数理解析研究所講究録 1356 巻 2004 年 108-115

2次元帯状領域におけるパターンの運動

横浜市立大学・総合理学研究科 栄 伸一郎 (Shin-Ichiro Ei)
Graduate School of Integrated Science, Yokohama City University

Abstract. Reaction-diffusion systems in an infinitely long strip-like domain with finite width in 2D are treated. We construct the solution connecting different types of stationary solutions at infinity by considering the neighborhood of Turing instability. We also derive 4th order equations of buckling type which shows the dynamics of the connecting solutions.

1 Introduction

In 1995, Kondo and Asai[1] showed some chemical waves are observed in the skin of real fishes. Their simulations by using reaction-diffusion systems reappear very well the growth of patterns on the skin. One of the typical patterns is the rearrangement of the stripe patterns. For example, when the width of skin are different in locations, the number of stripes are different depending on the width. Then patterns with different numbers of stripes are connected and some defects appear. According to the growth of skin, the location of the defects change. After their work, there have been many simulations related to such phenomena, but no theoretical results yet. To give a theoretical framework to it, we first consider a problem on a fixed domain and construct a solution connecting two different stable stripe patterns as follows:

Let $\Omega=(-\infty,\infty)\times(0,l)\in\boldsymbol{R}^2$ and consider a reaction-diffusion system

(1.1) $$\boldsymbol{u}_t=D\Delta\boldsymbol{u}+F(\boldsymbol{u}),\ t>0,\ \boldsymbol{x}=(x,y)\in\Omega$$

with the homogeneous Neumann boundary conditions, where $\boldsymbol{u}=(u_1,\cdots,u_N)\in\boldsymbol{R}^N$,$F$ is a smooth function on $\boldsymbol{R}^N$ and D denotes a diagonal matrix with positive elements d_j, that is, $D=diag(d_1,\cdots,d_N)$.

Let us also consider 1D problem of (1.1)

(1.2) $$\boldsymbol{u}_t=D\boldsymbol{u}_{yy}+F(\boldsymbol{u}),\ t>0,\ y\in(0,l)$$

with the boundary conditions $\dfrac{\partial\boldsymbol{u}}{\partial y}=0$ at $y=0,\ l$. We assume (1.2) have two different stationary solutions, say $U^{\pm}(y)$ with $U^+(y)\neq U^-(y)$. Under these assumptions, we look for the solutions $\boldsymbol{u}(t,\boldsymbol{x})$ of (1.1) satisfying

(1.3) $$\boldsymbol{u}(t,\pm\infty,y)=U^{\pm}(y).$$

There have been few works related to the problem (1.1) with (1.3) (e.g. [2], [3], [4]). Specially, traveling front solutions connecting $U^{\pm}(y)$ have been considered. If we take a solution in the form $\boldsymbol{u}(t,x,y) = U(z,y)$ for $z = x - ct$, then U satisfies

$$0 = D(U_{zz} + U_{yy}) + cU_z + F(U),\ U(\pm\infty, y) = U^{\pm}(y). \tag{1.4}$$

When the problem (1.1) is a gradient system or skew gradient system, we can know in a formal way how c is determined, which will be mentioned in the following sections. It gives the movement of the connecting solution. In fact, [4] showed rigorously the existence and its stability of connecting traveling wavefront solution for a scaler equation.

On the other hand, if there are no gradient structure in any sense for the system, there have been no theoretical results even in formal arguments.

In this paper, we consider the problem (1.1), (1.3) for general function F without any gradient structure, but in the neighborhood of Turing instability. By it, it is shown that a pitchfork type bifurcation occurs in the system (1.2) on 1D and we can get two different stable stationary solutions, say $U^{\pm}(y)$. We can construct approximate solutions of (1.1) connecting $U^{\pm}(y)$ at $x \to \pm\infty$. We can show the dynamics is essentially governed by the dynamics of

$$R_T = -\gamma_1 R_{\zeta\zeta\zeta\zeta} + R(M_2 - M_1 R^2),\ T > 0,\ \zeta \in \boldsymbol{R}, \tag{1.5}$$

where all coefficients are positive constants.

In the following sections, we give preliminaries and results.

2 traveling solutions

In this section, we consider the problem (1.4) and show how the velocity c is determined. All are by formal discussions.

2.1 Gradient systems

Suppose that $F(\boldsymbol{u}) = -\nabla W(\boldsymbol{u})$ holds for a function $W(\boldsymbol{u}) \in \boldsymbol{R}$ and that (1.4) has a solution $U(z,y)$. Then according to [4] we have

$$
\begin{aligned}
&-c\int_\Omega \langle U_z, U_z \rangle\, dzdy \\
&= \int_\Omega \langle D\Delta U, U_z \rangle\, dzdy - \int_\Omega W(U)_z dzdy \\
&= \int_\Omega \langle D\Delta U, U_z \rangle\, dzdy - \int_\Omega W(U)_z dzdy \\
&= \frac{1}{2}\int_\Omega \frac{\partial}{\partial z}\langle DU_z, U_z \rangle\, dzdy + \int_{-\infty}^{\infty}\left\{[\langle DU_y, U_z \rangle]_0^l - \int_0^l \langle DU_y, U_{zy} \rangle\, dy\right\} dz \\
&\qquad - \int_0^l [W(U)]_{-\infty}^{\infty} dy \\
&= \frac{1}{2}\left(\int_0^l [\langle DU_z, U_z \rangle]_{-\infty}^{\infty} dy - \int_0^l \int_{-\infty}^{\infty} \frac{\partial}{\partial z}\langle DU_y, U_y \rangle\, dzdy\right) - \int_0^l [W(U)]_{-\infty}^{\infty} dy \\
&= -\frac{1}{2}\int_0^l \{\langle DU_y^+, U_y^+ \rangle - \langle DU_y^-, U_y^- \rangle\}\, dy - \int_0^l \{W(U^+) - W(U^-)\} dy \\
&= -\frac{1}{2}\int_0^l \{\langle DU_y^+, U_y^+ \rangle - \langle DU_y^-, U_y^- \rangle\}\, dy - \int_0^l \{W(U^+) - W(U^-)\} dy \\
&= E(U^-) - E(U^+),
\end{aligned}
$$

where $E(U) := \displaystyle\int_0^l \left\{\frac{1}{2}\langle DU_y, U_y \rangle + W(U)\right\} dy$. Thus

$$
c\int_\Omega \langle U_z, U_z \rangle\, dzdy = E(U^+) - E(U^-) \tag{2.1}
$$

holds. This means the sign of c is determined the difference of energies $E(U^\pm)$ of $U^\pm(y)$ on the interval $(0,l)$.

2.2 Skew-gradient systems

Suppose that $F(\boldsymbol{u}) = -Q\nabla W(\boldsymbol{u})$ holds for a function $W(\boldsymbol{u}) \in \boldsymbol{R}$ and a matrix Q which is symmetric and invertible satisfying $DQ = QD$(e.g. [5]). Assume (1.4) has a solution $U(z,y)$. Then similarly to the previous subsection, we have

$$
\begin{aligned}
&-c\int_\Omega \langle U_z, Q^{-1}U_z \rangle\, dzdy \\
&= \int_\Omega \langle D\Delta U, Q^{-1}U_z \rangle\, dzdy - \int_\Omega \langle Q\nabla W(U), Q^{-1}U_z \rangle\, dzdy \\
&= E(U^+) - E(U^-),
\end{aligned} \tag{2.2}
$$

where

$$E(U) := \int_0^l \left\{ \frac{1}{2} \langle\, DU_y, Q^{-1}U_y \,\rangle + W(U) \right\} dy.$$

But the coefficient of c in the left hand side of (2.2) does not have fixed sign due to Q. Thus, we need more informations on the solution $U(z, y)$ in order to know the sign of c.

3 Preliminaries for Turing Instability

3.1 Bifurcation diagram in 1D problems

Let $\mathbf{0}$ be a linearly stable equilibrium of F, that is, $F(\mathbf{0}) = \mathbf{0}$ is satisfied and all eigenvalues of the linearized matrix $B := F'(\mathbf{0})$ are negative real parts. Consider the 1 dimensional problem (1.2) and L be a linearized operator with respect to $\mathbf{0}$. Expanding solutions in Fourier series $U = \sum_{n=0}^{\infty} \cos C_n y \boldsymbol{a}_n$ ($\boldsymbol{a}_n \in \boldsymbol{R}^N$) and substituting it to the eigenvalue problem $LU = \lambda U$, we have

$$\{-C_n^2 D + B\}\boldsymbol{a}_n = \lambda \boldsymbol{a}_n,$$

where $C_n := \frac{n\pi}{l}$. Therefore, we first consider the matrix $\Xi(\tau) := -\tau D + B$ parametrized by $\tau \geq 0$. Let $\lambda_j(\tau)$ ($j = 1, \cdots, N$) be eigenvalues of $\Xi(\tau)$. We assume there exist positive constants γ_0, τ_0 such that $Re(\lambda_j(\tau)) < -\gamma_0$ for $j = 2, \cdots, N$ and $\lambda_1(\tau) \in \boldsymbol{R}$, $\lambda_1(\tau) \leq 0$ for any $\tau \geq 0$ and $\lambda_1(\tau) = 0$ if and only if $\tau = \tau_0$(Fig.3.1). Moreover, we

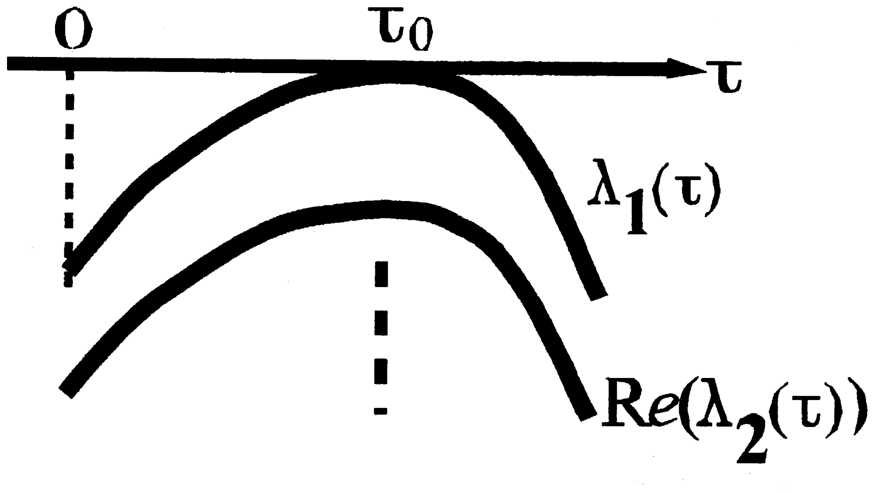

Figure 3.1: Eigenvalues $\lambda_1(\tau)$ and others.

assume $\lambda_0(\tau)$ is a simple eigenvalue of $\Xi(\tau)$ with the associated eigenvector $\boldsymbol{\alpha}(\tau) \in \boldsymbol{R}^N$.

If we take the width l of Ω such that $\left(\frac{\pi}{l}\right)^2 = \tau_0$ and take $\boldsymbol{a}_1 = \boldsymbol{\alpha}_0$, then $L\boldsymbol{\phi}_1 = \boldsymbol{0}$ holds, where $\boldsymbol{\phi}_n(y) := \cos C_n y \boldsymbol{a}_n$ and $\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}(\tau_0)$. Thus, in the system (1.2), only 1 mod solution close to $\boldsymbol{\phi}_1(y)$ can be unstable.

Under the above assumptions, we consider the system

$$\boldsymbol{u}_t = D\Delta\boldsymbol{u} + F(\boldsymbol{u}) + \eta G(\boldsymbol{u}),\ t > 0,\ \boldsymbol{x} = (x, y) \in \Omega \tag{3.1}$$

for small η and a function $G(\boldsymbol{u})$ with $G(\boldsymbol{0}) = \boldsymbol{0}$. Then we can show for the 1D problem of (3.1)

$$\boldsymbol{u}_t = \boldsymbol{u}_{yy} + F(\boldsymbol{u}) + \eta G(\boldsymbol{u}),\ t > 0,\ 0 < y < l \tag{3.2}$$

has a pitch-fork type bifurcation diagram in super critical as follows.

Proposition 3.1 *For sufficiently η, there exist constants M_1, M_2 and a function $\sigma = \sigma(r;\eta)(y)$ with $\|\sigma\| = O(|\eta| + r^2)$ such that*

$$\boldsymbol{u}(t, y) = r(t)\varphi_1(y) + \sigma(r(t);\eta)(y)$$

with

$$\dot{r} = -M_1 r^3 + M_2 r\eta + O(|\eta|^2 + r^4).$$

This proposition is easily proved by a standard center manifold theory.

Hereafter, we assume both M_1 and M_2 are positive, which means a super critically pitch-fork type bifurcation diagram occurs. In fact, there exist stable stationary solutions $U^{\pm}(y)$ corresponding to $r_{\pm} := \pm\sqrt{\dfrac{M_2\eta}{M_1}}$ for small $\eta > 0$. $U^{\pm}(y)$ are given by

$$U^{\pm}(y) = r_{\pm}\boldsymbol{\phi}_1(y) + \sigma(r_{\pm};\eta)(y) = \pm\sqrt{\frac{M_2\eta}{M_1}}\cos C_1 y\boldsymbol{\alpha}_0 + O(|\eta|).$$

3.2 Stability of planar solutions in 2D

In this subsection, we give the stability conditions of $U^{\pm}(y)$ as the solutions of (3.1) in 2D. Let $\boldsymbol{u}^{\pm}(x, y) := U^{\pm}(y)$ and let $\boldsymbol{\alpha}^*(\tau)$ be the eigenvectors satisfying ${}^t\Xi(\tau)\boldsymbol{\alpha}^*(\tau) = \lambda_1(\tau)\boldsymbol{\alpha}^*(\tau)$ and $\langle\ \boldsymbol{\alpha}(\tau), \boldsymbol{\alpha}^*(\tau)\ \rangle = 1$. Define $\boldsymbol{\alpha}_0^* := \boldsymbol{\alpha}^*(\tau_0)$. Then we have

Theorem 3.1 *Let ω_1 be the constant given by*

$$\omega_1 := \frac{1}{8}\langle\ F''(\boldsymbol{0})\boldsymbol{\alpha}_0 \cdot (\boldsymbol{v}_1 - \boldsymbol{b}_2), \boldsymbol{\alpha}_0^*\ \rangle + \frac{1}{32}\langle\ F'''(\boldsymbol{0})\boldsymbol{\alpha}_0^3, \boldsymbol{\alpha}_0^*\ \rangle,$$

where $\boldsymbol{v}_1$ and $\boldsymbol{b}_2$ are unique vectors determined by

$$(-4\tau_0 D + B)\boldsymbol{b}_2 + \frac{1}{4}F''(\boldsymbol{0})\boldsymbol{\alpha}_0^2 = \boldsymbol{0},\ (-2\tau_0 D + B)\boldsymbol{v}_1 + F''(\boldsymbol{0})\boldsymbol{\alpha}_0^2 = \boldsymbol{0}.$$

If $\omega_1 < 0$ (> 0) then $\boldsymbol{u}^{\pm}$ are stable (unstable) as the stationary solutions in 2D.

Corollary 3.1 *Assume* $\boldsymbol{u} = \begin{pmatrix} u \\ v \end{pmatrix} \in \boldsymbol{R}^2$ *and* $F(\boldsymbol{u})+\eta G(\boldsymbol{u}) = \begin{pmatrix} f(u) - v + \eta g(u,v) \\ \delta(u - \gamma v) \end{pmatrix}$ *with* $f(0) = g(0,0) = 0$ *and* $f''(0) = 0$, *where* δ *and* γ *are positive constants. Then* $\boldsymbol{u}^{\pm}(x,y)$ *are stable (or unstable) if* $f'''(0) < 0$ *(or* > 0*).*

Typical examples of f are cubic-like functions such as $f(u) = u(1-u^2)$. Then the conditions in Corollary 3.1 are easily satisfied and we find planar solutions $\boldsymbol{u}^{\pm}$ are stable.

4 Solutions of (3.1) connecting $\boldsymbol{u}^{\pm}$

Define $\Sigma(r)(y) := r\phi_1(y) + \sigma(r;\eta)(y)$. Then $\Sigma(r)(y)$ is invariant for the dynamics of (3.2), that is, there exists $H(r;\eta)$ such that

$$H\Sigma_r = \mathcal{L}(\Sigma),$$

where $\mathcal{L}(\boldsymbol{u}) := \boldsymbol{u}_{yy} + F(\boldsymbol{u}) + \eta G(\boldsymbol{u})$. Proposition3.1 suggests that $H(r;\eta) = -M_1 r^3 + M_2 r\eta + O(|\eta|^2 + r^4)$ holds. Especially, $U^{\pm}(y)$ are given by $\Sigma(r_{\pm};\eta)$.

For the eigenvalue $\lambda_1(\tau)$ of the matrix $\Xi(\tau)$ mentioned in Section 3.1, we may assume

$$\lambda_1(\tau_0 + \varepsilon) = -\gamma_1 \varepsilon^2 + O(\varepsilon^3) \tag{4.1}$$

for positive γ_1 and small ε (Fig.3.1).

Using the above, we define approximate functions by

$$\boldsymbol{u}^*(t,x,y) := \Sigma(\sqrt{\eta}R(T,\zeta))(y),\ T := \eta t,\ \zeta := \sqrt[4]{\eta}x$$

and define constants $R_{\pm} := \pm\sqrt{\dfrac{M_2}{M_1}}$. Then

Theorem 4.1

$$\boldsymbol{u}^*_t - \{D\Delta \boldsymbol{u}^* + F(\boldsymbol{u}^*) + \eta G(\boldsymbol{u}^*)\} = O(|\eta|^{3/2})$$

holds and $R(T,\zeta)$ *satisfies*

$$R_T = -\gamma_1 R_{\zeta\zeta\zeta\zeta} + R(M_2 - M_1 R^2) + O(\sqrt{\eta}) \tag{4.2}$$

uniformly for $0 \le T \le T^*$ *and* $(x,y) \in \Omega$, *where* T^* *is an arbitrarily given positive constant.*

Suppose there exists the solution, say $R_0(T,\zeta)$, *of (1.5), the equation (4.2) with* $\eta \to 0$, *satisfying* $R_0(T,\pm\infty) = R_{\pm}$ *for* $0 \le T \le T^*$. *Let* $\boldsymbol{u}^*_0(t,x,y) := \Sigma(\sqrt{\eta}R_0(T,\zeta))(y)$. *If the initial date* $\boldsymbol{u}(0,x,y)$ *is sufficiently close to* $\boldsymbol{u}^*_0(0,x,y)$, *then the solution of (3.1) satisfies*

$$\boldsymbol{u}(t,x,y) - \boldsymbol{u}^*_0(t,x,y) = o(1)$$

uniformly for $0 \le T \le T^*$ *and* $(x,y) \in \Omega$.

Remark 4.1

$$\boldsymbol{u}_0^*(t, \pm\infty, y) = \Sigma(\sqrt{\eta}R_0(T, \pm\infty))(y) = \Sigma(\sqrt{\eta}R_\pm)(y) = \Sigma(r_\pm)(y) = U^\pm(y)$$

holds. Thus, $\boldsymbol{u}_0^*(t, x, y)$ is the approximate function which connects $U^\pm(y)$ at $x \to \pm\infty$.

Remark 4.2 *For the symmetry of the equations (3.1) with respect to y-axis, $U^\pm(y)$ are symmetric each other for $y = l/2$. Hence $R_0(T, \zeta)$ may be taken as a stationary solution of (1.5), say $R_0 = R_0(\zeta)$.*

All coefficients mentioned hitherto such as M_j are explicitly given though we don't touch on it here because of the restriction of total pages. By using such informations, we can compare the approximate functions constructed here quantitatively with the solutions of (3.1). Fig4.1 denotes the stationary solution of (3.1) connecting $U^\pm(y)$ (in

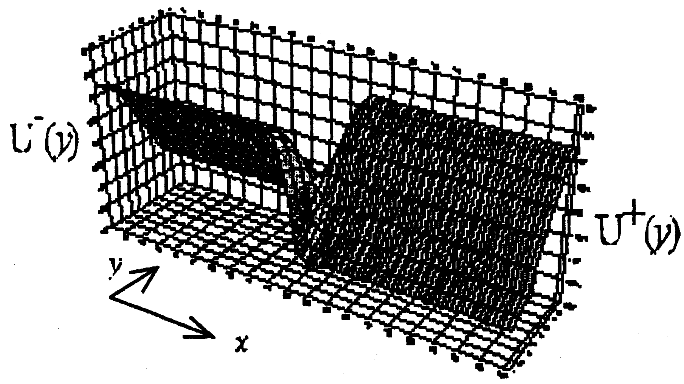

Figure 4.1: Solution of (3.1) connecting $U^\pm(y)$.

numerical sense), say $\boldsymbol{u}_0(x, y)$. Fig4.2 shows the graph of values of $\boldsymbol{u}_0(x, y)$ at $y = 0$.

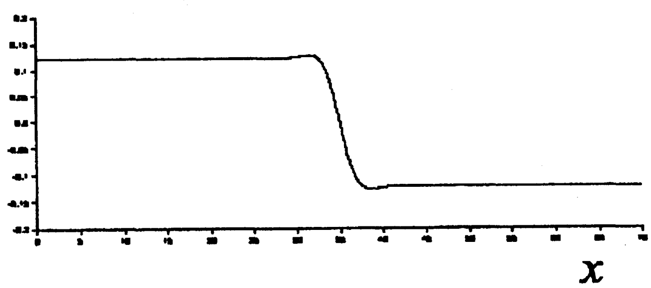

Figure 4.2: Graph of the edge $\boldsymbol{u}_0(x, 0)$.

It is quite coincident with the graph of $\sqrt{\eta}R_0(\sqrt[4]{\eta}x)$.

References

[1] S. Kondo and R. Asai, A reaction-diffusion wave on the skin of the marine angelfish pomacanthus, Nature 376-31 (1995), 765-768.

[2] A. Mielke, Essential manifolds for an elliptic problem in an infinite strip, J. D. E. 110 (1994), 322-355.

[3] , M. Mimura, K. Sakamoto and S.-I. Ei, Singular perturbation problems to a combustion equation in very long cylindrical domains, Studies in Advanced Math. 3 (1997), 75-84.

[4] J.M. Vega, Travelling wavefronts of reaction-diffusion equations in cylindrical domains, Comm. PDE 18 (1993), 505-530.

[5] E. Yanagida, Standing pulse solutions in reaction-diffusion systems with skew-gradient structure, J. Dyn. Diff. Eqs. 4(2002), 189-205.