数理解析研究所講究録 1358 巻 2004 年 77-97

Blow-up for nonlinear wave equations with multiple speeds

大阪大学・理 久保 英夫 (Hideo Kubo)
埼玉大学・理 太田 雅人 (Masahito Ohta)

1. Introduction

In this note we consider the following nonlinear system of wave equations with multiple speeds of propagation in three space dimensions:

$$\begin{cases} (\partial_t^2 - c_1^2\Delta)u_1 = |u_1|^{p_1}|u_2|^{p_2}, & (t,x) \in [0,\infty) \times \mathbb{R}^3, \\ (\partial_t^2 - c_2^2\Delta)u_2 = |u_1|^q, & (t,x) \in [0,\infty) \times \mathbb{R}^3 \end{cases} \tag{1.1}$$

with the initila data

$$u_j(0,x) = \varphi_j(x), \quad \partial_t u_j(0,x) = \psi_j(x), \quad x \in \mathbb{R}^3 \ (j = 1,2). \tag{1.2}$$

Here p_1, $p_2 \geq 1$, $q > 1$, $c_j > 0$ and $\varphi_j \in C^3(\mathbb{R}^3)$, $\psi_j \in C^2(\mathbb{R}^3)$ $(j = 1,2)$.

The main question here is formulated as follows.

Problem: Find sharp condition about the **small data global existence** and **blow-up** for (1.1). Here **small data global existence** means that the initial value problem (1.1)–(1.2) admits a unique global (mild) solution for all "small" initial data. On the contrary, we say **blow-up** occurs if **small data global existence** dose NOT hold. In other words, it means that one can find a pair of intial data (φ_j, ψ_j) such that the lifespan of the corresponding solution is finite.

We are going to answer the above problem based on the work [17]. Before going further, we recall several related results to our problem. The following system was studied by Del Santo, Georgiev and Mitidieri [5]:

$$\begin{cases} (\partial_t^2 - c_1^2\Delta)u_1 = |u_2|^p, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ (\partial_t^2 - c_2^2\Delta)u_2 = |u_1|^q, & (t,x) \in [0,\infty) \times \mathbb{R}^n \end{cases} \tag{1.3}$$

where p, $q > 1$ and $n \geq 2$. They found the **critical curve** $\Gamma(p,q) = 0$ in p-q plane when $c_1 = c_2$. Here **critical curve** means that if $\Gamma(p,q) > 0$, then **small data global**

existence holds, and otherewise **blow-up** occurs. The function $\Gamma(p,q)$ is defined as follows:

$$\Gamma(p,q)=\max\left\{\frac{q+2+p^{-1}}{pq-1},\frac{p+2+q^{-1}}{pq-1}\right\}-\frac{n-1}{2}. \tag{1.4}$$

The **blow-up** part was also established by Deng [6] independently. The critical case where $\Gamma(p,q)=0$ was treated independently by [2] for $n=3$ and by [15] for $n=2$, 3. In these works the **blow-up** result was obtained.

Next the authors studied the case of $c_1\neq c_2$ in [16]. This work is motivated by the results established by Kovalyov [14], Agemi and Yokoyama [3], Hoshiga and Kubo [11] and Yokoyama [27]. In those papers, **small data global existence** for systems of nonlinear wave equations with different propagation speeds has been well developed when the nonlinear terms depend only on the derivatives of unknown functions but not on unknown functions themselves (see also [24] and [1] for related results on nonlinear elastic wave equations, and [21] on Klein-Gordon-Zakharov equations). It was shown in [16] that even if $c_1\neq c_2$, the ctitical curve is the same as in the case of $c_1=c_2$ for $n=3$. Recently the authors extend the result to the two dimensional case in [18]. Therefore we see that the unequal propagation speeds dose not have major effect on the system (1.3).

On the cantrary, the following system has different structure according to the propagtion speeds:

$$\begin{cases}(\partial_t^2-c_1^2\Delta)u_1=\lambda_1|u_1|^{p_1}|u_2|^{p_2}, & (t,x)\in[0,\infty)\times\mathbb{R}^n,\\ (\partial_t^2-c_2^2\Delta)u_2=\lambda_2|u_1|^{q_1}|u_2|^{q_2}, & (t,x)\in[0,\infty)\times\mathbb{R}^n\end{cases} \tag{1.5}$$

where p_1, p_2, q_1, $q_2\geq 1$, λ_1, $\lambda_2\in\mathbb{R}$ and $n\geq 2$. Without losing such structure, we may assume that there is $\alpha\geq 2$ such that

$$p_1+p_2=q_1+q_2\equiv\alpha. \tag{1.6}$$

This condition means that the degree of the nonlinearlity of the first equation is the same as that of the second one.

When $c_1=c_2$, it follows from the result about the single wave equation

$$(\partial_t^2-c^2\Delta)u=|u|^p,\quad (t,x)\in[0,\infty)\times\mathbb{R}^n,\quad p>1, \tag{1.7}$$

that **small data global existence** holds if $\alpha > p_0(n)$ and that **blow-up** occurs if $2 \leq \alpha \leq p_0(n)$. Here $p_0(n)$ is the positive root of the following quadratic equation:

$$p[\frac{n-1}{2}p - \frac{n+1}{2}] = 1. \tag{1.8}$$

(For the detail about (1.7), see Section 2 below.)

Next we turn our attention to the case of $c_1 \neq c_2$. When $n = 3$, [19] firstly proved **small data global existence** for all $\alpha > 2$. Then [16] showed that the same is true for $\alpha = 2$. Let us compare these results with those for the case of $c_1 = c_2$. Since $p_0(3) = 1 + \sqrt{2}$, we find that there is a significant differenece among them when $2 \leq \alpha \leq 1 + \sqrt{2}$. Actually, for such α we have a global solution if $c_1 \neq c_2$, while **blow-up** occurs if $c_1 = c_2$. This obserbation exploits the effect of the discrepancy between the propagation speeds, which comes from the way of interaction in the nonlinearlties (recall that we don't have such effect for the system (1.3)). In fact, since the right hand side of the equations in (1.5) are involved by a product of u_1 and u_2, one can compensate the deficiency of the pointwise decaying order for the powers of u_1 and u_2 each other, based on the the discrepancy between the propagation speeds. Recently the following extention to the two spatial dimensional case was done by [18]: Let $c_1 \neq c_2$ and $n = 2$. If $\alpha > 3$, then **small data global existence** holds. On the contrary, if $2 \leq \alpha \leq 3$, then **blow-up** occurs. Therefore, when $3 \leq \alpha \leq p_0(2) = (3 + \sqrt{17})/2$, we have the effect of the unequal propagtion speeds as in the three spatial dimensional case.

Now the following question naturally arises: What will happen for the intermediate case between (1.3) and (1.5), like (1.1)? The point is that the right hand side of the first equation in (1.1) is involved by a product of u_1 and u_2, while that of the second one does not. For simplicity, we focus on the case where

$$p_1 = p_2 = 1. \tag{1.9}$$

The exposition for the general case where $p_1 \geq 1$, $p_2 \geq 2$ is complicated, although the real proof for large values of p_1 and p_2 is easier because of the "smallness" of solutions under our consideration. For this reason, we prefer to take $p_1 = p_2 = 1$. Our main result of this note is roughly stated as follows.

Theorem 1. (*Theorems 1.4 and 1.5 in* [17]) *Suppose that* $c_1 \neq c_2$ *and* $\varphi_j \in C^3(\mathbb{R}^3)$, $\psi_j \in C^2(\mathbb{R}^3)$ $(j = 1, 2)$. *Then for the initial value problem* (1.1)–(1.2) *with* (1.9) *we have:*

(i) *If* $1 < q < 3$, *then* **blow-up** *occurs.*

(ii) *If* $q > 3$, *then* **small data global existence** *holds.*

(iii) *Let* $q = 3$. *If* $c_1 > c_2$, *then* **blow-up** *occurs. While, when* $c_1 < c_2$, **small data global existence** *holds.*

Remark 1. *1) The statements of the theorem remains true, even if we replace the nonlinear terms* $|u_1||u_2|$, $|u_1|^q$ *in* (1.1) *by* u_1u_2, $|u_1|^{q-1}u_1$, *respectively.*

2) The case of common propagation speeds, i.e., $c_1 = c_2$ *can be treated analogously to the system* (1.3). *Notice that* $(p, q) = (2, 7/2)$ *is on the critical curve* $\Gamma(p, q) = 0$ *when* $n = 3$. *Therefore* **small data global existence** *holds if* $q > 7/2$, *while* **blow-up** *occurs if* $1 < q \leq 7/2$.

This note is organized as follows. In the next section we discuss the single wave equations in order to present a general idea to show blow-up result. Section 3 is devoted to a key lemma (Lemma 6) which provides a significant generalization of earlier estimates by John [12], Zhou [28] and the authors [16]. In Section 4, we prove the blow-up part of Theorem 1.

2. Single wave equation

This section is concerned with the initial value problem to (1.7) with

$$u(0, x) = \varphi(x), \quad \partial_t u(0, x) = \psi(x), \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R}^n)$. For the problem Strauss [25] introduced the number $p_0(n)$ which is the positive root of (1.8). The importance of this number is the fact that it plays the role as the critical exponet for the problem (1.7)–(2.1). Though the number seems to be strange at fisrt glance, one can understand it based on the **scaling invariance** of the semilinear equation. The **scaling invariance** means that if $u(t, x)$ is a solution of (1.7), then $D_{\lambda,p}u(t, x)$ also satisfies the same equation for all $\lambda > 0$, where we denoted by $D_{\lambda,p}u(t, x)$ the dilation of $u(t, x)$ defined by

$$D_{\lambda,p}u(t, x) = \lambda^{\frac{2}{p-1}}u(\lambda t, \lambda x) \quad (\lambda > 0). \tag{2.2}$$

Then the quadratic equation (1.8) follows from the self-similarity of the function

$$\dot{w}(r,t) = (t+r)^{-\frac{n-1}{2}}|ct-r|^{-(\frac{n-1}{2}p-\frac{n+1}{2})} \quad \text{for} \quad r,t \in [0,\infty).$$

Namely, if $p = p_0(n)$, then we have the dilation invariance $D_{\lambda,p_0(n)}\dot{w}(|x|,t) = \dot{w}(|x|,t)$ for all $\lambda > 0$.

Now we briefly mention known results. It was shown that **blowup** occurs for either $1 < p < p_0(n)$ or $p = p_0(n)$ and $n = 2, 3$ (see Sideris [23], Schaeffer [22]). Notice that due to the "bad" sign of the nonlinearlity, the solution likely blows up for small values of p.

On the other hand, the existence part was firstly solved by John [12] for $n = 3$. In the sequel, there are so many contribution on this issue. (See e.g., [9, 10, 20, 28] and the references cited therein). For general $n \geq 2$, Georgiev, Lindblad and Sogge [8] showed that **small data global existence** holds by proving the weighted version of Strichartz estimate, when $p_0(n) < p < (n+3)/(n-1)$ and the initial data is compactly supported. The proof of the weighted Strichartz estimate is simplified by Georgiev [7], Tataru [26] independently by using the Fourier transform on the hyperbolid. Finally, D'Ancona, Georgiev and Kubo [4] relaxed the assumption on the initial data.

In the rest of this section we sketch the proof of the **blow-up** result for the case of $n = 3$. Suppose that $u(t,x)$ is a classical solution of the problem (1.7)–(2.1). Then it satisfies the following integral equation:

$$u = K_c[\varphi,\psi] + L_c[|u|^p] \quad \text{in} \quad [0,\infty) \times \mathbb{R}^3, \tag{2.3}$$

where we put

$$K_c[\varphi,\psi](t,x) = J_c[\psi](t,x) + \partial_t J_c[\varphi](t,x), \tag{2.4}$$

$$L_c[F](t,x) = \int_0^t J_c[F(s,\cdot)](t-s,x)\,ds. \tag{2.5}$$

Here $J_c[\psi](t,x)$ is defined by

$$J_c[\psi](t,x) = \frac{t}{4\pi}\int_{|\omega|=1} \psi(x+ct\omega)\,d\omega, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3, \tag{2.6}$$

We take the initial data in such a way that

$$\varphi(x) = 0, \quad \psi(x) = \varepsilon g(x), \tag{2.7}$$

where $\varepsilon > 0$ and $g \in C(\mathbb{R}^3)$ satisfies

$$g(x) \geq 0 \quad \text{for} \quad \text{all } x \in \mathbb{R}^3, \quad g(0) > 0. \tag{2.8}$$

Then we have the following result.

Theorem 2. *Let $n = 3$ and $1 < p \leq p_0(3)$. Suppose that $\varepsilon \in (0, 1]$ and $g \in C(\mathbb{R}^3)$ satisfies (2.8). Then the solution of (2.3) with (2.7) blows up in a finite time $T^*(\varepsilon)$. Moreover, there exists a positive constant C^* independent of ε such that*

$$T^*(\varepsilon) \leq \begin{cases} \exp(C^* \varepsilon^{-p(p-1)}) & \text{if } p = p_0(3), \\ C^* \varepsilon^{-p(p-1)/(1-pp^*)} & \text{if } 1 < p < p_0(3). \end{cases} \tag{2.9}$$

In order to prove Theorem 2, we prepare a couple of estimates, and Lemma 2 and Proposition 1 below. By (2.3), (2.7) and (2.8), we have

$$u(t, x) \geq \varepsilon J_c[g](t, x), \qquad (t, x) \in [0, \infty) \times \mathbb{R}^3, \tag{2.10}$$

$$u(t, x) \geq L_c[|u|^p](t, x), \qquad (t, x) \in [0, \infty) \times \mathbb{R}^3, \tag{2.11}$$

Moreover, by (2.8), there exist $\delta > 0$ and $\phi_\delta \in C([0, \infty))$ such that

$$g(x) \geq \phi_\delta(|x|) \geq 0 \text{ for } x \in \mathbb{R}^3, \quad \phi_\delta(\rho) > 0 \text{ for } \rho \in [0, \delta]. \tag{2.12}$$

Note that we may assume that δ is sufficiently small.

In the sequel we shall make use of the following identity.

Lemma 2. *Let $n \geq 2$ and let $g \in C([0, \infty))$. Then we have*

$$\int_{|\omega|=1} g(|x + \rho\omega|) dS_\omega = \frac{2^{3-n}\omega_{n-1}}{(r\rho)^{n-2}} \int_{|\rho-r|}^{\rho+r} \lambda g(\lambda)[h(\lambda, \rho, r)]^{\frac{n-3}{2}} d\lambda \tag{2.13}$$

for $\rho > 0$ and $x \in \mathbb{R}^n$ with $r = |x| > 0$, where $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbb{R}^n$, and $h(\lambda, \rho, r)$ is defined by

$$h(\lambda, \rho, r) = \{\lambda^2 - (\rho - r)^2\}\{(\rho + r)^2 - \lambda^2\}. \tag{2.14}$$

Proof. We put

$$\lambda = |x + \rho\omega|, \quad x \cdot \omega = r\cos\theta \quad (0 \leq \theta \leq \pi).$$

Then we have

$$\lambda^2 = r^2 + 2r\rho\cos\theta + \rho^2, \quad \sin\theta = \frac{[h(\lambda, \rho, r)]^{1/2}}{2r\rho},$$

and

$$\int_{|\omega|=1} g(|x+\rho\omega|)dS_\omega = \int_0^\pi g(\lambda)\omega_{n-1}[\sin\theta]^{n-2}\,d\theta$$

$$= \omega_{n-1}\int_{|\rho-r|}^{\rho+r} g(\lambda)[\sin\theta]^{n-2}\frac{\lambda}{r\rho\sin\theta}\,d\lambda.$$

Thus we obtain (2.13). □

Proposition 1. *Let $G \in C(\mathbb{R}^3)$, $g \in C([0,\infty))$. If $G(x) \geq g(|x|) \geq 0$ for all $x \in \mathbb{R}^3$, then we have*

$$J_c[G](t,x) \geq \frac{1}{2cr}\int_{|r-ct|}^{r+ct} \lambda g(\lambda)\,d\lambda \tag{2.15}$$

for all $(t,x) \in [0,\infty)\times\mathbb{R}^3$, where $r = |x|$.

Moreover, let $F \in C([0,T)\times\mathbb{R}^3)$, $f \in C([0,\infty)\times[0,T))$ with $T > 0$ and suppose that $F(t,x) \geq f(|x|,t) \geq 0$ for all $(t,x) \in [0,T)\times\mathbb{R}^3$. Then we have

$$L_c[F](t,x) \geq \frac{1}{2cr}\iint_{D_c(r,t)} \lambda f(\lambda,s)\,d\lambda\,ds, \tag{2.16}$$

for all $(t,x) \in [0,T)\times\mathbb{R}^3$, where we put

$$\begin{aligned} D_c(r,t) = \{(\lambda,s) \in [0,\infty)^2 :\ & 0 \leq s \leq t, \\ & |r-c(t-s)| \leq \lambda \leq r+c(t-s)\}. \end{aligned} \tag{2.17}$$

Proof. First we prove (2.15). By $G(x) \geq g(|x|)$ for $x \in \mathbb{R}^3$, (2.6) implies $J_c[G](t,x) \geq J_c[g(|\cdot|)](t,x)$ for $(t,x) \in [0,\infty)\times\mathbb{R}^3$. Therefore it is easy to see from (2.6) and Lemma 2 that (2.15) holds for $n = 3$. Moreover, (2.16) follows from (2.5) and (2.15). This completes the proof. □

Now we shall give the proof of Theorem 2. In what follows, we put

$$p^* = p - 2. \tag{2.18}$$

Step 1. We see from (2.12) and Proposition 1 that

$$J_c[g](t,x) \geq \frac{1}{2cr}\int_{|r-ct|}^{r+ct} \lambda\phi_\delta(\lambda)\,d\lambda.$$

Therefore, if $|ct-r| \leq \delta/2$ and $ct+r \geq \delta$, then by (2.10) we have

$$u(t,x) \geq C_0\varepsilon r^{-1} \tag{2.19}$$

where we put $C_0 = (2c)^{-1} \int_{\delta/2}^{\delta} \lambda \phi_\delta(\lambda)\, d\lambda (> 0)$.

Step 2. We shall show that there is a positive constant $C_1 = C_1(g, \delta, c, p)$ such that

$$u(t,x) \geq \frac{C_1 \varepsilon^p}{(ct+r)(ct-r)^{p^*}} \tag{2.20}$$

holds for $c(t-\delta) \geq r = |x|$. Note that if $c(t-\delta) \geq r$, then we have $cs + \lambda \geq c\delta$ for $(\lambda, s) \in D_c(r,t)$. By (2.11), (2.19) and Proposition 1, for $c(t-\delta) \geq r$ we have

$$u(t,x) \geq \frac{C\varepsilon^p}{r} \iint_E \lambda^{1-p}\, d\lambda\, ds \geq \frac{C\varepsilon^p}{r} \iint_E (cs+\lambda)^{-p^*-1}\, d\lambda\, ds,$$

where we put $E = \{(\lambda, s) \in [0,\infty)^2 : \ |cs - \lambda| \leq \delta/2,\ ct - r \leq cs + \lambda \leq ct + r\}$. Changing the variables by

$$\xi = cs + \lambda, \quad \eta = \frac{cs - \lambda}{c}, \tag{2.21}$$

we have

$$u(t,x) \geq \frac{C\varepsilon^p}{r} \int_{-\delta/(2c)}^{\delta/(2c)} d\eta \int_{ct-r}^{ct+r} \frac{d\xi}{\xi^{p^*+1}} = \frac{C\varepsilon^p}{r} \int_{ct-r}^{ct+r} \frac{d\xi}{\xi^{p^*+1}}.$$

By (2.18) we have $p^* + 1 > 0$ for $p > 1$. Thus, using (2.22) below, we arrive at (2.20).

Lemma 3. *Let μ, a, $b > 0$. When $a < b$, there exista a positive constant $C = C(\mu)$ such that*

$$I := \int_{b-a}^{b+a} \frac{d\rho}{\rho^\mu} \geq \frac{Ca}{(b+a)(b-a)^{\mu-1}}. \tag{2.22}$$

Proof. We distinguish two cases $a < b < 3a$ and $b \geq 3a$. When $b < 3a$, we have $2(b-a) < b + a$. Therefore,

$$I \geq \int_{b-a}^{2(b-a)} \frac{d\rho}{\rho^\mu} \geq 2^{-\mu}(b-a)^{-\mu+1}.$$

Since $a + b > 2a$, we get (2.22) for this case.

While, if $b \geq 3a$, we have $2(b-a) \geq b + a$. Therefore it is easy to see from

$$I \geq 2a(b+a)^{-\mu}$$

that (2.22) holds. This completes the proof. □

<u>*Step 3.*</u> In view of (2.20), for c, $y > 0$ and $\kappa \in \mathbb{R}$, we introduce the following quantity:

$$\langle u\rangle_{c,\kappa}(y) = \inf\{(ct+|x|)(ct-|x|)^{\kappa}|u(t,x)| : (t,x) \in \tilde{\Sigma}(c,y)\}, \tag{2.23}$$

$$\tilde{\Sigma}(c,y) = \{(t,x) \in [0,\infty)\times\mathbb{R}^n : (|x|,t) \in \Sigma(c,y)\},$$

$$\Sigma(c,y) = \{(r,t) \in [0,\infty)^2 : r \le c(t-y)\}. \tag{2.24}$$

Since we may assume $0 < \delta \le 1$, (2.20) yields

$$\langle u\rangle_{c,p^*}(y) \ge C_1\varepsilon^p \quad \text{for } y \ge 1. \tag{2.25}$$

Next we shall show that there exists a constant $C_2 > 0$ such that

$$\langle u\rangle_{c,p^*}(y) \ge C_2\int_1^y \left(1-\frac{\eta}{y}\right)\frac{[\langle u\rangle_{c,p^*}(\eta)]^p}{\eta^{pp^*}}\,d\eta \quad \text{for } y \ge 1. \tag{2.26}$$

Let $y \ge 1$. By (2.11) and (2.16), for $(t,x) \in \tilde{\Sigma}(c,y)$, we have

$$\begin{aligned} u(t,x) &\ge L_c[|u|^p](t,x) \\ &\ge \frac{1}{2cr}\iint_{D_c(r,t)\cap\Sigma(c,1)} \frac{\lambda}{(cs+\lambda)^p(cs-\lambda)^{pp^*}}\left[\langle u\rangle_{c,p^*}\left(\frac{cs-\lambda}{c}\right)\right]^p d\lambda\,ds. \end{aligned}$$

Changing the variables by (2.21), we have

$$\begin{aligned} u(t,x) &\ge \frac{C}{r}\int_1^{(ct-r)/c}\left(\int_{ct-r}^{ct+r}\frac{(\xi-c\eta)[\langle u\rangle_{c,p^*}(\eta)]^p}{\xi^p\eta^{pp^*}}\,d\xi\right)d\eta \\ &\ge \frac{C}{r}\int_{ct-r}^{ct+r}\frac{d\xi}{\xi^p}\int_1^{(ct-r)/c}\frac{(ct-r-c\eta)[\langle u\rangle_{c,p^*}(\eta)]^p}{\eta^{pp^*}}\,d\eta. \end{aligned}$$

By (2.22), we get

$$\begin{aligned} u(t,x) &\ge \frac{C}{(ct+r)(ct-r)^{p-1}}\int_1^{(ct-r)/c}\frac{(ct-r-c\eta)[\langle u\rangle_{c,p^*}(\eta)]^p}{\eta^{pp^*}}\,d\eta \\ &= \frac{C}{(ct+r)(ct-r)^{p^*}}\int_1^{(ct-r)/c}\left(1-\frac{c\eta}{ct-r}\right)\frac{[\langle u\rangle_{c,p^*}(\eta)]^p}{\eta^{pp^*}}\,d\eta. \end{aligned}$$

Since the function

$$y \longmapsto \int_1^y\left(1-\frac{\eta}{y}\right)\frac{[\langle u\rangle_{c,p^*}(\eta)]^p}{\eta^{pp^*}}\,d\eta$$

is non-decreasing, we have for all $(t,x) \in \tilde{\Sigma}(c,y)$

$$(ct+r)(ct-r)^{p^*}u(t,x) \ge C\int_1^y\left(1-\frac{\eta}{y}\right)\frac{[\langle u\rangle_{c,p^*}(\eta)]^p}{\eta^{pp^*}}\,d\eta,$$

which implies (2.26).

Step 4. Now we are in a position to employ Lemma 4 below. Then we see that $\langle u\rangle_{c,p^*}(y)$ blows up in a finite time $y = T_*(\varepsilon)$, provided $pp^* \le 1$. The last condition is equivalent to $1 < p \le p_0(3)$ according to (1.8) with $n = 3$. Therefore the solution of (2.3) with (2.7) blows up in a finite time $T^*(\varepsilon) \le T_*(\varepsilon)$, if $1 < p \le p_0(3)$ and (2.8) hold. Moreover we have the upper bound (2.9) of the life span $T^*(\varepsilon)$.

Lemma 4. *Let $C_1,\ C_2 > 0,\ \alpha,\ \beta \ge 0,\ b > 0,\ \kappa \le 1,\ \varepsilon \in (0,1]$, and $p > 1$. Suppose that $f(y)$ satisfies*

$$f(y) \ge C_1\varepsilon^\alpha, \quad f(y) \ge C_2\varepsilon^\beta \int_1^y \left(1-\frac{\eta}{y}\right)^b \frac{f(\eta)^p}{\eta^\kappa}\,d\eta, \quad y \ge 1.$$

Then, $f(y)$ blows up in a finite time $T_*(\varepsilon)$. Moreover, there exists a constant $C^* = C^*(C_1, C_2, b, p, \kappa) > 0$ such that

$$T_*(\varepsilon) \le \begin{cases} \exp(C^*\varepsilon^{-\{(p-1)\alpha+\beta\}}) & \text{if } \kappa = 1, \\ C^*\varepsilon^{-\{(p-1)\alpha+\beta\}/(1-\kappa)} & \text{if } \kappa < 1. \end{cases}$$

Proof. First, we consider the case $\kappa = 1$. We put

$$F(z) = (C_1\varepsilon^\alpha)^{-1} f(\exp(\varepsilon^{-\mu}z)), \quad \mu = (p-1)\alpha + \beta.$$

Since the function $z \longmapsto (1-e^{-z})^b$ is increasing on $[0,\infty)$ and $0 < \varepsilon \le 1$, we have

$$F(z) \ge 1, \quad F(z) \ge C_1^{p-1}C_2 \int_0^z \left(1-e^{-(z-\zeta)}\right)^b F(\zeta)^p\,d\zeta, \quad z \ge 0. \tag{2.27}$$

Since it is easy to show that $F(z)$ blows up in a finite time, we obtain the desired estimate for the case $\kappa = 1$.

Next, we consider the case $\kappa < 1$. We put

$$G(z) = (C_1\varepsilon^\alpha)^{-1} f(\varepsilon^{-\nu}e^z), \quad \nu = \frac{(p-1)\alpha+\beta}{1-\kappa}.$$

Then we see that $G(z)$ satisfies (2.27). Thus we obtain the desired estimate for the case $\kappa < 1$. This completes the proof. □

3. Key Lemma

Fisrt we prepare the following lemma. We remark that the constant depends only on κ^* not on each κ.

Lemma 5. *Let $\kappa^* > 0$ and $\kappa \in (-\infty, \kappa^*]$. Then there exists a constant $C = C(\kappa^*) > 0$ such that*

$$\frac{1}{r}\int_{t-r}^{t+r} \frac{d\rho}{\rho^{1+\kappa}} \geq \frac{C}{(t+r)(t-r)^{\kappa}}, \quad t > r > 0.$$

Proof. For $\kappa \in (-\infty, \kappa^*]$, we put

$$I_\kappa(r,t) = \frac{(t+r)(t-r)^\kappa}{r}\int_{t-r}^{t+r} \frac{d\rho}{\rho^{1+\kappa}}.$$

Then by (2.22), there exists $C(\kappa^*) > 0$ such that $I_{\kappa^*}(r,t) \geq C(\kappa^*)$ for any $t > r > 0$. On one hand, for $t > r > 0$, we have

$$I_\kappa(r,t) = \frac{t+r}{r}\int_1^{(t+r)/(t-r)} \frac{d\lambda}{\lambda^{1+\kappa}} \geq \frac{t+r}{r}\int_1^{(t+r)/(t-r)} \frac{d\lambda}{\lambda^{1+\kappa^*}} = I_{\kappa^*}(r,t).$$

This completes the proof. □

The following lemma contains an essence to handle the problem for the unequal propagation speeds.

Lemma 6. *Let α, a_0, a_1, a_2, $\kappa^* > 0$, $\mu \in \mathbb{R}$, $\kappa \in [-\kappa^*, \kappa^*]$ and $a_1 \leq a_2$. Then, there exists a positive constant $C = C(a_0, a_1, a_2, \mu, \kappa^*)$ such that*

$$\langle L_{a_0}[R(f)]\rangle_{a_0,\mu+\kappa-2}(y) \geq C\int_\alpha^y \left(1 - \frac{\eta}{y}\right)^2 f(\eta)\, d\eta, \quad y \geq \alpha$$

holds for any non-negative function f, where we put

$$R(f)(t,x) = \frac{1}{(t+|x|)^\mu (a_2 t - |x|)^\kappa} f\left(\frac{a_1 t - |x|}{a_1}\right) \chi_{\Sigma(a_1,\alpha)}(t,x).$$

Here we denoted the characteristic function of a set A by χ_A.

Proof. Let $y \geq \alpha$. By (2.16), for any $(t,x) \in \Sigma(a_0, y)$ with $r = |x|$ we have

$$\begin{aligned} &L_{a_0}[R(f)](t,x) \geq I(r,t) \\ &:= \frac{1}{2a_0 r}\iint_{D_{a_0}(r,t)} \frac{\lambda}{(s+\lambda)^\mu (a_2 s - \lambda)^\kappa} f\left(\frac{a_1 s - \lambda}{a_1}\right) \chi_{[\alpha,\infty)}\left(\frac{a_1 s - \lambda}{a_1}\right) d\lambda\, ds. \end{aligned}$$

We distinguish two cases, $a_0 \leq a_1$ and $a_0 > a_1$, to show

$$I(r,t) \geq \frac{C}{(t+r)(a_0 t - r)^{\mu+\kappa-2}}\int_\alpha^{(a_0 t - r)/a_0} \left(1 - \frac{a_0 \eta}{a_0 t - r}\right)^2 f(\eta)\, d\eta. \tag{3.1}$$

First, we consider the case $a_0 \le a_1$. Changing the variables by $\xi = a_0 s + \lambda$, $\eta = (a_1 s - \lambda)/a_1$, by Lemma 5 we have

$$\begin{aligned} I(r,t) &\ge \frac{C}{r}\int_{\alpha}^{(a_0t-r)/a_0}\int_{a_0t-r}^{a_0t+r}\frac{(\xi-a_0\eta)f(\eta)}{\xi^{\mu}(a_2\xi/a_0)^{\kappa}}\,d\xi\,d\eta \\ &\ge \frac{C}{r}\int_{a_0t-r}^{a_0t+r}\frac{d\xi}{\xi^{\mu+\kappa}}\int_{\alpha}^{(a_0t-r)/a_0}(a_0t-r-a_0\eta)f(\eta)\,d\eta \\ &\ge \frac{C}{(t+r)(a_0t-r)^{\mu+\kappa-2}}\int_{\alpha}^{(a_0t-r)/a_0}\left(1-\frac{a_0\eta}{a_0t-r}\right)f(\eta)\,d\eta, \end{aligned}$$

which implies (3.1). Next, we consider the case $a_0 > a_1$. We divide further into two cases, $(t,x) \in \Sigma(a_1,\alpha)$ and $(t,x) \in \Sigma(a_0,\alpha)\setminus\Sigma(a_1,\alpha)$. In the case $(t,x) \in \Sigma(a_1,\alpha)$, we have $I(r,t) \ge C(I_1(r,t)+I_2(r,t))$, where

$$\begin{aligned} I_1(r,t) &= \frac{1}{r}\int_{\alpha}^{(a_1t-r)/a_1}\int_{a_0t-r}^{a_0t+r}\frac{(\xi-a_0\eta)f(\eta)}{\xi^{\mu+\kappa}}\,d\xi\,d\eta, \\ I_2(r,t) &= \frac{1}{r}\int_{(a_1t-r)/a_1}^{(a_0t-r)/a_0}\int_{a_0t-r}^{\xi^*(\eta)}\frac{(\xi-a_0\eta)f(\eta)}{\xi^{\mu+\kappa}}\,d\xi\,d\eta. \end{aligned}$$

While, in the case $(t,x) \in \Sigma(a_0,\alpha)\setminus\Sigma(a_1,\alpha)$, we have $I(r,t) \ge CI_3(r,t)$, where

$$I_3(r,t) = \frac{1}{r}\int_{\alpha}^{(a_0t-r)/a_0}\int_{a_0t-r}^{\xi^*(\eta)}\frac{(\xi-a_0\eta)f(\eta)}{\xi^{\mu+\kappa}}\,d\xi\,d\eta.$$

In the definitions of $I_2(r,t)$ and $I_3(r,t)$, we put

$$\xi^*(\eta) = \frac{a_0+a_1}{a_0-a_1}(a_0t-r) - \frac{2a_0a_1}{a_0-a_1}\eta.$$

As in the case $a_0 \le a_1$, we have

$$I_1(r,t) \ge \frac{C}{(t+r)(a_0t-r)^{\mu+\kappa-2}}\int_{\alpha}^{(a_1t-r)/a_1}\left(1-\frac{a_0\eta}{a_0t-r}\right)f(\eta)\,d\eta. \tag{3.2}$$

On the other hand, we have

$$I_j(r,t) \ge \frac{C}{r}\int_{\eta_j^*}^{(a_0t-r)/a_0}(a_0t-r-a_0\eta)f(\eta)\int_{a_0t-r}^{\xi^*(\eta)}\frac{d\xi}{\xi^{\mu+\kappa}}\,d\eta, \quad j=2,3,$$

where we put $\eta_2^* = (a_1t-r)/a_1$ and $\eta_3^* = \alpha$. Since

$$a_0t-r \le \xi^*(\eta) \le \frac{a_0+a_1}{a_0-a_1}(a_0t-r), \quad \xi^*(\eta)-(a_0t-r) = \frac{2a_0a_1}{a_0-a_1}(a_0t-r-\eta),$$

we have

$$\int_{a_0t-r}^{\xi^*(\eta)}\frac{d\xi}{\xi^{\mu+\kappa}} \ge C\frac{a_0t-r-a_0\eta}{(a_0t-r)^{\mu+\kappa}}.$$

Thus, for $j = 2$, 3, we obtain

$$I_j(r,t) \geq \frac{C}{(t+r)(a_0t-r)^{\mu+\kappa-2}} \int_{\eta_j^*}^{(a_0t-r)/a_0} \left(1 - \frac{a_0\eta}{a_0t-r}\right)^2 f(\eta)\,d\eta. \tag{3.3}$$

From (3.2) and (3.3), we see that (3.1) is also valid for the case $a_0 > a_1$. Since the function

$$y \longmapsto \int_\alpha^y \left(1 - \frac{\eta}{y}\right)^2 f(\eta)\,d\eta$$

is non-decreasing on $[\alpha, \infty)$, it follows from (3.1) that for any $(t,x) \in \Sigma(a_0, y)$

$$\begin{aligned} &(t+|x|)(a_0t-|x|)^{\mu+\kappa-2} L_{a_0}[R(f)](t,x) \\ &\geq C \int_\alpha^{(a_0t-|x|)/a_0} \left(1 - \frac{a_0\eta}{a_0t-|x|}\right)^2 f(\eta)\,d\eta \geq C \int_\alpha^y \left(1 - \frac{\eta}{y}\right)^2 f(\eta)\,d\eta. \end{aligned}$$

From the definition of $\langle \cdot \rangle_{a_0,\mu+\kappa-2}(y)$, we obtain the desired estimate. □

4. Main result

First of all, we precisely state the **blow-up** part of Theorem 1. Let us consider the system

$$\begin{cases} (\partial_t^2 - c_1^2\Delta)u_1 = |u_1||u_2|, & (t,x) \in [0,\infty) \times \mathbb{R}^3, \\ (\partial_t^2 - c_2^2\Delta)u_2 = |u_1|^q, & (t,x) \in [0,\infty) \times \mathbb{R}^3 \end{cases} \tag{4.1}$$

with the initila data

$$u_j(0,x) = 0, \quad \partial_t u_j(0,x) = \varepsilon g_j(x), \quad x \in \mathbb{R}^3 \ (j = 1,2). \tag{4.2}$$

Here $q > 1$, $c_j > 0$, $\varepsilon > 0$, and $g_j \in C(\mathbb{R}^3)$ $(j = 1,2)$ satisfies

$$g_j(x) \geq 0 \text{ for all } x \in \mathbb{R}^3, \quad g_1(0) > 0. \tag{4.3}$$

Then we have the following.

Theorem 3. *Let $c_1 \neq c_2$ and $1 < q \leq 3$. Suppose $g_j \in C(\mathbb{R}^3)$ $(j = 1,2)$ satisfies (4.3). Then for sufficiently small ε the solution (u_1, u_2) of (1.3)–(4.2) blows up in a* *finite time* $T(\varepsilon)$, *if either* $q=3$ *and* $c_1>c_2$ *or* $1<q<3$. *Moreover, there is a constant* $A>0$, *independent of* ε, *such that*

$$T(\varepsilon)\le\begin{cases}\exp(A\varepsilon^{-3}) & \textit{if}\ \ q=3\ \textit{and}\ c_1>c_2\\ A\varepsilon^{-q(2+q)/(3-q)^2} & \textit{if}\ \ 1<q<3\ \textit{and}\ c_1>c_2\\ A\varepsilon^{-2q/(3-q)^2} & \textit{if}\ \ 1<q<3\ \textit{and}\ c_1<c_2\end{cases}. \tag{4.4}$$

Remark 7. *As for the case where* $q=3$ *and* $c_1>c_2$, *Katayama and Matsumura* [13] *recently proved that there is a constant* $B>0$, *independent of* ε, *such that*

$$T(\varepsilon)\ge\exp(B\varepsilon^{-3}). \tag{4.5}$$

Proof. We treat the problem (4.1)–(4.2) in the integral form:

$$u_1=\varepsilon K_{c_1}[0,g_1]+L_{c_1}[|u_1||u_2|]\quad\text{in}\quad[0,\infty)\times\mathbb{R}^3, \tag{4.6}$$

$$u_2=\varepsilon K_{c_2}[0,g_2]+L_{c_2}[|u_1|^q]\quad\text{in}\quad[0,\infty)\times\mathbb{R}^3. \tag{4.7}$$

Basically we follow the argument in the previous section. In particular, the proof for the case where $1<q<3$ can be done analogously and less hard. For this reason, we concentrate on the case where $q=3$ and $c_1>c_2$. It is the most delicate one in the sense that the result depends not only on the exponent q but also on the propagation speeds c_1 and c_2.

By (4.6), (4.7) and (4.3), we have

$$u_1(t,x)\ge\varepsilon K_{c_1}[0,g_1](t,x),\quad(t,x)\in[0,\infty)\times\mathbb{R}^3, \tag{4.8}$$

$$u_1(t,x)\ge L_{c_1}[|u_1||u_2|](t,x),\quad(t,x)\in[0,\infty)\times\mathbb{R}^3, \tag{4.9}$$

$$u_2(t,x)\ge L_{c_2}[|u_1|^q](t,x),\quad(t,x)\in[0,\infty)\times\mathbb{R}^3. \tag{4.10}$$

We see from (4.3) that there is a constant $C>0$ such that

$$u_1(t,x)\ge C\varepsilon r^{-1}\quad\text{for }(t,x)\in E, \tag{4.11}$$

as in the proof of (2.19). Here we put

$$E:=\{(t,x)\in[0,\infty)\times\mathbb{R}^3:\ |c_1t-|x||\le\delta/2,\ c_1t+|x|\ge\delta\}.$$

Based on this estimate, we shall show

$$\langle u_1\rangle_{c_1,2}(y)\ge C_1\varepsilon^4,\quad\langle u_2\rangle_{c_2,1}(y)\ge C_2\varepsilon^3\quad\text{for}\quad y\ge1. \tag{4.12}$$

provided $0 < \delta \leq \min\{c_2, 2c_1(c_1 - c_2)/(5c_1 + c_2)\}$.

Since $\delta \leq c_2$, by (4.10), (2.16) and (4.11), we have

$$\begin{aligned} u_2(t,x) &\geq \frac{C\varepsilon^3}{r}\int_{-\delta/2}^{\delta/2} d\eta \int_{c_2t-r}^{c_2t+r}\frac{d\xi}{\xi^2} \\ &\geq \frac{C\varepsilon^3}{(t+r)(c_2t-r)}, \quad (t,x)\in\Sigma(c_2,1). \end{aligned} \tag{4.13}$$

Thus the second inequality in (4.12) holds true.

To prove the first one, we prepare the following estimate.

$$u_2(t,x) \geq \frac{C\varepsilon^3(c_1t-r)}{(t+r)^3}, \quad (t,x)\in\Omega, \tag{4.14}$$

where we set

$$\Omega = \{(t,x)\in[0,\infty)\times\mathbb{R}^3 : \ c_1t-|x|\geq 0, \ |x|-c_2t\geq\delta\}.$$

By (4.10), (2.16) and (4.11), we have

$$u_2(t,x) \geq \frac{C\varepsilon^{q_1}}{r}\int_{-\delta/2}^{0} d\eta \int_{\lambda_1(r,t)}^{\lambda_2(r,t)}\frac{d\lambda}{\lambda^2}, \quad (t,x)\in\Omega,$$

where we put

$$\lambda_1(r,t) = \frac{c_1}{c_1-c_2}(r-c_2t), \quad \lambda_2(r,t) = \frac{c_1}{c_1+c_2}(r+c_2t).$$

Since $\lambda_2(r,t) - \lambda_1(r,t) = 2c_1c_2(c_1t-r)/(c_1^2-c_2^2)$, we get (4.14).

By (4.11) and (4.14), we have

$$|u_1(t,x)||u_2(t,x)| \geq \frac{C\varepsilon^4(c_1t-r)}{r(c_1t+r)^3}, \quad (t,x)\in E\cap\Omega.$$

Since $\delta \leq 2c_1(c_1-c_2)/(5c_1+c_2)$, by (4.9) and (2.16), we have

$$\begin{aligned} u_1(t,x) &\geq \frac{C\varepsilon^4}{r}\int_0^{\delta/2}\eta\, d\eta \int_{c_1t-r}^{c_1t+r}\frac{d\xi}{\xi^3} \\ &\geq \frac{C\varepsilon^4}{(t+r)(c_1t-r)^2}, \quad (t,x)\in\Sigma(c_1,1), \end{aligned}$$

which implies the first inequality in (4.12).

Unfortunately, the first estimate in (4.12) is not enough to show the blow-up result because of the fast decay with respect to (c_1t-r). Thus our next step is to improve it. To this end, for $0 \leq \kappa \leq 2$ we set

$$U_{1,\kappa}(y) = \langle u_1\rangle_{c_1,\kappa}(y), \quad U_2(y) = \langle u_2\rangle_{c_2,1}(y).$$

Then (4.12) implies

$$U_{1,2}(y) \geq C_1\varepsilon^4, \quad U_2(y) \geq C_2\varepsilon^3, \quad y \geq 1. \tag{4.15}$$

To proceed further, we shall prove that for all $\kappa \in [0,2]$ there exist positive constants $C_3 = C_3(c_1, c_2)$ and $C_4 = C_4(c_1, c_2)$ such that

$$U_{1,\kappa}(y) \geq C_3 \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U_{1,\kappa}(\eta) U_2(\eta)}{\eta}\, d\eta, \quad y \geq 1, \tag{4.16}$$

$$U_2(y) \geq C_4 \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U_{1,\kappa}(\eta)^3}{\eta^{3\kappa}}\, d\eta, \qquad y \geq 1. \tag{4.17}$$

This can be done by the applications of propositions below.

Proposition 2. *Let α, a_0, $\kappa^* > 0$, μ_1, $\mu_2 \in \mathbb{R}$, κ_1, $\kappa_2 \in [-\kappa^*, \kappa^*]$ and $0 < a_1 \leq a_2$. Then, there exists a constant $C = C(a_0, a_1, a_2, \mu_1 + \mu_2, \kappa^*) > 0$ such that*

$$\begin{aligned} &\langle L_{a_0}[|fg|]\rangle_{a_0, \mu_1+\mu_2+\kappa_2-2}(y) \\ \geq\ & C \int_\alpha^y \left(1 - \frac{\eta}{y}\right)^2 F(\eta) G(\eta) \frac{d\eta}{\eta^{\kappa_1}} \quad y \in [\alpha, \infty), \end{aligned}$$

where for $\eta \geq \alpha$ we put

$$F(\eta) := \inf\{(t + |x|)^{\mu_1}(a_1 t - |x|)^{\kappa_1} |f(t,x)| \ :\ (t,x) \in \Sigma(a_1, \eta)\}$$

$$G(\eta) := \inf\{(t + |x|)^{\mu_2}(a_2 t - |x|)^{\kappa_2} |g(t,x)| \ :\ (t,x) \in \Sigma(a_2, \eta)\}$$

Proof. From the definition of $F(\eta)$, we have

$$|f(t,x)| \geq \frac{F((a_1 t - |x|)/a_1)}{(t + |x|)^{\mu_1}(a_1 t - |x|)^{\kappa_1}}, \quad (t,x) \in \Sigma(a_1, \alpha). \tag{4.18}$$

Since $a_1 \leq a_2$, if $(t,x) \in \Sigma(a_1, \alpha)$, then we have $(t,x) \in \Sigma(a_2, (a_1 t - |x|)/a_1)$. Thus, from the definition of $G(\eta)$, we have

$$|g(t,x)| \geq \frac{G((a_1 t - |x|)/a_1)}{(t + |x|)^{\mu_2}(a_2 t - |x|)^{\kappa_2}}, \quad (t,x) \in \Sigma(a_1, \alpha). \tag{4.19}$$

By (4.18), (4.19) and Lemma 6, we obtain the desired inequality. □

As a corollay of Proposition 2, we have the following proposition.

Proposition 3. *Let α, a, b, $\kappa^* > 0$, $\mu \in \mathbb{R}$ and $\kappa \in [-\kappa^*, \kappa^*]$. Then, there exists a constant $C = C(a, b, \mu, \kappa^*) > 0$ such that*

$$\langle L_a[|f|]\rangle_{a,\mu-2}(y) \geq C \int_\alpha^y \left(1 - \frac{\eta}{y}\right)^2 \frac{F(\eta)}{\eta^\kappa}\, d\eta, \quad y \in [\alpha, \infty),$$

where for $\eta \geq \alpha$ we put

$$F(\eta) := \inf\{(t + |x|)^\mu (a_1 t - |x|)^\kappa |f(t,x)| \ : \ (t,x) \in \Sigma(a_1, \eta)\}$$

We come back to the proof of (4.16) and (4.17). By (4.9) and Proposition 2, we have for $y \geq 1$

$$\begin{aligned}\langle u_1\rangle_{c_1,\kappa}(y) &\geq \langle L_{c_1}[|u_1||u_2|]\rangle_{c_1,\kappa}(y) \\ &\geq C \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{\langle u_1\rangle_{c_1,\kappa}(\eta)\langle u_2\rangle_{c_2,1}(\eta)}{\eta^{p_1\kappa}}\, d\eta,\end{aligned}$$

which shows (4.16).

Moreover, by (4.10) and Proposition 3, we have for $y \geq 1$

$$\begin{aligned}\langle u_2\rangle_{c_2,1}(y) &\geq \langle L_{c_2}[|u_1|^3]\rangle_{c_2,1}(y) \\ &\geq C \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{\langle u_1\rangle^3_{c_1,\kappa}(\eta)}{\eta^{3\kappa}}\, d\eta,\end{aligned}$$

which shows (4.17).

Now (4.15) and (4.16) yield

$$U_{1,\kappa}(y) \geq 16b \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U_{1,\kappa}(\eta)}{\eta}\, d\eta, \quad y \geq 1, \tag{4.20}$$

where $b = C_2 C_3 \varepsilon^3/16$. Especially (4.15) and (4.20) with $\kappa = 2$ give

$$U_{1,2}(y) \geq a, \quad U_{1,2}(y) \geq 16b \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U_{1,2}(\eta)}{\eta}\, d\eta, \quad y \geq 1 \tag{4.21}$$

with $a = C_1\varepsilon^4$. One can show that $U_{1,2}(y)$ grows in y, by using the following lemma.

Lemma 8. *Let $a > 0$, $0 < b \leq 1$ and $p \geq 1$. Assume that $f(y)$ satisfies*

$$f(y) \geq a, \quad f(y) \geq 16b \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{(f(\eta))^p}{\eta}\, d\eta, \quad y \geq 1.$$

If $p > 1$, then $f(y)$ blows up in a finite time. While, if $p = 1$, then we have

$$f(y) \geq \frac{a}{4} y^b, \quad y \geq 1.$$

Proof. When $p > 1$, the conclusion follows from Lemma 4 with $\alpha = \beta = 0$, $b = 2$ and $\kappa = 1$. Therefore it suffices to consider the case of $p = 1$.

Put $g(y) = (a/4)y^b$. Then we have $g(y) < f(y)$ for any $y \in [1, 4^{1/b})$. Moreover, since $0 < b \leq 1$ and

$$\int_1^y \left(1 - \frac{\eta}{y}\right)^2 \eta^{b-1}\, d\eta \geq \frac{1}{4}\int_1^{y/2} \eta^{b-1}\, d\eta = \frac{1}{4b}\left\{\left(\frac{y}{2}\right)^b - 1\right\},$$

we have

$$g(y) \leq 16b \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{g(\eta)}{\eta}\, d\eta, \quad y \geq 4^{1/b}.$$

By the comparison argument, we see that $f(y) \geq g(y)$ holds for any $y \geq 1$. This completes the proof. □

Applying the lemma with $p = 1$ to (4.21), we get

$$U_{1,2}(y) \geq \frac{a}{4} y^b, \quad y \geq 1. \tag{4.22}$$

For fixed $y \geq 1$, let $(t, x) \in \Sigma(c_1, y)$, so that $(c_1 t - |x|)/c_1 \geq 1$. Then (4.22) yields

$$|u_1(t,x)|(t + |x|)(c_1 t - |x|)^2 \geq \frac{a}{4}\left(\frac{c_1 t - |x|}{c_1}\right)^b, \quad \textit{i.e.}\ \ U_{1,2-b}(y) \geq \frac{a}{4{c_1}^b}$$

for $y \geq 1$. Repeating this procedure n times, we obtain

$$U_{1,2-nb}(y) \geq \frac{a}{4^n {c_1}^{nb}}, \quad y \geq 1. \tag{4.23}$$

Moreover, we have

$$U_{1,2-nb}(y) \geq \frac{a}{4^{2n} {c_1}^{nb}} y^{nb}, \quad y \geq 1. \tag{4.24}$$

In fact, for $(t, x) \in \Sigma(c_1, y)$, (4.23) with n replaced by $2n$ implies

$$|u_1(t,x)|(t + |x|)(c_1 t - |x|)^{2-2nb} \geq \frac{a}{4^{2n} {c_1}^{2nb}}, \quad y \geq 1.$$

Combining this with $c_1 t - |x| \geq c_1 y$, we get (4.24).

Let k be the smallest natural number satisfying $3(2 - kb) \leq 1$. Being $b = C_2 C_3 \varepsilon^3/16$, we see that $C_5 \varepsilon^{-3} \leq k \leq C_5 \varepsilon^{-3}$ with a positive constant C_5, independent of ε. Recalling $a = C_1 \varepsilon^4$, we get

$$\frac{a}{4^{2k} {c_1}^{kb}} y^{kb} \geq C y^{C_*} \exp(4 \log \varepsilon - 2C_5 \varepsilon^{-3} \log 4) \tag{4.25}$$

with $C_* = C_2C_3C_5/16$. Since $\varepsilon^3 \log \varepsilon$ has a minimum for $\varepsilon > 0$, we can take a positive constant C_6, so that for $0 < \varepsilon \le 1$

$$C \exp(4 \log \varepsilon - 2C_5\varepsilon^{-3} \log 4) \ge \exp(-C_6\varepsilon^{-3}).$$

Now taking $y \ge \alpha^* := \exp(C_6\varepsilon^{-3}/C_*)$, we see from (4.24) and (4.25) that $U_{1,2-kb}(y) \ge 1$. Therefore (4.17) with $\kappa = 2 - kb$ yileds

$$\begin{aligned} U_2(y) &\ge C_4 \int_{\alpha^*}^{y} \left(1 - \frac{\eta}{y}\right)^2 \frac{1}{\eta^{3(2-kb)}}\, d\eta, \\ &\ge C_4 \int_{\alpha^*}^{y/2} \left(1 - \frac{\eta}{y}\right)^2 \frac{1}{\eta}\, d\eta, \\ &\ge \frac{C_4}{4} \log \frac{y}{2\alpha^*}, \qquad y \ge \alpha^*. \end{aligned}$$

Thus $U_2(y) \ge 1$ for $y \ge \alpha := 2\alpha^* \exp(4/C_4)$.

Finally, rescaling as $U(z) = \min\{U_{1,2-kb}(\alpha z), U_2(\alpha z)\}$ and using $3(2 - kb) \le 1$, we find from (4.16) and (4.17) that

$$U(z) \ge 1, \quad U(z) \ge C_7 \int_1^z \left(1 - \frac{\zeta}{z}\right)^2 \frac{U(\zeta)^2}{\zeta}\, d\zeta$$

for $z \ge 1$, where $C_7 = \min\{C_3, C_4\}$. Emplying Lemma 8 with $p = 2$, we see that $U(z)$ blows up in a finite time. Hence the classical solution of (4.1)–(4.2) blows up in a finite time $T(\varepsilon)$. Moreover, $T(\varepsilon)$ is estimated from above by $\exp(C^*\varepsilon^{-3})$ with a suitable poistive constant C^*. This completes the proof. □

References

[1] R. Agemi, Global existence of nonlinear elastic waves, *Invent. Math.* **142** (2000), 225–250.

[2] R. Agemi, Y. Kurokawa and H. Takamura, Critical curve for *p-q* systems of nonlinear wave equations in three space dimensions, *J. Differential Equations.* **167** (2000), 87–133.

[3] R. Agemi and K. Yokoyama, The null conditions and global existence of solutions to systems of wave equations with different propagation speeds, *in* "Advances in nonlinear partial differential equations and stochastics" (S. Kawashima and T. Yanagisawa ed.), Series on Adv. in Math. for Appl. Sci., Vol. 48, 43–86, World Scientific, Singapore, 1998.

[4] D. D'Ancona, V. Georgiev and H. Kubo, Weighted decay estimates for the wave equation *J. Differential Equations* **163** (2001), 146–208.

[5] D. Del Santo, V. Georgiev and E. Mitidieri, Global existence of the solutions and formation of singularities for a class of hyperbolic systems, *in* "Geometric optics and related topics" (F.

Colombini and N. Lerner ed.), Progress in Nonlinear Differential Equations and Their Applications, Vol. 32, 117–140, Berkhäuser, Boston, 1997.

[6] K. Deng, Nonexistence of global solutions of a nonlinear hyperbolic system, *Trans. Amer. Math. Soc.* **349** (1997), 1685–1696.

[7] V. Georgiev, "Semilinear Hyperbolic equations", MSJ Memoirs, Vol. 7 (2000).

[8] V. Georgiev, H. Lindblad and C. Sogge, Weighted Strichartz estimate and global existence for semilinear wave equation, *Amer. J. Math.* **119** (1997), 1291–1319.

[9] R. T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, *Math. Z.* **177** (1981), 323–340.

[10] R. T. Glassey, Existence in the large for $\Box u = F(u)$ in two space dimensions, *Math. Z.* **178** (1981), 233–261.

[11] A. Hoshiga and H. Kubo, Global small amplitude solutions of nonlinear hyperbolic systems with a critical exponent under the null condition *SIAM J. Math. Anal.* **31** (2000), 486–513.

[12] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.* **28** (1979), 235–268.

[13] S. Katayama and A. Matsumura, private communication.

[14] M. Kovalyov, Resonance-type behaviour in a system of nonlinear wave equations, *J. Differential Equations* **77** (1989) 73–83.

[15] H. Kubo and M. Ohta, Critical blowup for systems of semilinear wave equations in low space dimensions, *J. Math. Anal. Appl.* **240** (1999), 340–360.

[16] H. Kubo and M. Ohta, Small data blowup for systems of semilinear wave equations with different propagation speeds in three space dimensions, *J. Differential Equations* **163** (2000), 475–492.

[17] H. Kubo and M. Ohta, On systems of semilinear wave equations with unequal propagation speeds in three space dimensions, *Preprint (Università di Pisa)* (2002).

[18] H. Kubo and M. Ohta, In preparation.

[19] H. Kubo and K. Tsugawa, Global solutions and self-similar solutions of the coupled system of semilinear wave equations in three space dimensions, *Discrete Contin. Dynam. Systems* **9** (2003), 471–482.

[20] H. Lindblad, Blow-up for solutions of $\Box u = |u|^p$ with small initial data, *Comm. Partial Differential Equations* **15** (1990) 757–821.

[21] T. Ozawa, K. Tsutaya and Y. Tsutsumi, Well-posedness in energy space for the Cauchy problem of the Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions, *Math. Ann.* **313** (1999) 127-140.

[22] J. Schaeffer, The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p, *Proc. Roy. Soc. Edinburgh* **101A** (1985), 31–44.

[23] T. C. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, *J. Differential Equations* **52** (1984), 378–406.

[24] T. C. Sideris, The null condition and global existence of nonlinear elastic waves, *Invent. Math.* **123** (1996), 323–342.

[25] W. Strauss, Nonlinear scattering theory at low energy, *J. Funct. Anal.* **41** (1981), 110–133.

[26] D. Tataru, Strichartz estimates in the hyperbolic space and global existence for semilinear wave equation *Trans. Amer. Math. Soc.* **353** (2000),795–807.

[27] K. Yokoyama, Global existence of classical solutions to systems of wave equations with critical nonlinearity in three space dimensions, *J. Math. Soc. Japan* **52** (2000), 609–632.

[28] Y. Zhou, Blow up of classical solutions to $\Box u = |u|^{1+\alpha}$ in three space dimensions, *J. Partial Differential Equations* **5** (1992), 21–32.

E-mail address: `kubo@math.sci.osaka-u.ac.jp`

E-mail address: `mohta@rimath.saitama-u.ac.jp`