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MAXIMAL ATTRACTOR AND INERTIAL SET FOR EGUCHI-OKI-MATSUMURA EQUATION

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1. INTRODUCTION

This is a joint work with Naoto Tanaka (Fukuoka University) and Atsusi Tani (Keio University). We consider following system of equations which was proposed by Eguchi-Oki-Matsumura ([7]):

$$
(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta(-\Delta u + 2u + uv^2), & (x,t) \in Q_T \equiv \Omega \times (0,T), \\ \dfrac{\partial v}{\partial t} = \beta \Delta v + \alpha v(a^2 - u^2 - b^2 v^2), & (x,t) \in Q_T, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0, \quad \dfrac{\partial v}{\partial n} = 0, & (x,t) \in \Gamma_T \equiv \Gamma \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases}
$$

where Ω is a bounded domain in $\mathbf{R}^3$ with smooth boundary $\partial\Omega \equiv \Gamma$ and $T > 0$. Here $u(x,t)$ is the local concentration of the solute atoms, $v(x,t)$ is the local degree of order, respectively. α, β, a, b are positive constants and $\dfrac{\partial}{\partial n}$ is exterior normal derivative to Γ.

It is well known that phase separation is described by so-called Cahn-Hilliard equation which is fourth order parabolic type [5], while the order-disorder transition is described by Allen-Cahn equation [1]. The system (1.1) is a model of simultaneous order-disorder and phase separation in binary alloys. T. Eguchi, K. Oki and S. Matsumura derived the system (1.1) assuming that the order-disorder transformation is second order and that phase separation can not take place in the disorder state, but can in the ordered state.

In our previous work [11], it was proved that there exist a unique local and global solution to problem (1.1). Many authors studied the dynamics of equations describing phase transition (for example, [2], [3], [4], [10], [12]). In this talk we show the existence of a maximal attractor and of an inertial set to problem (1.1). The main theorems are as follows:

Theorem 1. *Let* $H_{\bar{u}}=\{(u,v)\in (H^1(\Omega))' \times L^2(\Omega); \frac{1}{|\Omega|}\langle u,1\rangle = \bar{u}\}$. *For any* $\delta \geq 0$, *the semigroup* $S(t)$ *associated with problem* (1.1) *posesses in* $\mathcal{H}_\delta = \bigcup_{|\bar{u}|\leq\delta} H_{\bar{u}}$ *a maximal attractor* $\mathcal{A}_\delta$ *that is connected.*

Theorem 2. *Let* B_δ *be the absobing set in* $(H^1(\Omega))' \times L^2(\Omega)$ *and* $X_\delta = \overline{\cup_{t\geq t_0} S(t)B_\delta}$. *Then there exists an inertial set* M_δ *for* $(S(t)_{t\geq 0}, X_\delta)$ *which has fractal dimension.*

2. PRELIMINARIES

We shall summarize the results of [11]. First of all the existence theorem is as follows:

Theorem 3. *For any* $(u_0, v_0) \in (H^2(\Omega))^2$ *satisfying the compatibility conditions* $\frac{\partial u_0}{\partial n}\Big|_\Gamma = \frac{\partial v_0}{\partial n}\Big|_\Gamma = 0$, *problem* (1.1) *has a unique local solution* (u, v) *defined on* $Q_{T'}$ *for some* $T' \in (0, T)$ *such that*

$$(2.1)\qquad \begin{aligned} &u \in H^{4,1}(Q_{T'}) \cap C(0, T'; H^2(\Omega)),\\ &v \in L^2(0, T'; H^3(\Omega)) \cap H^1(0, T'; L^2(\Omega)) \cap C(0, T'; H^2(\Omega)). \end{aligned}$$

Here $H^{4,1}(Q_T) = H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^4(\Omega))$.

Theorem 4. *Under the same assumptions of Theorem* 3, *problem* (1.1) *admits a unique global solution* (u, v) *on* Q_T *for any* $T > 0$.

Problem (1.1) is a gradient flow and it has the Lyapunov functional

$$(2.2)\qquad J(u,v) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + \frac{\beta}{2\alpha}|\nabla v|^2 - \frac{a^2}{2}v^2 + \frac{b^2}{4}v^4 + u^2 + \frac{1}{2}u^2v^2\Big)dx,$$

which satisfies

$$(2.3)\qquad \frac{d}{dt}J(u,v) + \int_\Omega \Big(|\nabla K(u,v)|^2 + \frac{1}{\alpha}|v_t^2|\Big)dx = 0,$$

where $K(u, v) \equiv -\Delta u + 2u + uv^2$. From (2.3), we have

Lemma 5. *If* (u, v) *satisfies problem* (1.1), *then*

$$(2.4)\qquad \begin{aligned} &\frac{1}{2}\|\nabla u\|^2 + \|u\|^2 + \frac{\beta}{2\alpha}\|\nabla v\|^2 + \frac{b^2}{8}\|v\|_{L^4}^4 + \|uv\|^2\\ &+\int_0^t ds \int_\Omega \Big(|\nabla K(u,v)(s)|^2 + \frac{1}{\alpha}|v_t(s)|^2\Big)\, dx \le J(u_0, v_0) + \frac{a^4}{2b^2}|\Omega|. \end{aligned}$$

Moreover we can obtain the boundedness of the norm $\|v(t)\|_{L^\infty}$.

Lemma 6. *The estimate*

$$(2.5)\qquad \sup_{t>0}\|v(t)\|_{L^\infty} \le C \max\Big\{\|v_0\|_{L^\infty},\ \sup_{t>0}\|v(t)\|\Big\}$$

is valid for the solution (u, v) *to problem* (1.1).

3. The Maximal Attractor

Let $H = (H^1(\Omega))' \times L^2(\Omega)$. We define semigroup $S(t)$ associated to problem (1.1) by $(u(t), v(t)) = S(t)(u_0, v_0)$. Theorems 3 and 4 yield that $(u, v) = S(\cdot)(u_0, v_0) \in C(0, \infty; H)$, and that the mapping $(u_0, v_0) \mapsto (u, v)$ is a continuous operator from H to H. For $u \in (H^1(\Omega))'$ let Nu be the solution of boundary value problem

$$(3.1)\qquad \begin{cases} -\Delta\psi = u - \bar{u}, \quad x \in \Omega,\\ \dfrac{\partial\psi}{\partial n} = 0, \quad x \in \Gamma,\\ \displaystyle\int_\Omega \psi(x)dx = 0 \end{cases}$$

and put

$$\|u\|_{-1}^2 = \int_\Omega |\nabla\psi|^2 dx + |\Omega|\bar{u}^2.$$

To apply theorem I.1.1 in [13], it is necessary to show

Theorem 7. *For any* $\delta \ge 0$, *there exist absorbing sets in* $\mathcal{H}_\delta$ *and in* $(H^2(\Omega))^2 \cap \mathcal{H}_\delta$ *for semigroup* $S(t)$ *associated to problem* (1.1).

Proof of Theorem 7. We first consider the existence of an absorbing set in $\mathcal{H}_\delta$. Multiplying the equation for u by ψ and the equation for v by v and integrating by parts respectively, we get

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2}\|v\|^2) + \|\nabla u\|^2 + \frac{2\beta\bar{u}^2}{\alpha b^2}\|\nabla v\|^2 + \|u\|^2 + (1+\frac{2\bar{u}^2}{b^2})\int_\Omega u^2v^2dx + \frac{3}{4}\bar{u}^2\|v\|_{L^4}^4 \le (2+\frac{a^4}{b^4})\bar{u}^2|\Omega|.$$

And we use the inequalities

$$\frac{\lambda_2}{2}\|u\|_{-1}^2 \le \frac{1}{2}\|u\|^2 + \frac{\lambda_2}{2}\bar{u}^2|\Omega|,$$
$$\frac{\lambda_2}{2}\frac{2\bar{u}^2}{\alpha b^2}\|v\|^2 \le \frac{\bar{u}^2}{4}\|v\|_{L^4}^4 + (\frac{\lambda_2}{\alpha b^2})^2\bar{u}^2|\Omega|,$$

where λ_2 is the least positive eigenvalue of the $-\Delta$ with homogeneous Neumann boundary condition to obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2}\|v\|^2) + \frac{\lambda_2}{2}(\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2}\|v\|^2) + \|\nabla u\|^2 + \frac{2\beta\bar{u}^2}{\alpha b^2}\|\nabla v\|^2 + \frac{1}{2}\|u\|^2$$
$$+(1+\frac{2\bar{u}^2}{b^2})\int_\Omega u^2v^2dx + \frac{1}{2}\bar{u}^2\|v\|_{L^4}^4 \le \left(2+\frac{a^4}{b^4}+\frac{\lambda_2}{2}+(\frac{\lambda_2}{\alpha b^2})^2\right)\bar{u}^2|\Omega| \equiv \frac{C_1}{2}.$$

And we have

$$\frac{d}{dt}(\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2}\|v\|^2) + \lambda_2(\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2}\|v\|^2) \le C_1. \tag{3.2}$$

Applying Gronwall's lemma to (3.2) we deduce for all

$$\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2}\|v\|^2 \le \left(\|u_0\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2}\|v_0\|^2\right)e^{-\lambda_2 t} + \frac{C_1}{\lambda_2}(1-e^{-\lambda_2 t}).$$

We have obtained an absorbing set in $\mathcal{H}_\delta$.

Next we show the existence of an absorbing set in $(H^2(\Omega))^2 \cap \mathcal{H}_\delta$. Multiplying the equation for u by $\Delta^2 u$ and integrating by parts yield

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \frac{1}{2}\|\Delta^2 u\|^2 + 2\|\nabla\Delta u\|^2 \le \frac{1}{2}\|\Delta(uv^2)\|^2. \tag{3.3}$$

Multiplying the equation for v by $\Delta^2 v$ and integrating by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta v\|^2 + \beta\|\nabla\Delta u\|^2 &= -\alpha\int_\Omega \nabla[v(a^2-u^2-b^2v^2)]\cdot\nabla\Delta v dx\\ &\le \frac{\beta}{2}\|\nabla\Delta v\|^2 + \frac{3\alpha^2}{2\beta}(a^4\|\nabla v\|^2 + \|\nabla(vu^2)\|^2 + 9b^4\|v^2\nabla v\|^2).\end{aligned} \tag{3.4}$$

Using Lemmas 5 and 6, the right hand side of (3.3) leads to

$$\begin{aligned}\|\Delta(uv^2)\|^2 &= \|v^2\Delta u + 2uv\Delta v + 4v\nabla u\cdot\nabla v + 2u|\nabla v|^2\|^2\\ &\le C(\|v\|_{L^\infty}^4\|\Delta u\|^2 + \|u\|_{L^\infty}^2\|v\|_{L^\infty}^2\|\Delta v\|^2 + \|v\|_{L^\infty}^2\|\nabla u\|_{L^4}^2\|\nabla v\|_{L^4}^2 + \|u\|_{L^\infty}^2\|\nabla v\|_{L^4}^4)\\ &\le C(\|\Delta u\|^2 + \|u\|_{L^\infty}^2\|\Delta v\|^2 + \|\nabla u\|_{L^4}^2\|\nabla v\|_{L^4}^2 + \|u\|_{L^\infty}^2\|\nabla v\|_{L^4}^4)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\|\nabla(vu^2)\|^2 &= \|u^2\nabla v + 2uv\nabla u\|^2\\ &\le C(\|\nabla v\|^2\|u\|_{L^\infty}^4 + \|v\|_{L^\infty}^2\|u\|_{L^\infty}^2\|\nabla u\|^2)\\ &\le C(\|u\|_{L^\infty}^4 + \|u\|_{L^\infty}^2).\end{aligned} \tag{3.6}$$

Using the inequalities (see, [13] p.161 and p.52)

$$\|\nabla v\|_{L^4} \le C\|\nabla v\|_{H^{\frac{3}{4}}} \le C\|\nabla v\|^{\frac{1}{4}}\|\nabla^2 v\|^{\frac{3}{4}},$$
$$\|u\|_{L^\infty} \le C\|u\|_{H^1}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}},$$

we find, for example,

$$\begin{aligned}\|u\|_{L^\infty}^2\|\nabla v\|_{L^4}^4 &\le C(\|u\|_{H^1}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}})^2(\|\nabla v\|^{\frac{1}{4}}\|\nabla^2 v\|^{\frac{3}{4}})^4\\ &\le C\|\Delta u\|\|\Delta v\|^3 + C'\\ &\le C(\|\Delta u\|^4 + \|\Delta v\|^4) + C'.\end{aligned}$$

After some similar calculation, we obtain that

$$\frac{d}{dt}(\|\Delta u\|^2 + \beta\|\Delta v\|^2) \le C_2(\|\Delta u\|^2 + \beta\|\Delta v\|^2)(\|\Delta u\|^2 + \beta\|\Delta v\|^2) + C_3. \tag{3.7}$$

Multiplying the equations for u by u and the equations for v by Δv and integrating by parts respectively, we have

$$\frac{d}{dt}(\|u\|^2 + \|\nabla v\|^2) + \|\Delta u\|^2 + \beta\|\Delta v\|^2 \le C_4. \tag{3.8}$$

Here we have used Lemmas 5 and 6. By integrating (3.8), we find that the conditions of the uniform Gronwall lemma ([13] p.91) hold. Therefore we have

$$\|\Delta u(t)\|^2 + \beta\|\Delta v(t)\|^2 \le (C_5 + C_4 + C_3)e^{C_2(C_5+C_4)} \tag{3.9}$$

for $t \ge 1$. From (3.9) we conclude the existence of an absorbing set in $(H^2(\Omega))^2 \cap \mathcal{H}_\delta$. □

4. The Inertial Set

Let B_δ be the absorbing set in $(H^2(\Omega))^2 \cap \mathcal{H}_\delta$ from Theorem 7 and put $X_\delta = \overline{\cup_{t\ge t_0} S(t)B_\delta}$. We note that X_δ is bounded in $(C(\bar{\Omega}))^2$.

Lemma 8. *The semigroup $S(t) : X_\delta \to X_\delta$ is Lipschitz continuous, i.e.,*

$$\|(u_1 - u_2, v_1 - v_2)\|_H^2 \le \|(u_{01} - u_{02}, v_{01} - v_{02})\|_H^2 e^{2dt}, \tag{4.1}$$

where (u_i, v_i) is the solutions of (1.1) *with the initial conditions $(u_{0i}, v_{0i}), i = 1, 2$, and $\|(u,v)\|_H^2 = \|u\|_{-1}^2 + \|v\|^2$.*

Proof of Lemma 8. The difference of solution $(u_1 - u_2, v_1 - v_2)$ satisfies

(4.2)
$$\begin{cases}\dfrac{\partial(u_1 - u_2)}{\partial t} = \Delta(-\Delta(u_1 - u_2) + 2(u_1 - u_2) + u_1 v_1^2 - u_2 v_2^2),\\ \dfrac{\partial(v_1 - v_2)}{\partial t} = \beta\Delta(v_1 - v_2) + \alpha(v_1 - v_2)(a^2 - u_1^2 - b^2 v_1^2) - \alpha v_2\{(u_1^2 - u_2^2) + b^2(v_1^2 - v_2^2)\},\\ \dfrac{\partial(u_1 - u_2)}{\partial n} = \dfrac{\partial\Delta(u_1 - u_2)}{\partial n} = 0, \quad \dfrac{\partial(v_1 - v_2)}{\partial n} = 0, \qquad (x,t) \in \Gamma_T,\\ u_1(x,0) - u_2(x,0) = u_{01}(x) - u_{02}(x), \quad v_1(x,0) - v_2(x,0) = v_{01}(x) - v_{02}(x), \quad x \in \Omega,\end{cases}$$

Let ψ_i be the solution of (3.1) with replacing $u - \bar{u}$ by $u_i - \bar{u}, i = 1, 2$. Multiplying the first equation of (4.2) by $\psi = \psi_1 - \psi_2$ and multiplying the second equation of (4.2) by $v_1 - v_2$ and integrating by parts respectivly, we get

$$\begin{aligned}(4.3)\qquad &\frac{1}{2}\frac{d}{dt}\|(u_1-u_2,v_1-v_2)\|_H^2+\|\nabla(u_1-u_2)\|^2+\|u_1-u_2\|^2+\beta\|\nabla(v_1-v_2)\|^2\\ &+\int_\Omega v_1^2(u_1-u_2)^2dx+\alpha\int_\Omega(v_1-v_2)^2(u_1^2+b^2v_1^2)dx\le\ d\|v_1-v_2\|^2,\end{aligned}$$

where $d=\alpha a^2+2\alpha b^2M^2+\frac{1}{4}M^2(\alpha+4)(1+4\alpha)$ and M is a constant such that $\|(u_i,v_i)\|_{C(\bar\Omega\times[0,\infty))}\le M$ Applying Gronwall lemma leads to (4.1). □

Moreover we find from (4.3)

Corollary 9.

$$(4.4)\qquad \int_0^t\|u_1-u_2\|_{H^1}^2e^{Ds}ds\le\frac{1}{2}(1+e^{(D+2d)}t)\|(u_{10}-u_{20},v_{10}-v_{20})\|_H^2,$$

where D is a constant.

Proof of Corollary 9. From (4.3), we have

$$(4.5)\qquad \frac{1}{2}\frac{d}{dt}\|(u_1-u_2,v_1-v_2)\|_H^2+\|u_1-u_2\|_{H^1}^2\le d\|v_1-v_2\|.$$

Multiplying (4.5) by e^{Ds} and integrating, we have

(4.6)

$$\int_0^t\|u_1-u_2\|_{H^1}^2e^{Ds}ds\le\frac{1}{2}\|(u_{10}-u_{20},v_{10}-v_{20})\|_H^2+(\frac{1}{2}D+d)\int_0^t\|(u_1-u_2,v_1-v_2)\|_H^2e^{Ds}ds.$$

By using (4.1),

$$\begin{aligned}(4.7)\qquad \int_0^t\|u_1-u_2\|_{H^1}^2e^{Ds}ds\le&\frac{1}{2}\|(u_{10}-u_{20},v_{10}-v_{20})\|_H^2\\ &+(\frac{1}{2}D+d)\|(u_{10}-u_{20},v_{10}-v_{20})\|_H^2\int_0^te^{(D+2d)s}ds\\ \le&\frac{1}{2}(1+e^{(D+2d)t})\|(u_{10}-u_{20},v_{10}-v_{20})\|_H^2.\end{aligned}$$

□

Next we shall show the squeezing property of $S(t)$. We denote by $\lambda_i,(0=\lambda_1<\lambda_2\le\cdots\le\lambda_i\le\cdots)$ the eigenvalue of the operator $-\Delta$ with homogeneous Neumann boundary conditions and w_i the coressponding eigenfunctions such that $\|w_i\|_{L^2}=1,i=1,2,\cdots$. It is well-known that $\{w_i\}_{i=1}^\infty$ are a complete orthogonal basis in $L^2(\Omega)$. Let $H_n=\text{span}\{w_1,\cdots,w_n\}$ and the operator $p_n:(H^1(\Omega))'\longmapsto H_n$ be orthogonal projection and $q_n=I-p_n$, where I is identity on $(H^1(\Omega))'$. Then it holds that

$$(4.8)\qquad \|\varphi\|_{-1}^2\le\frac{1}{\lambda_{n+1}}\|\varphi\|^2\le\frac{1}{\lambda_{n+1}^2}\|\nabla\varphi\|^2$$

for any $\varphi\in q_n((H^1(\Omega))')$. Furthermore we define the corresponding product projection $P_n(u,v)$, $Q_n(u,v)$ on H such that $P_n(u,v)=(p_nu,p_nv),Q_n=I-P_n$.

Theorem 10. *Semigroup $S(t)$ for problem (1.1) possesses the squeezing property, i.e., for any $t^*>0$ there exists number $n_0=n_0(t^*)$ such that for any $\Psi_1=(u_1,v_1),\Psi_2=(u_2,v_2)\in X_\delta$ satisfying that if*

$$(4.9)\qquad \|p_{n_0}(S(t^*)\Psi_1-S(t^*)\Psi_2)\|_H\le\|(I-p_{n_0})(S(t^*)\Psi_1-S(t^*)\Psi_2)\|_H,$$

then

$$(4.10)\qquad \|S(t^*)\Psi_1-S(t^*)\Psi_2\|_H\le\frac{1}{8}\|\Psi_1-\Psi_2\|_H.$$

Proof of Theorem 10. We set $(U,V) \equiv Q_n(u_1-u_2, v_1-v_2)$. Operating the equation (4.2) by Q_n, it hold that

$$\begin{cases} \dfrac{\partial U}{\partial t} = \Delta(-\Delta U + 2U + q_n(u_1v_1^2 - u_2v_2^2)), \\ \dfrac{\partial V}{\partial t} = \beta\Delta V + \alpha a^2 V - \alpha q_n(v_1(u_1^2+b^2v_1^2) - v_2(u_2^2+b^2v_2^2)). \end{cases} \tag{4.11}$$

Multiplying the first equation of (4.11) by $NU \in q_n(H^2(\Omega))$ and the second equation of (4.11) by V and integrating, we have

(4.12)
$$\frac{1}{2}\frac{d}{dt}\|(U,V)\|_H + \|\nabla U\|^2 + 2\|U\|^2 + \beta\|\nabla V\|^2 \le \|U\|^2 + \alpha(a^2+1)\|V\|^2 + \frac{1}{4}\int_\Omega (u_1v_1^2 - u_2v_2^2)^2 dx$$
$$+\frac{\alpha}{4}\int_\Omega [v_1(u_1^2+b^2v_1^2) - v_2(u_2^2+b^2v_2^2)]^2 dx.$$

Using the inequalities

$$\begin{cases} (\lambda_{n+1}^2 + \lambda_{n+1})\|U\|_{-1}^2 \le \|\nabla U\|^2 + \|U\|^2, \\ \beta\lambda_{n+1}\|V\|^2 \le \beta\|\nabla V\|^2, \end{cases} \tag{4.13}$$

it yields that

$$\frac{d}{dt}\|(U,V)\|_H^2 + (D_1\lambda_{n+1} - D_2)\|(U,V)\|_H^2 \le C(\|u_1-u_2\|^2 + \|v_1-v_2\|^2), \tag{4.14}$$

where $D_1 = 2\min\{1,\beta\}$, $D_2 = 2\alpha(a^2+1)$. Applying Gronwall's lemma leads to

$$\begin{aligned} \|(U,V)\|_H^2 \le & \|(U,V)(0)\|_H^2 e^{-Dt} + \varepsilon e^{-Dt}\int_0^t \|u_1-u_2\|_{H^1}^2 e^{Ds} ds \\ & + C_\varepsilon e^{-Dt}\int_0^t (\|u_1-u_2\|_{-1}^2 + \|v_1-v_2\|^2) e^{Ds} ds, \end{aligned} \tag{4.15}$$

where $D = D_1\lambda_{n+1} - D_2$. Here we use the inequality $\|u_1-u_2\|^2 \le C\|u_1-u_2\|_{-1}\|u_1-u_2\|_{H^1}$. From Lemma 8 and Corollary 9, it holds that

$$\|(U,V)(t)\|_H^2 \le \|(u_{10}-u_{20}, v_{10}-v_{20})\|_H^2 \left\{ e^{-Dt} + \frac{\varepsilon}{2}(e^{-Dt} + e^{2dt}) + C_\varepsilon \frac{e^{2dt}}{D+2d} \right\}. \tag{4.16}$$

Now assume that

$$\|p_{n_0}(u_1-u_2, v_1-v_2)(t^*)\|_H \le \|Q_{n_0}(u_1-u_2, v_1-v_2)(t^*)\|_H \tag{4.17}$$

for $t^* > 0$, then by using (4.16)

(4.18)
$$\begin{aligned} \|(u_1-u_2, v_1-v_2)(t^*)\|_H^2 &\le 2\|Q_{n_0}(u_1-u_2, v_1-v_2)(t^*)\|_H^2 \\ &\le \|(u_{10}-u_{20}, v_{10}-v_{20})\|_H^2 \left\{ e^{-D_3t^*} + \frac{\varepsilon}{2}(e^{-D_3t^*} + e^{2dt^*}) + C_\varepsilon \frac{e^{2dt^*}}{D_3+2d} \right\}, \end{aligned}$$

where $D_3 = D_1\lambda_{n_0+1} - D_2$. Taking $\varepsilon > 0$ so small that

$$\varepsilon(e^{-D_3t^*} + e^{2dt^*}) \le \frac{1}{128} \tag{4.19}$$

and choosing a number n_0 sufficiently large so as to satisfy

$$2(e^{-D_3t^*} + \frac{C_\varepsilon e^{2dt^*}}{D_3+2d}) \le \frac{1}{128}, \tag{4.20}$$

then we conclude

$$\|(u_1 - u_2, v_1 - v_2)(t^*)\|_H^2 \le \left(\frac{1}{8}\right)^2 \|(u_{10} - u_{20}, v_{10} - v_{20})\|_H^2. \tag{4.21}$$

□

Therefore the proof of Theorem 2 is completed if we apply Theorem 2.1 in [6].

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