数理解析研究所講究録 1359 巻 2004 年 85-92

Parametric extensions of Shannon inequality and its reverse one in Hilbert space operators via characterizations of operator concave functions

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Abstract. We shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators. *Let $p \in [0,1]$ and also let $\{A_1, A_2, ..., A_n\}$ and $\{B_1, B_2, ..., B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^{n} A_j \sharp_p B_j \leq I$. Then*

$$\sum_{j=1}^{n} S_{p+1}(A_j|B_j) \geq \Big[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + (I - \sum_{j=1}^{n} A_j \sharp_p B_j)\Big] \log\Big[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + (I - \sum_{j=1}^{n} A_j \sharp_p B_j)\Big]$$
$$\geq \log\Big[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + (I - \sum_{j=1}^{n} A_j \sharp_p B_j)\Big] \geq \sum_{j=1}^{n} S_p(A_j|B_j) \geq -\log\Big[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + (I - \sum_{j=1}^{n} A_j \sharp_p B_j)\Big]$$
$$\geq -\Big[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + (I - \sum_{j=1}^{n} A_j \sharp_p B_j)\Big] \log\Big[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + (I - \sum_{j=1}^{n} A_j \sharp_p B_j)\Big] \geq \sum_{j=1}^{n} S_{p-1}(A_j|B_j)$$

where $S_q(A|B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$ for $A > 0$, $B > 0$ and any real number q and $A \natural_q B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q A^{\frac{1}{2}}$ for $A > 0$, $B > 0$ and any real number q.

In particular, if $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, then

$$\sum_{j=1}^{n} S_2(A_j|B_j) \geq \Big[\sum_{j=1}^{n} B_j A_j^{-1} B_j\Big] \log\Big[\sum_{j=1}^{n} B_j A_j^{-1} B_j\Big] \geq \log\Big[\sum_{j=1}^{n} B_j A_j^{-1} B_j\Big] \geq \sum_{j=1}^{n} S_1(A_j|B_j) \geq 0$$
$$\geq \sum_{j=1}^{n} S(A_j|B_j) \geq -\log\Big[\sum_{j=1}^{n} A_j B_j^{-1} A_j\Big] \geq -\Big[\sum_{j=1}^{n} A_j B_j^{-1} A_j\Big] \log\Big[\sum_{j=1}^{n} A_j B_j^{-1} A_j\Big] \geq \sum_{j=1}^{n} S_{-1}(A_j|B_j)$$

where $S(A|B) = S_0(A|B) = A^{\frac{1}{2}}(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$ which is the relative operator entropy of $A > 0$ and $B > 0$.

Our results can be considered as parametric extensions of the following celebrated Shannon inequality ([7],[9] and [233 p ,1]) which is very useful and so famous in information theory. *Let $\{a_1, a_2, ..., a_n\}$ and $\{b_1, b_2, ..., b_n\}$ be two probability vectors. Then*

$0 \geq \sum_{j=1}^{n} a_j \log b_j - \sum_{j=1}^{n} a_j \log a_j$ (see inequalities (2.4) of Corollary 2.4).

§1 Introduction

First the Shannon inequality asserts: *Let* $\{a_1, a_2, ..., a_n\}$ *and* $\{b_1, b_2, ..., b_n\}$ *be two probability vectors. Then*

$$0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j}. \tag{1.1}$$

We remark that $0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j}$ in (1.1) is equivalent to $D = \sum_{j=1}^{n} a_j \log \frac{a_j}{b_j} \geq 0$ which is the original number type Shannon inequality and this D is called "divergence" in [7] and [9].

In this paper we shall state parametric extensions of Shannon inequality and its reverse one in Hilbert space operators.

A bounded linear operator T on a Hilbert space H is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

Definition 1.1. $S_q(A|B)$ for $A > 0$, $B > 0$ and any real number q is defined by

$$S_q(A|B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}.$$

We recall that $S_0(A|B) = A^{\frac{1}{2}}(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} = S(A|B)$ is the relative operator entropy in [2] and $S(A|I) = -A\log A$ is the usual operator entropy in [8].

Definition 1.2. $A\natural_q B$ *for* $A > 0$ *and* $B >$ *and any real number* q *is defined by*

$$A\natural_q B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q A^{\frac{1}{2}}$$

and $A\natural_p B$ *for* $p \in [0,1]$ *just coincides with* $A\sharp_p B$ *which is well known as p-power mean.*

We remark that $S_1(A|B) = -S(B|A)$ and moreover $S_q(A|B) = -S_{1-q}(B|A)$ for any q.

Following after Definition 1.1, The original Shannon inequality can be expressed as follows:

$$0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} = \sum_{j=1}^{n} a_j^{\frac{1}{2}}(\log a_j^{\frac{-1}{2}} b_j a_j^{\frac{-1}{2}})a_j^{\frac{1}{2}} = \sum_{j=1}^{n} S(a_j|b_j).$$

Consequently $0 \geq \sum_{j=1}^{n} S(a_j|b_j)$ in the original Shannon inequality can be extented to $0 \geq \sum_{j=1}^{n} S(A_j|B_j)$ in operator version case (2.4) of Corollary 2.4, so that the form of (1.1) is convenient for operator type extension. We can summarize the following contrast:

The original Shannon inequality and its reverse one

$$0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} \geq -\log \sum_{j=1}^{n} \frac{a_j^2}{b_j}.$$

for $a_j, b_j > 0$ *with* $1 = \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j$.

The operator version Shannon inequality and its reverse one

$$0 \geq \sum_{j=1}^{n} S(A_j|B_j) \geq -\log \sum_{j=1}^{n} A_j B_j^{-1} A_j.$$

for $A_j, B_j > 0$ with $I = \sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j$.

§2 Parametric extensions of operator reverse type Shannon inequality derived from two operator concave functions $f_1(t) = \log t$ and $f_2(t) = -t \log t$

Firstly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function $f(t) = \log t$.

Theorem 2.1. *Let* $p \in [0,1]$ *and also let* $\{A_1, A_2, ..., A_n\}$ *and* $\{B_1, B_2, ..., B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* H *such that* $\sum_{j=1}^{n} A_j \sharp_p B_j \leq I$, *where* I *means the identity operator on* H. *Then*

$$\begin{aligned}
(2.1) \qquad & \log\Big[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + t_0(I - \sum_{j=1}^{n} A_j \sharp_p B_j)\Big] - \log t_0(I - \sum_{j=1}^{n} A_j \sharp_p B_j) \\
& \geq \sum_{j=1}^{n} S_p(A_j|B_j) \\
& \geq -\log\Big[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + t_0(I - \sum_{j=1}^{n} A_j \sharp_p B_j)\Big] + \log t_0(I - \sum_{j=1}^{n} A_j \sharp_p B_j)
\end{aligned}$$

for fixed real number $t_0 > 0$, *where* $S_p(A|B)$ *is defined in Definition 1.1 and* $A \natural_q B$ *is defined in Definition 1.2.*

Secondly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function $f(t) = -t \log t$.

Theorem 2.2. *Let* $p \in [0,1]$ *and also let* $\{A_1, A_2, ..., A_n\}$ *and* $\{B_1, B_2, ..., B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* H *such that* $\sum_{j=1}^{n} A_j \sharp_p B_j \leq I$, *where* I *means the identity operator on* H. *Then*

$$(2.2) \qquad \sum_{j=1}^{n} S_{p+1}(A_j|B_j)$$

$$\geq \Big[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j) + t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big] \log\Big[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j) + t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$
$$-t_0 \log t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j) \quad \textit{for fixed real number } t_0 > 0,$$

and

(2.2') $$\sum_{j=1}^{n} S_{p-1}(A_j|B_j)$$
$$\leq -\Big[\sum_{j=1}^{n}(A_j\natural_{p-1}B_j) + t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big] \log\Big[\sum_{j=1}^{n}(A_j\natural_{p-1}B_j) + t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$
$$+t_0 \log t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j) \quad \textit{for fixed real number } t_0 > 0,$$

where $S_q(A|B)$ *is defined in Definition 1.1 and* $A\natural_q B$ *is defined in Definition 1.2.*

We shall state the following result which can be shown by combining Theorem 2.1 with Theorem 2.2.

Corollary 2.3. *Let* $p \in [0,1]$ *and also let* $\{A_1, A_2, ..., A_n\}$ *and* $\{B_1, B_2, ..., B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* H *such that* $\sum_{j=1}^{n} A_j\sharp_p B_j \leq I$, *where* I *means the identity operator on* H. *Then*

(2.3) $$\sum_{j=1}^{n} S_{p+1}(A_j|B_j)$$
$$\geq \Big[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j) + (I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big] \log\Big[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j) + (I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$
$$\geq \log\Big[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j) + (I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$
$$\geq \sum_{j=1}^{n} S_p(A_j|B_j)$$
$$\geq -\log\Big[\sum_{j=1}^{n}(A_j\natural_{p-1}B_j) + (I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$
$$\geq -\Big[\sum_{j=1}^{n}(A_j\natural_{p-1}B_j) + (I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big] \log\Big[\sum_{j=1}^{n}(A_j\natural_{p-1}B_j) + (I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$
$$\geq \sum_{j=1}^{n} S_{p-1}(A_j|B_j)$$

where $S_q(A|B)$ *is defined in Definition 1.1 and* $A\natural_q B$ *is defined in Definition 1.2.*

Corollary 2.3 easily implies the following result which can be considered as *operator version of Shannon inequality and its reverse one.*

Corollary 2.4. *Let* $\{A_1, A_2, ..., A_n\}$ *and* $\{B_1, B_2, ..., B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* H. *If* $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, *then*

$$(2.4)\quad \sum_{j=1}^{n} S_2(A_j|B_j) \geq \Big[\sum_{j=1}^{n} B_j A_j^{-1} B_j\Big] \log\Big[\sum_{j=1}^{n} B_j A_j^{-1} B_j\Big] \geq \log\Big[\sum_{j=1}^{n} B_j A_j^{-1} B_j\Big]$$
$$\geq \sum_{j=1}^{n} S_1(A_j|B_j) \geq 0 \geq \sum_{j=1}^{n} S(A_j|B_j)$$
$$\geq -\log\Big[\sum_{j=1}^{n} A_j B_j^{-1} A_j\Big] \geq -\Big[\sum_{j=1}^{n} A_j B_j^{-1} A_j\Big] \log\Big[\sum_{j=1}^{n} A_j B_j^{-1} A_j\Big]$$
$$\geq \sum_{j=1}^{n} S_{-1}(A_j|B_j).$$

Remark 2.1. We recall $S_q(A|B)$ for $A > 0$, $B > 0$ and any real number q as follows:

$$S_q(A|B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^q (\log A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}.$$

By an easy calculation we have

$$\tfrac{d}{dq}[S_q(A|B)] = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^q [\log A^{\frac{-1}{2}} B A^{\frac{-1}{2}}]^2 A^{\frac{1}{2}} \geq 0,$$

so that $S_q(A|B)$ is an increasing function of q, and it is interesting to point out that the decreasing order of the positions of $\sum_{j=1}^{n} S_2(A_j|B_j)$, $\sum_{j=1}^{n} S_1(A_j|B_j)$, $\sum_{j=1}^{n} S(A_j|B_j)$, and $\sum_{j=1}^{n} S_{-1}(A_j|B_j)$ in (2.4) of Corollary 2.4 is quite reasonable since $\sum_{j=1}^{n} S(A_j|B_j) = \sum_{j=1}^{n} S_0(A_j|B_j)$.

§3 Propositions needed to give proofs of the results in §2

By careful scrutinizing nice proofs in [5, Theorem 2.1] and [4, Theorem], we have the following parallel result to [5, Theorem 2.1].

Proposition 3.1. *If* f *is a continuous, real function on an interval* J, *the following conditions are equivalent:*

(i) f *is operator concave.*

(ii) $f(C^*AC + t_0(I - C^*C)) \geq C^*f(A)C + f(t_0)(I - C^*C)$

for operator C *with* $||C|| \leq 1$ *and self-adjoint operator* A *with* $\sigma(A) \subseteq J$ *and for fixed real number* $t_0 \in J$.

(iii) $f\Big(\sum_{j=1}^{n} C_j^*A_jC_j + t_0(I - \sum_{j=1}^{n} C_j^*C_j)\Big) \geq \sum_{j=1}^{n} C_j^*f(A_j)C_j + f(t_0)(I - \sum_{j=1}^{n} C_j^*C_j)$

for operators C_j *with* $\sum_{j=1}^{n} C_j^*C_j \leq I$ *and self-adjoint operators* A_j *with* $\sigma(A_j) \subseteq J$ *for* $j = 1, 2, ..., n$ *and for fixed real number* $t_0 \in J$.

(iv) $f(\sum_{j=1}^{n} C_j^*A_jC_j) \geq \sum_{j=1}^{n} C_j^*f(A_j)C_j$

for operators C_j *with* $\sum_{j=1}^{n} C_j^*C_j = I$ *and self-adjoint operators* A_j *with* $\sigma(A_j) \subseteq J$ *for* $j = 1, 2, ..., n$, *where* $n \geq 2$.

(v) $f(PAP + t_0(I - P)) \geq Pf(A)P + f(t_0)(I - P)$

for projection P *and self-adjoint operator* A *with* $\sigma(A) \subseteq J$ *and for fixed real number* $t_0 \in J$.

Corollary 3.2. *If* f *is continuous operator concave function on the half open interval* $[0, \alpha)$ *to* $[0, \alpha)$ *with* $\alpha \leq \infty$, *then*

$$\begin{aligned} f(\sum_{j=1}^{n} C_j^*A_jC_j) &\geq \sum_{j=1}^{n} C_j^*f(A_j)C_j + f(0)(I - \sum_{j=1}^{n} C_j^*C_j) \\ &\geq \sum_{j=1}^{n} C_j^*f(A_j)C_j \end{aligned}$$

for operators C_j *with* $\sum_{j=1}^{n} C_j^*C_j \leq I$ *and self-adjoint operators* A_j *with* $\sigma(A_j) \subseteq [0, \alpha)$ *for* $j = 1, 2, ..., n$.

We recall the following obvious Proposition 3.3.

Proposition 3.3. *Let* $A > 0$ *and* $B > 0$. *Then*

(i) $A\natural_{-1}B = AB^{-1}A$, (ii) $A\natural_2B = BA^{-1}B$, (iii) $A\natural_0B = A$, (iv) $A\natural_1B = B$, *and*

(v) $A \log A \geq \log A$ *for any* $A > 0$.

Remark 3.1. If (i') f is continuous operator concave on J containing 0 and $f(0) \geq 0$, then the following (ii') holds by (i) and (ii) of Proposition 5.1

(ii') $$f(C^*AC) \geq C^*f(A)C + f(0)(I - C^*C) \geq C^*f(A)C$$

for operator C with $||C|| \leq 1$ and self-adjoint operator A with $\sigma(A) \subseteq J$ since $f(0) \geq 0$ and $I - C^*C \geq 0$.

As "*f is continuous operator concave function and $f(0) \geq 0$*" just essentially corresponds to "*f is continuous operator convex function and $f(0) \leq 0$*" in (i) of [5, Theorem 2.1], it turns out that Proposition 3.1 is essentially shown under an additional condition $f(0) \geq 0$ in [5, Theorem 2.1],*briefly speaking, Proposition 3.1 with $f(0) \geq 0$ becomes Theorem 2.1 in* [5].

Remark 3.2. It is shown in [6 ,Theorem 6] that if f is operator monotone function, (iv) of Proposition 3.1 holds. Also Corollary 3.2 implies that if f is an operator monotone function on the half open interval $[0, \alpha)$ to $[0, \alpha)$ with $\alpha \leq \infty$, then $f(\sum_{j=1}^{n} C_j^* A_j C_j) \geq \sum_{j=1}^{n} C_j^* f(A_j) C_j$ for operators C_j with $\sum_{j=1}^{n} C_j^* C_j \leq I$ and self-adjoint operators A_j with $\sigma(A_j) \subseteq [0, \alpha)$ for $j = 1, 2, ..., n$, which is shown in [6 , Corollary 7], bacause f is operator concave on $[0, \alpha)$ to $[0, \alpha)$ with $\alpha \leq \infty$ if and only if f is operator monotone on $[0, \alpha)$ to $[0, \alpha)$ with $\alpha \leq \infty$.

Addendum. After we have written this manuscript, we know that quite similar results to Proposition 5.1 are shown in the following recent paper: F.Hansen and G.K.Pedersem, Jensen's operator inequality, Bull. London Math. Soc., **35**(2003), 553-564.

This paper will appear elsewhere with complete proofs.

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