数理解析研究所講究録 1357 巻 2004 年 48-52

Examples of splendid equivalent blocks with non-abelian defect groups

Naoko Kunugi(功刀 直子)
Aichi Univerisity of Education(愛知教育大学)

1 Introduction

Let G be a finite group. Let k be an algebraically closed field of characteristic $p>0$. We denote by $B_0(G)$ the principal block of kG.

We say that two finite groups G and H have the same p-local structure if they have a common Sylow p-subgroup P such that whenever Q_1 and Q_2 are subgroups of P and $f: Q_1 \rightarrow Q_2$ is an isomorphism, then there is an element $g\in G$ such that $f(x)=x^{g}$ for all $x\in Q_1$ if and only if there is an element $h\in H$ such that $f(x)=x^{h}$ for all $x\in Q_1$.

Conjecture 1.1 (Broué [1],[2] and Rickard [10]) Let G and H be finite groups having the same p-local structure with common Sylow p-subgroup P. If P is abelian then the principal blocks $B_0(G)$ and $B_0(H)$ would be splendid equivalent.

If a finite group G has an abelian Sylow p-subgroup P then G and $N_G(P)$ have the same p-local structure. So we normally take $N_G(P)$ as H.

There is a counterexmaple to the conjecture if P is not abelian. However it would be meaningful to investigate other cases of non abelian defect groups. The purpose of this note is to present some examples of splendid equivalent blocks with non-abelian defect groups.

2 $PGL(3,2^2)$ and $PGU(3,2^2)$

Throught the rest of this note, let k be an algebraically closed field of characteristic 3.

Set

$$G=PGL(3,2^{2})\rhd G'=PSL(3,2^{2})$$

and

$$H=PGU(3,2^{2})\rhd H'=PSU(3,2^{2}).$$

Let Q be a common Sylow 3-subgroup of G' and H', and let P be a common Sylow 3-subgroup of G and H. Then $Q \cong Z_3 \times Z_3$, an elementary abelian 3-group of order 9, and $P \cong M(3)$, an extraspecial 3-group of order 27 of exponent 3. Note that $H' \cong N_{G'}(Q) \cong (C_3 \times C_3) \rtimes Q_8$ and $H \cong N_G(Q) \cong (C_3 \times C_3) \rtimes SL(2,3)$. In particular G and H have the same 3-local structure.

The principal blocks $B_0(G')$ and $B_0(H')$ have 5 simple modules $\{k_{G'}, T'_1, T'_2, T'_3, S'\}$ and $\{k_{H'}, 1'_1, 1'_2, 1'_3, 2'\}$ respectively. The principal blocks $B_0(G)$ and $B_0(H)$ have 3 simple modules $\{k_{G'}, T, S\}$ and $\{k_H, 3, 2\}$ respectively. We have

$$T\downarrow_{G'} = T'_1 \oplus T'_2 \oplus T'_3, \quad T'_i \uparrow^G = T, \quad S' \downarrow_{G'} = S,$$

and

$$3\downarrow_{H'} = 1'_1 \oplus 1'_2 \oplus 1'_3, \quad 1'_i \uparrow^H = 3, \quad 2' \downarrow_{H'} = 2.$$

Theorem 2.1 (Kunugi-Usami) *The principal blocks of $B_0(G)$ and $B_0(H)$ are splendid equivalent.*

In [7] and [8], Okuyama proved that the principal blocks $B_0(G')$ and $B_0(H')$ are splendid equivalent. However we reconstruct a splendid equivalence between $B_0(G')$ and $B_0(H')$, since the equivalence constructed in [7] does not lift to any derived equivalence between $B_0(G)$ and $B_0(H)$. Let

$$F' = Res^{G'}_{H'} : \mathrm{stmod}B_0(G') \to \mathrm{stmod}B_0(H')$$

be the restriction functor. Then F' gives a stable equivalence of Morita type since the Sylow 3-subgroup Q of G' and H' is TI. Then we have the following lemma.

Lemma 2.2 *There exist exact sequences*

$$(1)\ 0 \longrightarrow \Omega^{-1}\begin{pmatrix} k_{H'} \\ 2' \\ 1'_i \end{pmatrix} \longrightarrow \Omega^2 F'(T'_i) \longrightarrow k_{H'} \longrightarrow 0$$

$$(2)\ 0 \longrightarrow \Omega^{-1}\begin{pmatrix} k_{H'} \\ 2' \end{pmatrix} \longrightarrow \Omega F'(S') \longrightarrow k_{H'} \oplus k_{H'} \longrightarrow 0.$$

We easily know the structure of the projective indecomposable kH'-modules. Therefore, using the above lemma, we can conclude that the tilting complex defined by a sequence $\{1'_1, 1'_2, 1'_3\}$, $\{1'_1, 1'_2, 1'_3, 2'\}$, $\{1'_1, 1'_2, 1'_3, 2'\}$ of subsets of $\{k_{H'}, 1'_1, 1'_2, 1'_3, 2'\}$ (see [7]) gives a derived equivalence between $B_0(H')$ and $B_0(G')$.

Now we consider the case in Theorem 2.1. The restriction functor Res_H^G induces a stable equivalence, but does not lift to any derived equivalences. Therefore what we have to do next is to construct a suitable stable equivalence of Morita type between $B_0(G)$ and $B_0(H)$.

Let

$$M \xrightarrow{\pi} k_{G\times H} \longrightarrow 0$$

be a $\Delta(P)$-projective cover of $k_{G\times H}$, and let

$$N \xrightarrow{\iota} \Omega_{\Delta(P)}(k_{G\times H}) \longrightarrow 0$$

be a $\Delta(Q_0)$-projective cover of $\Omega_{\Delta(P)}(k_{G\times H})$, where Q_0 is a unique subgroup of P (up to G-conjugate) such that $B_0(C_G(Q_0)) \not\cong B_0(C_H(Q_0))$. Define a complex

$$M^\bullet : 0 \longrightarrow N \xrightarrow{\phi} M \longrightarrow 0,$$

where $\phi = \iota \circ \pi$. Then, $\mathrm{Br}_{\Delta(R)}(M^\bullet)$ is a splendid tilting complex for $C_G(R)$ and $C_H(R)$ for any subgroup R of P, so that the functor $F = - \otimes_{B_0(G)} M^\bullet$ induces a stable equivalence of Morita type between $B_0(G)$ and $B_0(H)$ by a result of Rouquier (Theorem 5.6 in [11]).

Lemma 2.3 *There exist exact sequences*

$$(1)\ 0 \longrightarrow \Omega^{-1}\left(\begin{pmatrix} k_{H'} \\ 2' \\ 1'_i \end{pmatrix}^{\uparrow H}\right) \longrightarrow \Omega^2 F(T_i) \longrightarrow k_{H'}{}^{\uparrow H} \longrightarrow 0$$

$$(2)\ 0 \longrightarrow \Omega^{-1}\begin{pmatrix} k_H \\ 2 \end{pmatrix} \longrightarrow \Omega F(S) \longrightarrow \begin{pmatrix} k_H \\ k_H \end{pmatrix} \longrightarrow 0.$$

It follows from Lemma 2.3 that the tilting complex defined by $\{3\}$, $\{2,3\}$, $\{2,3\}$ gives a derived equivalence between $B_0(G)$ and $B_0(H)$, and actually this equivalence is splendid, as desired.

Combining results in [6], [3], [4] and Theorem 2.1 we have the following.

Corollary 2.4 *Let q be a power of a prime such that 3 divides $q+1$ and 3^2 does not divide $q+1$. Then the principal blocks $B_0(PGL(3,q^2))$ and $B_0(PGU(3,q^2))$ are splendid equivalent.*

3 $GL(3,q^2)$ and $GU(3,q^2)$

Let q be a power of a prime such that 3^2 divides $q+1$.

Theorem 3.1 (Kunugi-Okuyama)
(1) *The blocks $B_0(PSL(3,q^2))$ and $B_0(PSU(3,q^2))$ are splendid equivalent.*
(2) *The blocks $B_0(SL(3,q^2))$ and $B_0(SU(3,q^2))$ are splendid equivalent.*

Let P be a common Sylow 3-subgroup of $SL(3,q^2)$ and $SU(3,q^2)$. Let Q_0 be a unique subgroup of P of order 3^a(up to conjugate) such that $B_0(C_{SL(3,q^2)}(Q_0))$ is not Morita equivalent to $B_0(C_{SU(3,q^2)}(Q_0))$, where 3^a is the highest power of 3 dividing $q+1$. As in §2, we construct a complex

$$M^\bullet : 0 \longrightarrow N \xrightarrow{\phi} M \longrightarrow 0$$

where ϕ is a composition of $\pi : M \to k_{SL(3,q^2)\times SU(3,q^2)}$, a $\Delta(P)$-projective cover of $k_{SL(3,q^2)\times SU(3,q^2)}$, and $\iota : N \to \Omega_{\Delta(P)}(k_{SL(3,q^2)\times SU(3,q^2)})$, a $\Delta(Q_0)$-projective cover of $\Omega_{\Delta(P)}(k_{SL(3,q^2)\times SU(3,q^2)})$. Then,

$$M^\bullet \otimes M^{\bullet *} \cong 0 \to B_0(SL(3,q^2)) \oplus X \to 0$$

where X is a $\Delta(Z(P))$-projective p-permutation module. Put $F' = - \otimes \overline{M}^\bullet$, where $\overline{M}^\bullet = \mathrm{Inv}_{Z(P)\times 1}(M^\bullet)$. Then F' induces a stable equivalence between $B_0(PSL(3,q^2))$ and $B_0(PSU(3,q^2))$. To show (1), we need to show the same statement as in Lemma 2.2. The statement for (2) follows from (1) and a fact that the functor $Inv_{Z(P)\times 1}(-)$ induces a one to one correspondence between the set of the trivial source $k[SL(3,q^2)\times SU(3,q^2)]$- modules with vertex $\Delta(Z(P))$ and the set of the indecomposable projective $k[PSL(3,q^2)\times PSU(3,q^2)]$-modules.

We also have the following result.

Theorem 3.2 (Kunugi-Okuyama)
(1) *The blocks $B_0(PGL(3,q^2))$ and $B_0(PGU(3,q^2))$ are splendid equivalent.*
(2) *The blocks $B_0(GL(3,q^2))$ and $B_0(GU(3,q^2))$ are splendid equivalent.*

Remark 3.3 If a characteristic p of k is bigger than 3 and p divides $q+1$, then $GL(3,q^2)$ and $GU(3,q^2)$ have an abelian Sylow p-subgroup. The corresponding results to Theorem 3.1 and 3.2 have been obtained from results by [5] and [9]

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