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Relations between two operator inequalities via operator means

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Abstract

Let A and B be (not necessarily invertible) positive operators. Recently, the author and Yamazaki discussed relations between

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \quad \text{and} \quad A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$$

for $p \geq 0$ and $r \geq 0$, and also Yamazaki and Yanagida discussed relations between

$$\tfrac{p}{p+r}I + \tfrac{r}{p+r}B^{\frac{r}{2}}A^pB^{\frac{r}{2}} \geq B^r \quad \text{and} \quad A^p \geq \frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\frac{r}{p+r}A^{\frac{p}{2}}B^rA^{\frac{p}{2}} + \frac{p}{p+r}I}$$

for $p \geq 0$ and $r \geq 0$.

In this report, as a generalization of their results via the representing functions of operator means, we shall show relations between two operator inequalities

$$f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \quad \text{and} \quad A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}}),$$

where f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$.

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator T is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$. We denote the set of positive operators by $\mathcal{B}(\mathcal{H})_+$.

Kubo-Ando [8] investigated an axiomatic approach for operator means (see also [5]). A binary operation $\sigma : \mathcal{B}(\mathcal{H})_+ \times \mathcal{B}(\mathcal{H})_+ \to \mathcal{B}(\mathcal{H})_+$ is called an operator connection if it satisfies the following conditions (i), (ii) and (iii) for $A, B, C, D \in \mathcal{B}(\mathcal{H})_+$:

(i) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$,

(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$,

(iii) $A_n, B_n \in \mathcal{B}(\mathcal{H})_+$, $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$,
where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \cdots$ and A_n converges strongly to A.

An operator connection σ is called an *operator mean* if

(iv) $I\sigma I = I$.

There exists a one-to-one correspondence between an operator connection σ and an operator monotone function $f \geq 0$ on $[0, \infty)$. The operator connection σ can be defined via the corresponding function f, which is called the *representing function* of σ, by

$$A\sigma B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$$

if A is invertible, and σ is an operator mean if and only if $f(1) = 1$.

The following are typical examples of operator means. For positive invertible operators A and B, and for $\alpha \in [0,1]$,

(i) Arithmetic mean: $A \nabla_\alpha B = (1-\alpha)A + \alpha B$,

(ii) Geometric mean (α-power mean): $A \sharp_\alpha B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$,

(iii) Harmonic mean: $A !_\alpha B = \{(1-\alpha)A^{-1} + \alpha B^{-1}\}^{-1}$.

The representing functions of ∇_α, $\sharp_\alpha$ and $!_\alpha$ are $(1-\alpha)+\alpha t$, t^α and $\{(1-\alpha)+\alpha t^{-1}\}^{-1} = \frac{t}{(1-\alpha)t+\alpha}$, respectively. On these operator means, the following relations are known. We remark that (1.1) was shown in [4], and (1.1) and (1.2) can be proved without using properties of operator means. *Let A and B be positive invertible operators. For each $p \geq 0$ and $r \geq 0$,*

$$B^{-r} \sharp_{\frac{r}{p+r}} A^p \geq I \iff I \geq A^{-p} \sharp_{\frac{p}{p+r}} B^r \tag{1.1}$$

and

$$B^{-r} \nabla_{\frac{r}{p+r}} A^p \geq I \iff I \geq A^{-p} !_{\frac{p}{p+r}} B^r. \tag{1.2}$$

(1.1) is closely related to Furuta inequality [3], and a mean theoretic approach to Furuta inequality was disscussed in [1][7] and others. We remark the following ralations on inequalities in (1.1) and (1.2): *Let A and B be positive invertible operators. For each $p \geq 0$ and $r \geq 0$,*

$$A \geq B \Longrightarrow \log A \geq \log B \Longrightarrow \begin{cases} B^{-r} \sharp_{\frac{r}{p+r}} A^p \geq I, \\ I \geq A^{-p} \sharp_{\frac{p}{p+r}} B^r \end{cases} \Longrightarrow \begin{cases} B^{-r} \nabla_{\frac{r}{p+r}} A^p \geq I, \\ I \geq A^{-p} !_{\frac{p}{p+r}} B^r. \end{cases}$$

The first relation holds since $\log t$ is operator monotone, the second was shown in [2][4], and the third holds since $(1-\alpha) + \alpha t \geq t^\alpha \geq \frac{t}{(1-\alpha)t+\alpha}$ for $t \geq 0$ and $\alpha \in [0,1]$. We remark that it was shown in [2][4] that

$$\begin{aligned} \log A \geq \log B &\iff B^{-r} \sharp_{\frac{r}{p+r}} A^p \geq I \quad \text{for all } p \geq 0 \text{ and } r \geq 0 \\ &\iff I \geq A^{-p} \sharp_{\frac{p}{p+r}} B^r \quad \text{for all } p \geq 0 \text{ and } r \geq 0. \end{aligned}$$

In this report, firstly we attempt a mean theoretic approach to (1.1) and (1.2). In other words, we shall state a result corresponding to (1.1) and (1.2) on a general operator mean for invertible operators. Secondly we shall show relations between

$$f(B^{\frac{1}{2}}AB^{\frac{1}{2}})\geq B \quad \text{and} \quad A\geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$$

for (not necessarily invertible) positive operators A and B, where f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$. This result is a further generalization of the former argument via the representing functions of operator means. Moreover this result includes the ones by the author and Yamazaki [6] and by Yamazaki and Yanagida [11].

2 A result on a general operator mean

In this section, we shall state a result corresponding to (1.1) and (1.2) on a general operator mean for invertible operators. At first we state definitions and propreties of some operator means via a operator mean σ.

Definition ([8]). *Let σ be the operator mean with a representing function f.*

(i) *σ' is said to be the transpose of σ if σ' is the operator mean with a representing function $tf(t^{-1})$.*

(ii) *σ^* is said to be the adjoint of σ if σ^* is the operator mean with a representing function $\{f(t^{-1})\}^{-1}$.*

(iii) *$\sigma^{\perp}$ is said to be the dual of σ if $\sigma^{\perp}$ is the operator mean with a representing function $\frac{t}{f(t)}$.*

We remark that these representing functions can be defined on $[0,\infty)$ by setting the value on 0 by the limit to $+0$ since f is operator monotone.

Proposition 2.A ([8]). *Let σ be an operator mean and $A,B\in\mathcal{B}(\mathcal{H})_+$.*

(i) $A\sigma'B=B\sigma A$.

(ii) $A\sigma^*B=(A^{-1}\sigma B^{-1})^{-1}$ *if A and B are invertible.*

(iii) $(\sigma')'=(\sigma^*)^*=(\sigma^{\perp})^{\perp}=\sigma$.

(iv) $\sigma^{\perp}=(\sigma')^*=(\sigma^*)'$, $\sigma'=(\sigma^*)^{\perp}=(\sigma^{\perp})^*$ *and* $\sigma^*=(\sigma^{\perp})'=(\sigma')^{\perp}$.

By using Prposition 2.A, we shall show a generalization of (1.1) and (1.2).

Proposition 2.1. *Let A and B be positive invertible operators. For every operator mean σ,*

$$B^{-1}\sigma A \geq I \iff I \geq A^{-1}\sigma^{\perp}B. \tag{2.1}$$

Proof. By (i) of Proposition 2.A,

$$B^{-1}\sigma A = A\,\sigma' B^{-1} \geq I. \tag{2.2}$$

By (ii) and (iv) of Proposition 2.A, (2.2) is equivalent to

$$I \geq (A\,\sigma' B^{-1})^{-1} = A^{-1}(\sigma')^{*}B = A^{-1}\sigma^{\perp}B.$$

Hence the proof is complete. □

Since $(\sharp_\alpha)^\perp = \sharp_{1-\alpha}$ and $(\nabla_\alpha)^\perp = !_{1-\alpha}$, Proposition 2.1 leads (1.1) (resp. (1.2)) by replacing A and B with A^p and B^r and by putting $\sigma = \sharp_{\frac{r}{p+r}}$ (resp. $\sigma = \nabla_{\frac{r}{p+r}}$). We remark that (2.1) can be rewritten by

$$f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \iff A \geq \frac{A^{\frac{1}{2}}BA^{\frac{1}{2}}}{f(A^{\frac{1}{2}}BA^{\frac{1}{2}})} \tag{2.3}$$

with the representing function f of σ.

3 Main results

In this section, we shall show a further generalization of Proposition 2.1 via the representing functions of operator means.

When we rewrite (1.1) and (1.2) for positive invertible operators A and B by

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \iff A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \tag{3.1}$$

and

$$\tfrac{p}{p+r}I + \tfrac{r}{p+r}B^{\frac{r}{2}}A^pB^{\frac{r}{2}} \geq B^r \iff A^p \geq \frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\frac{r}{p+r}A^{\frac{p}{2}}B^rA^{\frac{p}{2}} + \frac{p}{p+r}I} \tag{3.2}$$

with the representing functions, we can consider non-invertible operators on this argument. On relations between two inequalities in (3.1) and (3.2) for (not necessarily invertible) positive operators A and B, the following results were obtained in [6] and [11].

Theorem 3.A ([6]). *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$, the following assertions hold:*

(i) *If* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, *then* $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$.

(ii) *If* $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ *and* $N(A) \subseteq N(B)$, *then* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.

Theorem 3.B ([11]). *Let A and B be positive operators. Then for each $p > 0$ and $r \geq 0$, the following assertions hold:*

(i) *If* $\frac{p}{p+r}I + \frac{r}{p+r}B^{\frac{r}{2}}A^pB^{\frac{r}{2}} \geq B^r$, *then* $A^p \geq \dfrac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\frac{r}{p+r}A^{\frac{p}{2}}B^rA^{\frac{p}{2}} + \frac{p}{p+r}I}$.

(ii) *If* $A^p \geq \dfrac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\frac{r}{p+r}A^{\frac{p}{2}}B^rA^{\frac{p}{2}} + \frac{p}{p+r}I}$ *and* $N(A) \subseteq N(B)$, *then* $\frac{p}{p+r}I + \frac{r}{p+r}B^{\frac{r}{2}}A^pB^{\frac{r}{2}} \geq B^r$.

Here we shall obtain a generalization of Proposition 2.1 via the form of (2.3). This result is also an extension of Theorems 3.A and 3.B.

Theorem 3.1. *Let A and B be positive operators, and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying*

$$f(t)g(t) = t. \tag{3.3}$$

(i) *If* $g(0) = 0$ *or* $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$, *then* $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$ *ensures* $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$.

(ii) *If* $N(A) \subseteq N(B)$, *then* $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$ *ensures* $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$.

In Theorem 3.1, f and g are not necessarily operator monotone functions. We also remark that if $f(0) > 0$, then automatically $g(0) = 0$ by (3.3).

If A and B are positive invertible operators and σ is the operator mean with a representing function f, Theorem 3.1 ensures Proposition 2.1 since (2.1) is equivalent to (2.3). Theorem 3.1 also leads Theorem 3.A (resp. Theorem 3.B) by replacing A and B with A^p and B^r and by putting $f(t) = t^{\frac{r}{p+r}}$ and $g(t) = t^{\frac{p}{p+r}}$ (resp. $f(t) = \frac{p}{p+r} + \frac{r}{p+r}t$ and $g(t) = \frac{t}{\frac{r}{p+r}t + \frac{p}{p+r}}$). We remark that $g(0) = 0$ in these cases.

We need some lemmas in order to prove Theorem 3.1.

Lemma 3.C. *Let T be a positive operator. Then*

$$\lim_{\varepsilon \to +0} T^{\frac{1}{2}}(T + \varepsilon I)^{-1}T^{\frac{1}{2}} = \lim_{\varepsilon \to +0}(T + \varepsilon I)^{-1}T = P_{N(T)^{\perp}},$$

where $P_{\mathcal{M}}$ is a projection onto a closed subspace $\mathcal{M}$.

Lemma 3.C is a well-known result. For example, it was shown in [9] and [6].

Lemma 3.2. *Let f be a non-negative continuous function on $[0,\infty)$ such that $f(0)=0$ and $f(t)>0$ for $t>0$. Then $N(f(T))=N(T)$ for every positive operator T.*

Proof. Let $T=\int_0^{\|T\|} tdE_t$ be the spectral decomposition of a positive operator T. Then

$$(f(T)x,y)=\int_0^{\|T\|} f(t)d(E_tx,y) \quad \text{for } x,y\in\mathcal{H}. \tag{3.4}$$

We remark that $E_{-0}=0$.

Assume that $x\in N(T)$. Then $E_0x=(E_0-E_{-0})x=P_{N(T)}x=x$, and $(f(T)x,y)=f(0)(x,y)=0$ for any $y\in\mathcal{H}$ by (3.4). Therefore $f(T)x=0$, so that $x\in N(f(T))$.

Conversely, assume that $x\in N(f(T))$. Then for $\varepsilon>0$,

$$0=(f(T)x,x)=\int_0^{\varepsilon} f(t)d(E_tx,x)+\int_{\varepsilon}^{\|T\|} f(t)d(E_tx,x)$$

by (3.4). Since $f(t)>0$ for $t>0$, $E_\varepsilon x=x$ for $\varepsilon>0$. By tending $\varepsilon\to+0$, we have $P_{N(T)}x=E_0x=x$, so that $x\in N(T)$. □

Lemma 3.3. *Let $T=U|T|$ be the polar decomposition of an operator T, and let f be a continuous function on $[0,\infty)$. Then*

$$Uf(|T|)U^*=f(|T^*|)-f(0)(I-UU^*).$$

Proof. First we shall show the case $f(0)=0$ by the same way to [10, Lemma]. Since $U|T|^nU^*=|T^*|^n$ for each positive integer n, $Up(|T|)U^*=p(|T^*|)$ holds for any polynomials p such that $p(0)=0$. By taking a sequence $\{p_n\}$ of polynomials with $p_n(0)=0$ which convarges uniformly to f on $[0,\|T\|]$, we obtain $Uf(|T|)U^*=f(|T^*|)$ for general f with $f(0)=0$.

Next, let $g(t)=f(t)-f(0)$. Then $g(0)=0$, so that

$$\begin{aligned}Uf(|T|)U^*&=U\{g(|T|)+f(0)I\}U^*=Ug(|T|)U^*+f(0)UU^*\\&=g(|T^*|)+f(0)I-f(0)(I-UU^*)=f(|T^*|)-f(0)(I-UU^*).\end{aligned}$$

Hence the proof is complete. □

Proof of Theorem 3.1. Let $\varepsilon>0$.
Proof of (i). Since $f(B^{\frac{1}{2}}AB^{\frac{1}{2}})\geq B$, we obtain

$$(B+\varepsilon I)^{-1}\geq\left\{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)+\varepsilon I\right\}^{-1}.$$

Let $A^{\frac{1}{2}}B^{\frac{1}{2}}=U|A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition of $A^{\frac{1}{2}}B^{\frac{1}{2}}$. Then we have

$$\begin{aligned}
&A^{\frac{1}{2}}B^{\frac{1}{2}}(B+\varepsilon I)^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}\\
&\geq A^{\frac{1}{2}}B^{\frac{1}{2}}\{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)+\varepsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}\\
&=U|A^{\frac{1}{2}}B^{\frac{1}{2}}|\{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)+\varepsilon I\}^{-1}|A^{\frac{1}{2}}B^{\frac{1}{2}}|U^*\\
&=U\{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)+\varepsilon I\}^{-1}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2U^*\\
&=U\{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)+\varepsilon I\}^{-1}f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)g(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)U^* \quad \text{by (3.3)}.
\end{aligned} \tag{3.5}$$

In (3.5), by tending $\varepsilon\to+0$ and Lemma 3.C, we obtain

$$A^{\frac{1}{2}}P_{N(B)^\perp}A^{\frac{1}{2}}\geq UP_{N\left(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)\right)^\perp}g(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)U^*=Ug(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)U^* \tag{3.6}$$

by the following: If $f(0)>0$, then $f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)$ is invertible and $P_{N\left(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)\right)^\perp}=I$. If $f(0)=0$, then $UP_{N\left(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)\right)^\perp}=UP_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)^\perp}=UP_{N(A^{\frac{1}{2}}B^{\frac{1}{2}})^\perp}=U$ by Lemma 3.2.

Therefore, noting that $UU^*=P_{N(B^{\frac{1}{2}}A^{\frac{1}{2}})^\perp}=P_{N(A^{\frac{1}{2}}BA^{\frac{1}{2}})^\perp}=I$ if $N(A^{\frac{1}{2}}BA^{\frac{1}{2}})=\{0\}$, we have

$$\begin{aligned}
A&\geq A^{\frac{1}{2}}P_{N(B)^\perp}A^{\frac{1}{2}}\\
&\geq Ug(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2)U^* && \text{by (3.6)}\\
&=g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)-g(0)(I-UU^*) && \text{by Lemma 3.3}\\
&=g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2) && \text{since } g(0)=0 \text{ or } N(A^{\frac{1}{2}}BA^{\frac{1}{2}})=\{0\}\\
&=g(A^{\frac{1}{2}}BA^{\frac{1}{2}}).
\end{aligned}$$

Proof of (ii). Since $A\geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$, we obtain

$$(A+\varepsilon I)^{-1}\leq\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)+\varepsilon I\}^{-1}.$$

Let $B^{\frac{1}{2}}A^{\frac{1}{2}}=V|B^{\frac{1}{2}}A^{\frac{1}{2}}|$ be the polar decomposition of $B^{\frac{1}{2}}A^{\frac{1}{2}}$. Then we have

$$\begin{aligned}
&B^{\frac{1}{2}}A^{\frac{1}{2}}(A+\varepsilon I)^{-1}A^{\frac{1}{2}}B^{\frac{1}{2}}\\
&\leq B^{\frac{1}{2}}A^{\frac{1}{2}}\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)+\varepsilon I\}^{-1}A^{\frac{1}{2}}B^{\frac{1}{2}}\\
&=V|B^{\frac{1}{2}}A^{\frac{1}{2}}|\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)+\varepsilon I\}^{-1}|B^{\frac{1}{2}}A^{\frac{1}{2}}|V^*\\
&=V\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)+\varepsilon I\}^{-1}|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2V^*\\
&=V\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)+\varepsilon I\}^{-1}g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)f(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)V^* \quad \text{by (3.3)}.
\end{aligned} \tag{3.7}$$

In (3.7), by tending $\varepsilon\to+0$ and Lemma 3.C, we obtain

$$B^{\frac{1}{2}}P_{N(A)^\perp}B^{\frac{1}{2}}\leq VP_{N\left(g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)\right)^\perp}f(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)V^*=Vf(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)V^* \tag{3.8}$$

by the following: If $g(0) > 0$, then $g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)$ is invertible and $P_{N\left(g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)\right)^\perp} = I$. If $g(0) = 0$, then $VP_{N\left(g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)\right)^\perp} = VP_{N(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)^\perp} = VP_{N(B^{\frac{1}{2}}A^{\frac{1}{2}})^\perp} = V$ by Lemma 3.2.

Therefore, noting that $N(A) \subseteq N(B)$ is equivalent to $P_{N(A)^\perp} \geq P_{N(B)^\perp}$, we have

$$\begin{aligned} B &= B^{\frac{1}{2}} P_{N(B)^\perp} B^{\frac{1}{2}} \\ &\leq B^{\frac{1}{2}} P_{N(A)^\perp} B^{\frac{1}{2}} && \text{since } N(A) \subseteq N(B) \\ &\leq V f(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2) V^* && \text{by (3.8)} \\ &= f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) - f(0)(I - VV^*) && \text{by Lemma 3.3} \\ &\leq f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) \\ &= f(B^{\frac{1}{2}}AB^{\frac{1}{2}}). \end{aligned}$$

Hence the proof is complete. □

Corollary 3.4. *Let A and B be positive operators, and let f and g be positive continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$. If $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$, then $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$ is equivalent to $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$.*

Proof. Since $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$ ensures $\{0\} = N(A) \subseteq N(B)$, $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$ is equivalent to $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$ by Theorem 3.1. □

Of course $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$ if A and B are invertible.

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