteristic functions.

Proposition 5.1. *For any stationary interface Γ, there exists a complex-valued characteristic function $F = F(\mu)$ of a complex variable μ with parameters σ_i, L_i, $(\gamma_i \in \Gamma)$ and h_i $(\gamma_i \in B)$ satisfying the following properties:*

(i) *For $\mu \neq 0$, $F(\mu) = 0$ if and only if $\lambda = \mu^2$ is an eigenvalue of $\mathcal{L}$.*

(ii) *$F(\mu)$ is analytic in μ, σ_i, L_i, and h_i. Further, $F(\mu)$ is real-valued if μ is restricted to real numbers.*

(iii) *$F(\mu)$ is odd with respect to μ. In particular, $F(0) = 0$ for any σ_i, L_i, h_i.*

(iv) *Any zero of $F(\mu)$ lies on the real axis or imaginary axis, and it depends on σ_i, L_i, h_i continuously.*

(v) *$F(\mu) \to +\infty$ as $\mu \to +\infty$.*

(vi) *For each $\gamma_i \in B$, F is written as $F = P(\mu)h_i + Q(\mu)$, where P and Q are independent of h_i.*

We prove this by induction. First notice that the assertion was proved in [8] if Γ has only one triple junction.

Next, let Γ be a stationary interface with two or more triple junctions. Take an edge $\gamma_k \in \Gamma \setminus B$. We divide Γ into two parts Γ^α and Γ^β by introducing a virtual boundary C which intersects γ_k orthogonally (See Fig. 2). Denote by h the curvature of C at the intersection point with Γ^α. Then the curvature of C is $-h$ for Γ^β at the same intersection point. Suppose that the assertion is true for Γ^α and Γ^β, and denote by F^α and F^β the characteristic functions for Γ^α and Γ^β, respectively, satisfying the properties (i)$\sim$(vi). By (vi), we can write them as

$$
\begin{aligned}
F^\alpha(\mu) &= P^\alpha(\mu)h + Q^\alpha(\mu), \\
F^\beta(\mu) &= -P^\beta(\mu)h + Q^\beta(\mu),
\end{aligned} \tag{6}
$$

where $P^\alpha, Q^\alpha, P^\beta, Q^\beta$ are independent of h. From $F^\alpha(\mu) = 0$ and $F^\beta(\mu) = 0$, we can eliminate h to define a function $F^{\alpha+\beta}(\mu)$ by

$$
F^{\alpha+\beta}(\mu) := \frac{P^\alpha(\mu)Q^\beta(\mu) + Q^\alpha(\mu)P^\beta(\mu)}{\sigma_i \mu} \qquad \text{for } \mu \neq 0 \tag{7}
$$

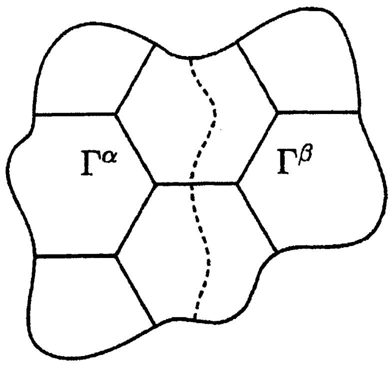

Figure 2: The interface divided into two parts. The dotted line stands for the virtual boundary.

and $F^{\alpha+\beta}(0) = 0$. We can prove this function satisfies the desired properties.

Moreover, from (7), we obtain an explicit expression of $(dF/d\mu)|_{\mu=0}$.

Proposition 5.2. *Let F be the characteristic function constructed as above. Then the derivative of $F(\mu)$ at $\mu = 0$ is given by*

$$D := \left.\frac{dF}{d\mu}\right|_{\mu=0} = \sum_{\gamma_i\in\Gamma} \sigma_i L_i \times \prod_{\gamma_i\in B} h_i + \sum_{\gamma_i\in B} \left\{\sigma_i \prod_{\gamma_j\in B\setminus\{\gamma_i\}} h_j\right\}.$$

6 Outline of the Proof

We have the following lemma on the nondegeneracy of zero eigenvalues.

Lemma 6.1. *If at most one of h_i ($\gamma_i \in B$) is zero, then any zero eigenvalue is simple.*

First notice that we can deform Ω without changing the shape of a given stationary interface. Hence we may regard h_i ($\gamma_i \in B$) as variable parameters. Without loss of generality, we put $B = \{\gamma_1, \dots, \gamma_k\}$.

We count the number of positive eigenvalues as follows. Assume first that h_1, h_2, $\dots$, $h_k > 0$. Then (5) implies $N_U = 0$. Next, we decrease the values of $h_1, h_2, \dots, h_m$ one by one to negative values. By this procedure, the index D can change its sign at most m times and hence $N_U \leq m$. On the other hand, Proposition 4.2 shows $N_U \geq m-1$. Hence $N_U = m-1$ or m. Since $D > 0$ if $h_1, h_2, \dots, h_k > 0$, N_U is even if $D > 0$ and is odd if $D < 0$. Thus (i) is proved.

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