数理解析研究所講究録 1360 巻 2004 年 24-42

GENERALIZED PRIKRY FORCING AND ITERATION OF GENERIC ULTRAPOWERS

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1. Introduction

In this paper we shall generalize the following well-known theorem on a relation between Prikry forcing and iterated ultrapowers.

Theorem 1.1 (Solovay). *Assume κ is a measurable cardinal and U is a normal ultrafilter on κ. Let $\langle M_n, j_{m,n} \mid m \leq n \leq \omega\rangle$ be the iteration of ultrapowers of V by U. Then the sequence $\langle j_{0,n}(\kappa) \mid n \in \omega\rangle$ is a Prikry generic sequence for M_ω with respect to $j_{0,\omega}(U)$.*

We generalize the above theorem for normal filters which are not necessarily maximal. Of course, the above theorem can be restated using the dual ideal of U. In this paper we argue with ideals instead of filters. To generalize the above theorem, we must generalize Prikry Forcing and the iteration of ultrapowers for normal ideals which are not necessarily maximal. The iteration of ultrapowers has an obvious generalization, i.e. the iteration of generic ultrapowers. On the other hand, there are two natural generalizations of Prikry Forcing.

Let I be a normal ideal on κ. If I is maximal then tree type Prikry Forcing, PR_I, consists of all pairs $\langle t, T\rangle$ such that $t \in {}^{<\omega}\kappa$ and $T \subseteq {}^{<\omega}\kappa$ is a tree in which every node has I-measure 1 immediate successors, i.e. for each $s \in T$, $\{\xi \in \kappa \mid s^\frown\langle\xi\rangle\}$ is in the dual filter of I. The order is defined by $\langle t_1, T_1\rangle \leq \langle t_2, T_2\rangle$ if for each $s_1 \in T_1$, there is $s_2 \in T_2$ such that $t_1{}^\frown s_1 = t_2{}^\frown s_2$. In this case "$I$-measure 1" and "$I$-positive" coincide, but if I is not maximal then this is not the case. So if I is not maximal then there are two natural generalizations of Prikry Forcing, PR_I^* and PR_I^+. PR_I^* consists of all $\langle t, T\rangle$ such that $t \in {}^{<\omega}\kappa$ and T is a tree in which every node has I-measure 1 immediate successors. PR_I^+ consists of all $\langle t, T\rangle$ such that $t \in {}^{<\omega}\kappa$ and T is a tree in which every node has I-positive immediate successors. In both PR_I^* and PR_I^+, order is defined in the same way as Prikry Forcing. In this paper we generalize the above theorem for both PR_I^* and PR_I^+. (Theorem 3.3 and 3.5).

In Section 2 we study basic facts on the finite step iteration of generic ultrapowers. In particular, we show that the n-th iterate of generic ultrapowers by an ideal I can be represented as a one-step generic ultrapower by the n-th Fubini power of I. In Section 3 we generalize Solovay's theorem for both PR_I^* and PR_I^+.

Basic Definitions and facts about embeddings between partial orderings:

Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orderings.

$\sigma : \mathbb{P} \to \mathbb{Q}$ is a complete embedding if

(1) σ is order preserving, i.e. $\forall p_1, p_2 \in \mathbb{P}$, $p_1 \le p_2 \rightarrow \sigma(p_1) \le \sigma(p_2)$.
(2) If $A \subseteq \mathbb{P}$ is a maximal antichain of $\mathbb{P}$, $\sigma[A]$ is a maximal antichain of $\mathbb{Q}$.

$\pi : \mathbb{Q} \rightarrow \mathbb{P}$ is a projection if

(1) π is order preserving, i.e. $\forall q_1, q_2 \in \mathbb{Q}$, $q_1 \le q_2 \rightarrow \pi(q_1) \le \pi(q_2)$.
(2) $\forall q \in \mathbb{Q} \forall p \in \mathbb{P}$, if $p \le \pi(q)$ then there is a $q^* \le q$ such that $\pi(q^*) = p$.

Projections which appear in this paper have the following additional property:

(3) $\forall q \in \mathbb{Q} \forall p \in \mathbb{P}$, if $p \ge \pi(q)$ then there is a $q^* \ge q$ such that $\pi(q^*) = p$.

We call π a good projection if π satisfies (1)-(3).

If $\sigma : \mathbb{P} \rightarrow \mathbb{Q}$ is a complete embedding and G is $\mathbb{P}$-generic then the quotient $\mathbb{Q}/_\sigma G$ is the p.o. obtained from restricting $\mathbb{Q}$ to $\{q \in \mathbb{Q} \mid \forall p \in G, q$ is compatible with $\sigma(p)\}$. If $\pi : \mathbb{Q} \rightarrow \mathbb{P}$ is a projection and G is $\mathbb{P}$-generic then the quotient $\mathbb{Q}/_\pi G$ is the p.o. obtained from restricting $\mathbb{Q}$ to $\pi^{-1}[G]$. If σ or π is clear from the context, we just write $\mathbb{Q}/G$ for $\mathbb{Q}/_\sigma G$ or $\mathbb{Q}/_\pi G$.

We present basic facts on complete embeddings and projections without proof.

Fact . *Let $\mathbb{P}$ and $\mathbb{Q}$ be p.o..*

(1) *Assume that $\sigma : \mathbb{P} \rightarrow \mathbb{Q}$ is a complete embedding. Then H is $(V, \mathbb{Q})$-generic iff $G := \sigma^{-1}[H]$ is $(V, \mathbb{P})$-generic and H is $(V[G], \mathbb{Q}/_\sigma G)$-generic.*
(2) *Assume that $\pi : \mathbb{Q} \rightarrow \mathbb{P}$ is a good projection. Then H is $(V, \mathbb{Q})$-generic iff $G := \pi[H]$ is $(V, \mathbb{P})$-generic and H is $(V[G], \mathbb{Q}/_\pi G)$-generic.*
(3) *Assume that $\sigma : \mathbb{P} \rightarrow \mathbb{Q}$ is a complete embedding, $\pi : \mathbb{Q} \rightarrow \mathbb{P}$ is a projection and $\pi \circ \sigma = id$.*
   (a) *If H is $(V, \mathbb{Q})$-generic then $\sigma^{-1}[H] = \pi[H]$.*
   (b) *If G is $(V, \mathbb{P})$-generic then $\mathbb{Q}/_\sigma G = \mathbb{Q}/_\pi G$.*

2. Finite step Iteration of generic ultrapowers.

In this section we study basics on the finite step iteration of generic ultrapowers. This is a natural generalization of Kunen's theory of iterated ultrapower. If $n \in \omega$ and U is an ultrafilter on κ then the n-th power of U, U^n, can be defined as an ultrafilter on ${}^n\kappa$ and the n-th iterate of ultrapowers of V by U can be represented as a one-step ultrapower of V by U^n. In this section we generalize this for the iteration of generic ultrapowers.

2.1. Fubini powers of ideals.

In this subsection we introduce the Fubini powers of ideals and their basic properties. Throughout this subsection, let κ be an uncountable regular cardinal and I be a κ-complete ideal on κ.

For each $n \in \omega$, the n-th Fubini power of I, I^n, is the ideal on ${}^n\kappa$ defined as follows: Let $I^0 := \{\emptyset\}$. Note that ${}^0\kappa = \{\langle\rangle\}$, where $\langle\rangle$ is the empty sequence. So I^0 is an ideal on ${}^0\kappa$ and $(I^0)^+ = (I^0)^* = \{\{\langle\rangle\}\}$. Assuming I^n was defined as an ideal on ${}^n\kappa$, let I^{n+1} be the ideal on ${}^{n+1}\kappa$ such that for each $A \subseteq {}^{n+1}\kappa$

$$A \in I^{n+1} \;\Leftrightarrow\; \{s \in {}^n\kappa \mid \{\xi < \kappa \mid s ^\frown \langle\xi\rangle \in A\} \in I\ \} \in (I^n)^*.$$

It can be easily seen that I^{n+1} is a κ-complete ideal on ${}^{n+1}\kappa$. Note that I^1 and I are the same if we identify κ with ${}^1\kappa$ in the obvious way. (For each sequence s, $\langle\rangle ^\frown s = s ^\frown \langle\rangle = s$.)

The following lemma is basic:

Lemma 2.1. *Assume $m \le n \in \omega$. Then for each $A \subseteq {}^n\kappa$*

(1) $A \in I^n \Leftrightarrow \{s \in {}^m\kappa \mid \{t \in {}^{n-m}\kappa \mid s^\frown t \in A\} \in I^{n-m}\} \in (I^m)^*$,
(2) $A \in (I^n)^+ \Leftrightarrow \{s \in {}^m\kappa \mid \{t \in {}^{n-m}\kappa \mid s^\frown t \in A\} \in (I^{n-m})^+\} \in (I^m)^+$,
(3) $A \in (I^n)^* \Leftrightarrow \{s \in {}^m\kappa \mid \{t \in {}^{n-m}\kappa \mid s^\frown t \in A\} \in (I^{n-m})^*\} \in (I^m)^*$.

Proof. By induction on the lexicographical order of (n, m), we show (1)-(3) simultaneously. If $n = m = 0$ then (1)-(3) are trivial. Assume $m \leq n \in \omega$ and (1)-(3) are true for each pair m', n' such that $m' \leq n'$ and $(n', m') < (n, m)$. Because (2) and (3) follow from (1), it suffices to show (1) for m, n. If $m = n$ then (1) is trivial and if $m = n - 1$ then (1) is the definition of I^n. So we may assume $m < n - 1$.

Take an arbitrary $A \subseteq {}^n\kappa$. For each $s \in {}^{n-1}\kappa$, let A_s be $\{\xi < \kappa \mid s^\frown\langle\xi\rangle \in A\}$. Then

$$\begin{aligned} & A \in I^n \\ \Leftrightarrow\ & \{s \in {}^{n-1}\kappa \mid A_s \in I\} \in (I^{n-1})^* \\ \Leftrightarrow\ & \{t \in {}^m\kappa \mid \{u \in {}^{n-1-m}\kappa \mid A_{t^\frown u} \in I\} \in (I^{n-1-m})^*\} \in (I^m)^* \\ \Leftrightarrow\ & \{t \in {}^m\kappa \mid \{v \in {}^{n-m}\kappa \mid t^\frown v \in A\} \in I^{n-m}\} \in (I^m)^*. \end{aligned}$$

The first and third equivalences follow from the definition of I^n and I^{n-m}. The second equivalence follows from the induction hypothesis. □

If $m \leq n < \omega$, there are a natural complete embedding and a projection between $\mathbb{P}_{I^m}$ and $\mathbb{P}_{I^n}$.

Let $\sigma_{m,n} : \mathcal{P}({}^m\kappa) \to \mathcal{P}({}^n\kappa)$ be the function such that for each $A \subseteq {}^m\kappa$,

$$\sigma_{m,n}(A) := \{s \in {}^n\kappa \mid s \restriction m \in A\}$$

and let $\pi_{n,m} : \mathcal{P}({}^n\kappa) \to \mathcal{P}({}^m\kappa)$ be the function such that for each $B \subseteq {}^n\kappa$,

$$\pi_{n,m}(B) := \{s \in {}^m\kappa \mid \{t \in {}^{n-m}\kappa \mid s^\frown t \in B\} \in (I^{n-m})^+\}.$$

Note that if $m = n$ then $\sigma_{m,n} = \pi_{n,m} = id$.

By Lemma 2.1, if $m \leq n$ then $\sigma_{m,n}[(I^m)^+] \subseteq (I^n)^+$ and $\pi_{n,m}[(I^n)^+] \subseteq (I^m)^+$. Moreover, as we show in the next lemma, $\sigma_{m,n} \restriction (I^m)^+$ is a complete embedding from $\mathbb{P}_{I^m}$ to $\mathbb{P}_{I^n}$ and $\pi_{n,m} \restriction (I^n)^+$ is a projection from $\mathbb{P}_{I^n}$ to $\mathbb{P}_{I^m}$. We call $\sigma_{m,n}$ the natural complete embedding associated with I and call $\pi_{n,m}$ the natural projection associated with I.

Lemma 2.2. *Assume $l \leq m \leq n \in \omega$. Then the following hold:*

(1) $\sigma_{m,n} \circ \sigma_{l,m} = \sigma_{l,n}$.
(2) $\pi_{m,l} \circ \pi_{n,m} = \pi_{n,l}$.
(3) $\pi_{n,m} \circ \sigma_{m,n} = id \restriction \mathcal{P}({}^m\kappa)$.
(4) $A \in (I^m)^+ \Leftrightarrow \sigma_{m,n}(A) \in (I^n)^+$, *for each* $A \subseteq {}^n\kappa$.
(5) $A \in (I^n)^+ \Leftrightarrow \pi_{n,m}(A) \in (I^m)^+$, *for each* $A \subseteq {}^m\kappa$.
(6) $\sigma_{m,n} \restriction (I^m)^+ : \mathbb{P}_{I^m} \to \mathbb{P}_{I^n}$ *is a complete embedding.*
(7) $\pi_{n,m} \restriction (I^n)^+ : \mathbb{P}_{I^n} \to \mathbb{P}_{I^m}$ *is a good projection.*

Proof. (1) and (3) are clear by the definition of σ and π. (2),(4) and (5) are clear by Lemma 2.1. So we show (6) and (7). We can assume $m < n$.

(6). Clearly $\sigma_{m,n}$ is order preserving and $A \perp B \to \sigma_{m,n}(A) \perp \sigma_{m,n}(B)$ for each $A, B \in \mathbb{P}_{(I^m)}$. So it suffices to show that if $M \subseteq \mathbb{P}_{I^m}$ is predense then $\sigma_{m,n}[M]$ is predense in $\mathbb{P}_{I^n}$. Assume $M \subseteq \mathbb{P}_{I^m}$ is predense. Take an arbitrary $A \in \mathbb{P}_{I^n}$. We must find $B \in M$ such that $\sigma_{m,n}(B) \cap A \in (I^n)^+$. Because $\pi_{n,m}(A) \in \mathbb{P}_{I^m}$ we can take $B \in M$ such that $B \cap \pi_{n,m}(A) \in (I^m)^+$. Then for each $s \in B \cap \pi_{n,m}(A)$,

$$\{t \in {}^{n-m}\kappa \mid s^\frown t \in \sigma_{m,n}(B) \cap A\} = \{t \in {}^{n-m}\kappa \mid s^\frown t \in A\} \in (I^{n-m})^+.$$

So, by Lemma 2.1, $\sigma_{m,n}(B) \cap A \in (I^n)^+$.

(7). Clearly $\pi_{n,m}$ is order preserving. By (3), $\pi_{n,m}$ is surjective. Assume $A \in \mathbb{P}_{I^n}$, $B \in \mathbb{P}_{I^m}$ and $B \leq \pi_{n,m}(A)$. Then, for each $s \in B$,

$$\{t \in {}^{n-m}\kappa \mid s^\frown t \in \sigma_{m,n}(B) \cap A\} = \{t \in {}^{n-m}\kappa \mid s^\frown t \in A\} \in (I^{n-m})^+.$$

So $C := \sigma_{m,n}(B) \cap A \in (I^n)^+$. Moreover, clearly, $C \leq A$ and $\pi_{n,m}(C) = B$. So $\pi_{n,m}$ is a projection. It is easy to see that $\pi_{n,m}$ is good. □

Lemma 2.3. *Assume I is normal. Let $n \in \omega$. Then $A \in (I^n)^*$ if and only if there is an $X \in I^*$ such that $A \subseteq [X]^n$, where $[X]^n$ is the set of all strictly increasing sequences of elements of X of length n.*

Proof. If $X \in I^*$ then it can be easily seen that $[X]^n \in (I^n)^*$. So ($\Leftarrow$) is true. We show ($\Rightarrow$) by induction on $n \in \omega$. If $n = 0$ or $n = 1$ then this is clear. So, assuming $n > 1$ and ($\Rightarrow$) is true for $n - 1$, we show this for n.

Assume $A \in (I^n)^*$. For each $t \in {}^{n-1}\kappa$, let $A_t := \{\xi < \kappa \mid t^\frown\langle\xi\rangle \in A\}$. Then $B := \{t \in {}^{n-1}\kappa \mid A_t \in I^*\} \in (I^{n-1})^*$. By the induction hypothesis, there is a $Y \in I^*$ such that $B \supseteq [Y]^{n-1}$. For each $\xi < \kappa$, let $A_\xi := \bigcap\{A_t \mid t \in B \wedge max(t) < \xi\}$. Because I is κ-complete $A_\xi \in I^*$. Let $Z := \Delta_{\xi\in\kappa} A_\xi \in I^*$. Then let $X := Y \cap Z \cap Lim(\kappa) \in I^*$. We show that if $s \in [X]^n$ then $s \in A$. Assume $s \in [X]^n$. Then, because $s\restriction n-1 \in [Y]^{n-1}$, $s\restriction n-1 \in B$. Let $\xi := max(s\restriction n-1)+1$. Then $max(s\restriction n-1) < \xi < s(n-1)$. Because $s(n-1) \in Z$, $s(n-1) \in A_\xi$ and so $s(n-1) \in A_{s\restriction n-1}$. This means $s \in A$. □

2.2. **Representation of a finite step iterated generic ultrapower.**

In this subsection we see that a finite step iterated generic ultrapower of V by some ideal I can be represented as a one step generic ultrapower of V by Fubini powers of I.

All through this subsection, in V, fix κ, I, $\langle \mathbb{P}_n \mid n \in \omega\rangle$, $\langle \sigma_{m,n} \mid m \leq n < \omega\rangle$ and $\langle \pi_{n,m} \mid m \leq n < \omega\rangle$ so that

- κ is a regular uncountable cardinal,
- I is a normal precipitous ideal on κ,
- $\sigma_{m,n} : \mathcal{P}({}^m\kappa) \to \mathcal{P}({}^n\kappa)$ is the natural complete embedding associated with I,
- $\pi_{n,m} : \mathcal{P}({}^n\kappa) \to \mathcal{P}({}^m\kappa)$ is the natural projection associated with I,
- $\mathbb{P}_n := \mathbb{P}_{I^n}$.

Our first aim is to show:

- I^n is precipitous for each $n \in \omega$.
- Assume G_n is $(V, \mathbb{P}_n)$-generic and, for each $m \leq n$, G_m is the $(V, \mathbb{P}_m)$-generic filter naturally obtained from G_n, i.e. $G_m = \pi_{n,m}[G_n]$ $(= \sigma_{m,n}^{-1}[G_n])$. In $V[G_n]$, let M_m be the transitive collapse of $Ult(V, G_m)$ for each $m \leq n$. Then $\langle M_m \mid m \leq n\rangle$ is an iteration of generic ultrapowers of V by I.

We begin with the factor lemma for $\mathbb{P}_n$.

Lemma 2.4. *Assume that $m, k \in \omega$, I^m is precipitous and G_m is a $(V, \mathbb{P}_m)$-generic filter. Let $j_m : V \to M_m \cong Ult(V, G_m)$ be the generic elementary embedding and κ_m, I_m, $\mathbb{P}^m_k$ be $j_m(\kappa)$, $j_m(I)$, $j_m(\mathbb{P}_k)$ respectively. Then, in $V[G_m]$, there is a surjective dense embedding from $\mathbb{P}_{m+k}/G_m$ to $\mathbb{P}^m_k$.*

Note: In M_m, I_m is a normal ideal on κ_m and $\mathbb{P}^m_k = \mathbb{P}_{(I_m)^k}$.

Notation: Assume $m, k \in \omega$. For each $A \subseteq {}^{m+k}\kappa$ which is in V, let f^A_m be the function on ${}^m\kappa$ such that

$$f^A_m(s) = \{t \in {}^k\kappa \mid s^\frown t \in A\}$$

for each $s \in {}^m\kappa$. (Note that $f^A_m \in V$.)

Proof. In $V[G_m]$, let $d^m_k : \mathbb{P}_{m+k}/G_m \to \mathbb{P}^m_k$ be the function such that

$$d^m_k(A) := [f^A_m]_{G_m}$$

for each $A \in \mathbb{P}_{m+k}$. Recall that $|\mathbb{P}_{m+k}/G_m| = \pi^{-1}_{m+k,m}[G_m]$. So if $A \in \mathbb{P}_{m+k}/G_m$ then $d^m_k(A) \in \mathbb{P}^m_k$. Moreover it is clear that d^m_k is surjective and order preserving. So it suffices to show that if $A \perp B$ in $\mathbb{P}_{m+k}/G_n$ then $d^m_k(A) \perp d^m_k(B)$ in $\mathbb{P}^m_k$.

Assume $d^m_k(A)$ and $d^m_k(B)$ are compatible. Then, by Łoś's theorem,

$$X := \{s \in {}^m\kappa \mid f^A_m(s) \cap f^B_m(s) \in (I^k)^+\} \in G_m .$$

So, by the definition of f^A_m and f^B_m, $\pi_{m+k,m}(A \cap B) = X \in G_m$. This means that $A \cap B \in \mathbb{P}_{m+k}/G_m$ and so A and B are compatible in $\mathbb{P}_{m+k}/G_m$. □

Next we show the factor lemma for a generic ultrapower of V by I^n. If G_{m+k} is $(V, \mathbb{P}_{m+k})$-generic and $G_m = \pi_{m+k,m}[G_{m+k}]$ then $d^m_k[G_{m+k}]$ is $(V[G_m], \mathbb{P}^m_k)$-generic, where $\mathbb{P}^m_k$ and d^m_k are as in the previous lemma. (Note that d^m_k is surjective.) Because $\mathbb{P}^m_k \in M_m \subseteq V[G_m]$, $d^m_k[G_{m+k}]$ is $(M_m, \mathbb{P}^m_k)$-generic. So, in $V[G_{m+k}]$, we can construct $Ult(M_m, d^m_k[G_{m+k}])$. We see that this model is isomorphic to $Ult(V, G_{m+k})$.

Lemma 2.5. *Assume $m, k \in \omega$, I^m is precipitous and G_{m+k} is $(V, \mathbb{P}_{m+k})$-generic. Let $G_m := \pi_{m+k,m}[G_{m+k}]$. In $V[G_m]$, let j_m, M_m, κ_m, I_m, $\mathbb{P}^m_k$ be as in Lemma 2.4 and let $d^m_k : \mathbb{P}_{m+k}/G_m \to \mathbb{P}^m_k$ be the dense embedding defined as in the proof of Lemma 2.4. In $V[G_{m+k}]$, let $G^m_k = d^m_k[G_{m+k}]$. Then $Ult(V, G_{m+k}) \cong Ult(M_m, G^m_k)$.*

Notation: Assume $m, k \in \omega$. For each function $g \in V$ on ${}^{m+k}\kappa$, let $f^g_m \in V$ be the function on ${}^m\kappa$ such that

$$f^g_m(s) = \text{the function on } {}^k\kappa \text{ such that } \forall t \in {}^k\kappa,\ f^g_m(s)(t) = g(s^\frown t)$$

for each $s \in {}^m\kappa$.

Proof. In $V[G_{m+k}]$, define $\tau : Ult(V, G_{m+k}) \to Ult(M_m, G^m_k)$ as

$$\tau((g)_{G_{m+k}}) := ([f^g_m]_{G_m})_{G^m_k}$$

for each $(g)_{G_{m+k}} \in Ult(V, G_{m+k})$. We show that τ is isomorphic.

First we see that τ is well-defined, injective and elementary. Let $\varphi(v_1, ..., v_l)$ be a formula and $g_1, ..., g_l \in V$ be functions on ${}^{m+k}\kappa$. Then, by Łoś's theorem,

$$Ult(M_m, G^m_k) \models \varphi(([f^{g_1}_m]_{G_m})_{G^m_k}, ..., ([f^{g_l}_m]_{G_m})_{G^m_k}) \quad (1)$$

$$\Leftrightarrow \{t \in {}^k\kappa_m \mid M_m \models \varphi([f^{g_1}_m]_{G_m}(t), ..., [f^{g_l}_m]_{G_m}(t))\} \in G^m_k. \quad (2)$$

Now, in V, let $A \subseteq {}^{m+k}\kappa$ be such that

$$A := \{u \in {}^{m+k}\kappa \mid V \models \varphi(g_1(u), ..., g_l(u))\}.$$

Then, for each $s \in {}^m\kappa$

$$f^A_m(s) = \{t \in {}^k\kappa \mid V \models \varphi(f^{g_1}_m(s)(t), ..., f^{g_l}_m(s)(t))\}$$

holds in V. So, by Łoś's theorem, in M_m,

$$[f_m^A]_{G_m} = \{t \in {}^k\kappa_m \mid M_m \models \varphi([f_m^{g_1}]_{G_m}(t), \ldots, [f_m^{g_l}]_{G_m}(t))\}.$$

Thus

$$\begin{aligned}(2) \;&\Leftrightarrow\; [f_m^A]_{G_m} \in G_k^m \;\Leftrightarrow\; A \in G_{m+k} \\ &\Leftrightarrow\; Ult(V, G_{m+k}) \models \varphi((g_1)_{G_{m+k}}, \ldots, (g_l)_{G_{m+k}}). \qquad (3)\end{aligned}$$

For the second equivalence, recall that $d_k^m(A) = [f_m^A]_{G_m}$ and $G_k^m = d_k^m[G_{m+k}]$. The equivalence between (1) and (3) implies that τ is well-defined, injective and elementary. (For the well-definedness and injectivity, let φ be the formula "$v_1 = v_2$".)

Finally it is clear from the definition that τ is surjective. So τ is isomorphic. □

Remark: If $Ult(V, G_{m+k})$ and $Ult(M_m, G_k^m)$ are well-founded then, because the above τ is isomorphic,

$$[g]_{G_{m+k}} = [[f_m^g]_{G_m}]_{G_k^m}.$$

Lemma 2.6. *For each $m \in \omega$, I^m is precipitous.*

Proof. We show this by induction on $m \in \omega$. If $m = 1$, this is clear by the precipitousness of I. Assume I^m is precipitous. Assume G_{m+1} is $(V, \mathbb{P}_{m+1})$-generic. Let $G_m = \pi_{m+1,m}[G_{m+1}]$ and M_m, j_m, I_m, G_1^m be as in Lemma 2.5. (Let $k = 1$.) Then G_1^m is $(M_m, \mathbb{P}_{I_m})$-generic. On the other hand, by the elementarity of j_m, $M_m \models$ "I_m is precipitous". So $Ult(M_m, G_1^m)$ is well-founded. So, by Lemma 2.5, $Ult(V, G_{m+1})$ is well-founded. This shows I^{m+1} is precipitous. □

In the following lemma, note that if $m_1 \le m_2 \le n \in \omega$, G_n is $(V, \mathbb{P}_n)$-generic and $G_{m_j} = \pi_{n,m_j}[G_n]$ $(j = 1, 2)$ then $G_{m_1} = \pi_{m_2,m_1}[G_{m_2}]$.

Lemma 2.7. *Assume $n \in \omega$ and G_n is $(V, \mathbb{P}_n)$-generic. For each $m \le n$, let $G_m := \pi_{n,m}[G_n]$ and M_m be the transitive collapse of $Ult(V, G_m)$. For each $m < n$, let G_1^m be as in Lemma 2.5. Then $\langle M_m, G_1^l \mid m \le n,\ l < m\rangle$ is an iteration of generic ultrapowers of V by I.*

Proof. Clear by Lemma 2.5. □

In the rest of this subsection, we show basic facts needed in the next section. From now on, let W be an outer model of V in which there is a sequence $\langle G_n \mid n \in \omega\rangle$ such that if $m \le n \in \omega$ then G_n is a $(V, \mathbb{P}_n)$-generic filter and $G_m = \pi_{n,m}[G_n]$. Basically we work in W. For each $m, k \in \omega$, let $j_m, M_m, \mathbb{P}_k^m$, e.t.c. be as before, i.e.

- $j_m : V \to M_m \cong Ult(V, G_m)$ is the generic elementary embedding,
- $\kappa_m := j_m(\kappa)$, $I_m := j_m(I)$,
- $\mathbb{P}_k^m := j_m(\mathbb{P}_k) = (\mathbb{P}_{(I_m)^k})^{M_m}$,
- $d_k^m : \mathbb{P}_{m+k}/G_m \to \mathbb{P}_k^m$ is a dense embedding such that for each $A \in \mathbb{P}_{m+k}/G_m$,
$$d_k^m(A) := [f_m^A]_{G_m},$$
- $G_k^m := d_k^m[G_{m+k}]$.

First we give a representation for the map from M_m to M_n associated with the iteration of generic ultrapowers. For each $m \le n \in \omega$, let $j_{m,n} : M_m \to M_n$ be the function defined as

$$j_{m,n}([g]_{G_m}) := [\bar{g}]_{G_n}$$

for each $[g]_{G_m} \in M_m$, where $\bar{g} \in V$ is the function on ${}^n\kappa$ such that $\bar{g}(s) = g(s \restriction m)$ for each $s \in {}^n\kappa$. It is easy to see that if $l \leq m \leq n \in \omega$ then $j_{0,n} = j_n$ and $j_{l,n} = j_{m,n} \circ j_{l,m}$.

Lemma 2.8. *Assume $m \leq n \in \omega$. Then $j_{m,n} : M_m \to M_n$ is the generic elementary embedding associated with $Ult(M_m, G^m_{n-m})$.*

Proof. Take an arbitrary $x \in M_m$ and assume $x = [g]_{G_m}$. We show that $j_{m,n}(x) = [c_x]_{G^m_{n-m}}$, where $c_x \in M_m$ is the constant function on ${}^{n-m}\kappa_m$ with the value x. Let $\bar{g}$ be as above, i.e. the function on ${}^n\kappa$ such that $\bar{g}(s) = g(s \restriction m)$ for each $s \in {}^n\kappa$. Then, in M_m, $[f^{\bar{g}}_m]_{G_m} = c_x$. By the Remark after Lemma 2.5, $[[f^{\bar{g}}_m]_{G_m}]_{G^m_{n-m}} = [\bar{g}]_{G_n}$. So

$$[c_x]_{G^m_{n-m}} \;=\; [[f^{\bar{g}}_m]_{G_m}]_{G^m_{m-n}} \;=\; [\bar{g}]_{G_n} \;=\; j_{m,n}(x)\,.$$

□

In particular, $j_{m,m+1} : M_m \to M_{m+1}$ is the ultrapower map associated with $Ult(V, G^m_1)$. Then, because $\langle M_n, j_{m,n} \mid m \leq n \in \omega \rangle$ is a directed system, $j_{m,n} : M_m \to M_n$ is the map associated with the iteration of generic ultrapowers, i.e. $\langle M_n, G^m_1, j_{m,n} \mid m \leq n \in \omega \rangle$ is an iteration of generic ultrapowers of V by I.

Next we give the representation for the sequence of critical points. Because I is normal, the sequence of critical points have a good representation.

Lemma 2.9. *Assume $m < n \in \omega$. Then $\langle \kappa_k \mid m \leq k < n \rangle = [id \restriction {}^{n-m}\kappa_m]_{G^m_{n-m}}$. So, for each $A \subseteq {}^{n-m}\kappa_m$ which is in M_m, $A \in G^m_{n-m}$ if and only if $\langle \kappa_k \mid m \leq k < n \rangle \in j_{m,n}(A)$.*

Proof. For each $k < n$, let $i_k \in V$ be the function on ${}^n\kappa$ such that $i_k(s) = s(k)$ for each $s \in {}^n\kappa$. First we show $[i_k]_{G_n} = \kappa_k$. Let h_k be the function on ${}^{k+1}\kappa$ such that $h_k(s) = s(k)$ for each $s \in {}^{k+1}\kappa$. Then $j_{k+1,n}([h_k]_{G_{k+1}}) = [i_k]_{G_n}$. Because $j_{k+1,n}$ does not move κ_k, it suffices to show $[h_k]_{G_{k+1}} = \kappa_k$.

In V, $f^{h_k}_k(s) = id \restriction \kappa$ for each $s \in {}^k\kappa$. (Here we identified ${}^1\kappa$ with κ.) So, in M_k, $[f^{h_k}_k]_{G_k} = id \restriction \kappa_k$. Then, by normality of I_k, $\kappa_k = [[f^{h_k}_k]_{G_k}]_{G^k_1}$. Then, by the remark after Lemma 2.5, $[h_k]_{G_{k+1}} = \kappa_k$.

Now let $g \in V$ be the function on ${}^n\kappa$ such that $g(s) = \langle s(m), s(m+1), ..., s(n-1) \rangle$ for each $s \in {}^n\kappa$. Then, because $f^g_m(s) = id \restriction {}^{n-m}\kappa$ for each $s \in {}^m\kappa$, $[f^g_m]_{G_m} = id \restriction {}^{n-m}\kappa_m$. So

$$[[f^g_m]_{G_m}]_{G^m_{n-m}} \;=\; [id \restriction {}^{n-m}\kappa_m]_{G^m_{n-m}}.$$

On the other hand,

$$[g]_{G_n} \;=\; \langle\, [i_m]_{G_n}\,,\, [i_{m+1}]_{G_n}\,, ..., [i_{n-1}]_{G_n} \,\rangle \;=\; \langle \kappa_m, \kappa_{m+1}, ..., \kappa_{n-1} \rangle.$$

So, by the remark after Lemma 2.5,

$$[id \restriction {}^{n-m}\kappa_m]_{G^m_{n-m}} \;=\; \langle \kappa_m, \kappa_{m+1}, ..., \kappa_{n-1} \rangle.$$

□

For each $m, k, l \in \omega$ such that $k \leq l$, let

- $\sigma^m_{k,l} := j_m(\sigma_{k,l})$,
- $\pi^m_{l,k} := j_m(\pi_{l,k})$.

Note that if m, k, l is as above then, in M_m,

- $\sigma^m_{k,l} : \mathcal{P}({}^k\kappa_m) \to \mathcal{P}({}^l\kappa_m)$ is the natural complete embedding associated with I_m,

- $\pi^m_{l,k} : \mathbb{P}({}^l\kappa_m) \to \mathcal{P}({}^k\kappa_m)$ is the natural projection associated with I_m.

Lemma 2.10. *Assume $m \in \omega$ and $k \le l \in \omega$. Then the following diagrams commute. So $G^m_k = \pi^m_{l,k}[G^m_l] = (\sigma^m_{k,l})^{-1}[G^m_l]$.*

$$(1)\qquad \begin{array}{ccc} \mathbb{P}_{m+k}/G_m & \xrightarrow{\sigma_{m+k,m+l}} & \mathbb{P}_{m+l}/G_m \\ {\scriptstyle d^m_k}\downarrow & & \downarrow{\scriptstyle d^m_l} \\ \mathbb{P}^m_k & \xrightarrow{\sigma^m_{k,l}} & \mathbb{P}^m_l \end{array}$$

$$(2)\qquad \begin{array}{ccc} \mathbb{P}_{m+k}/G_m & \xleftarrow{\pi_{m+l,m+k}} & \mathbb{P}_{m+l}/G_m \\ {\scriptstyle d^m_k}\downarrow & & \downarrow{\scriptstyle d^m_l} \\ \mathbb{P}^m_k & \xleftarrow{\pi^m_{l,k}} & \mathbb{P}^m_l \end{array}$$

Proof.

(1): Assume $B \in \mathbb{P}_{m+k}/G_m$. Let $A := \sigma_{m+k,m+l}(B)$. Then $f^A_m(s) = \sigma_{k,l}(f^B_m(s))$ for each $s \in {}^m\kappa$. So, by Łoŝ's Theorem,

$$d^m_l(A) \;=\; [\,f^A_m\,]_{G_m} \;=\; \sigma^m_{k,l}([\,f^B_m\,]_{G_m}) \;=\; \sigma^m_{k,l}(d^m_k(B))\,.$$

(2): Assume $A \in \mathbb{P}_{m+l}/G_m$. Let $B := \pi_{m+l,m+k}(A)$. Then $f^B_m(s) = \pi_{l,k}(f^A_m(s))$ for each $s \in {}^m\kappa$. So

$$d^m_k(B) \;=\; [\,f^B_m\,]_{G_m} \;=\; \pi^m_{l,k}([\,f^A_m\,]_{G_m}) \;=\; \pi^m_{l,k}(d^m_l(A))\,.$$

□

We end this subsection with a definition. By Lemma 2.7, $\langle M_n, G^m_1, j_{m,n} \mid m \le n \in \omega\rangle$ is an iteration of generic ultrapowers of V by I. Then, because of the iterability of generic ultrapowers, the direct limit of $\langle M_n, j_{m,n} \mid m \le n \in \omega\rangle$ is well-founded. Let M_ω be the transitive collapse of the direct limit of $\langle M_n, j_{m,n} \mid m \le n \in \omega\rangle$ and, for each $m \in \omega$, let $j_{m,\omega} : M_m \to M_\omega$ be the associated elementary embedding. Then we call $\langle M_n, G^m_1, j_{m,n} \mid m \le n \le \omega, m < \omega\rangle$ *the iteration of generic ultrapowers of V by I associated with $\langle G_n \mid n \in \omega\rangle$.*

3. Generalized Prikry Forcing and Iteration of Generic Ultrapower.

3.1. PR^* and PR^+.

In this subsection, we define two generalizations, PR^* and PR^+, of Prikry Forcing and show their basic properties.

First we give some definitions involving trees. Let α be an ordinal and $T \subseteq {}^{<\omega}\alpha$ be a tree. Then for each $t \in {}^{<\omega}\alpha$, let

- $t^\frown T := \bigcup\{t \restriction k \mid k < |t|\} \cup \bigcup\{t^\frown s \mid s \in T\}$, (so $t^\frown T$ is a tree whose stem is t and t^T is isomorphic with T above its stem),
- $T/t := \{s \in {}^{<\omega}\alpha \mid t^\frown s \in T\}$,
- $Suc_T(t) := \{\xi < \alpha \mid t^\frown\langle\xi\rangle \in T\}$.

Next we generalize Prikry Forcing. Assume J is an ideal on some infinite ordinal α. For each tree $T \subset {}^{<\omega}\alpha$,

- T is called a *J^*-tree* if $T \ne \emptyset \wedge \forall t \in T,\ Suc_T(t) \in J^*$,
- T is called a *J^+-tree* if $T \ne \emptyset \wedge \forall t \in T,\ Suc_T(t) \in J^+$.

Let PR^*_J be the p.o. such that

$$|PR^*_J| \;=\; \{\langle t,T\rangle \mid t \in {}^{<\omega}\alpha \;\wedge\; T \subseteq {}^{<\omega}\alpha \text{ is a } J^*\text{-tree}\}$$

and, for each $\langle t_1,T_1\rangle, \langle t_2,T_2\rangle \in |PR^*_J|$, $\langle t_1,T_1\rangle \leq \langle t_2,T_2\rangle$ iff $t_1{}^\frown T_1 \subseteq t_2{}^\frown T_2$. Let PR^+_J be the p.o. such that

$$|PR^+_J| \;=\; \{\langle t,T\rangle \mid t \in {}^{<\omega}\alpha \;\wedge\; T \subseteq {}^{<\omega}\alpha \text{ is a } J^+\text{-tree}\}$$

and, for each $\langle t_1,T_1\rangle, \langle t_2,T_2\rangle \in |PR^+_J|$, $\langle t_1,T_1\rangle \leq \langle t_2,T_2\rangle$ if $t_1{}^\frown T_1 \subseteq t_2{}^\frown T_2$.

In Shelah [2], PR^+ is treated as a variant of Namba Forcing and studied in detail. Note that if J is a prime ideal then $PR^*_J = PR^+_J$ and this p.o. is Prikry Forcing. As is the case with Prikry Forcing, if Γ is PR^*_J-generic (or PR^*_J-generic) then $\bigcup\{t \mid \exists T, \langle t,T\rangle \in \Gamma\}$ becomes an ω-sequence of ordinals in α and Γ can be recovered from this sequence. First we show this.

Lemma 3.1. *Assume W is a transitive model of ZFC, $\alpha \in W$ is an infinite ordinal and $J \in W$ is such that $W \models$ "J is an ideal on α". Let $\mathbb{P}^* := (PR^*_J)^W$ and $\mathbb{P}^+ := (PR^+_J)^W$.*

1. *Assume Γ is a $(W,\mathbb{P}^*)$-generic filter. Let $b := \bigcup\{t \mid \exists T, \langle t,T\rangle \in \Gamma\}$ and $\Gamma_b := \{\langle t,T\rangle \in \mathbb{P}^* \mid \forall n \in \omega, b{\restriction}n \in t^\frown T\}$. Then $\Gamma_b = \Gamma$.*
2. *Assume Γ is a $(W,\mathbb{P}^+)$-generic filter. Let $b := \bigcup\{t \mid \exists T, \langle t,T\rangle \in \Gamma\}$ and $\Gamma_b := \{\langle t,T\rangle \in \mathbb{P}^+ \mid \forall n \in \omega, b{\restriction}n \in t^\frown T\}$. Then $\Gamma_b = \Gamma$.*

Proof. We show only (1). (2) can be shown in the same way. Clearly $\Gamma \subseteq \Gamma_b$. So it suffices to show that $\Gamma_b \subseteq \Gamma$.

Assume $\langle s,S\rangle \notin \Gamma$. Because

$$D \;:=\; \{\langle t,T\rangle \in \mathbb{P}^* \mid \langle t,T\rangle \leq \langle s,S\rangle \;\text{ or }\; t \notin s^\frown S\}$$

is in W and dense in $\mathbb{P}^*$, there is a $\langle t,T\rangle \in D \cap \Gamma$. Because $\langle s,S\rangle \notin \Gamma$, $t \notin s^\frown S$. Then, because t is an initial segment of b, $b \notin [s^\frown S]$. So $\langle s,S\rangle \notin \Gamma_b$. □

We call the above b's a PR^*_J-sequence or a PR^+_J-sequence. More precisely we make the following definitions.

Assume W is a transitive model of ZFC, $\alpha \in W$ is an infinite ordinal and $J \in W$ is such that $W \models$ "J is an ideal on α". Let $b \in {}^\omega\alpha$. Then we say:

- b is *a PR^*_J-sequence over W* if there is a $(W,(PR^*_J)^W)$-generic filter Γ such that $b = \bigcup\{t \mid \exists T, \langle t,T\rangle \in \Gamma\}$.
- b is *a PR^+_J-sequence over W* if there is a $(W,(PR^+_J)^W)$-generic filter Γ such that $b = \bigcup\{t \mid \exists T, \langle t,T\rangle \in \Gamma\}$.

By Lemma 3.1, b is a PR^*_J-sequence over W if and only if $\Gamma_b := \{\langle t,T\rangle \in (PR^*_J)^W \mid \forall n \in \omega, b{\restriction}n \in t^\frown T\}$ is a $(W,(PR^*_J)^W)$-generic filter. (For the backward direction, note that if Γ_b is a generic filter then $b = \bigcup\{t \mid \exists T, \langle t,T\rangle \in \Gamma_b\}$.) This is also true for PR^+_J.

The following lemma is useful.

Lemma 3.2. *Assume W, α and J are as in Lemma 3.1 and $b,c \in {}^\omega\alpha$ have a common tail, i.e. $\exists m,n \in \omega\, \forall k \in \omega$, $b(m+k) = c(n+k)$. Then:*

1. *b is a PR^*_J-sequence over W iff c is a PR^*_J-sequence over W.*
2. *b is a PR^+_J-sequence over W iff c is a PR^+_J-sequence over W.*

Proof. We show only (1). Let W, α, J, b, c be as above. Let $m, n \in \omega$ be such that $\forall k \in \omega$, $b(m+k) = c(n+k)$ and let $u, v \in {}^{<\omega}\alpha$ be $b \restriction m$, $c \restriction n$ respectively.

Assume that b is a PR^*_J-sequence over W and Γ witnesses this. In W, let $\mathbb{P}_u$ be $PR^*_J \restriction \langle u, {}^{<\omega}\alpha\rangle$, i.e. the p.o. obtained from restricting PR^*_J to $\{\langle t, T\rangle \mid \langle t, T\rangle \leq \langle u, {}^{<\omega}\alpha\rangle\}$. Let $\mathbb{P}_v$ be $PR^*_J \restriction \langle v, {}^{<\omega}\alpha\rangle$. Then let $d : \mathbb{P}_u \to \mathbb{P}_v$ be such that

$$d(\langle u^\frown s, S\rangle) = \langle v^\frown s, S\rangle$$

for each $\langle u^\frown s, S\rangle \in \mathbb{P}_u$. Then $d \in W$ and d is an isomorphism. Because u is an initial segment of b, $\langle u, {}^{<\omega}\alpha\rangle \in \Gamma$. So $\Gamma \cap \mathbb{P}_u$ is $(W, \mathbb{P}_u)$-generic. So $d[\Gamma \cap \mathbb{P}_u]$ is $(W, \mathbb{P}_v)$-generic. So the filter Ω on $(PR^*_J)^W$ which is generated by $d[\Gamma \cap \mathbb{P}_u]$ is generic over W. Moreover,

$$\begin{aligned}\bigcup\{t \mid \exists T,\ \langle t, T\rangle \in \Omega\} &= \bigcup\{t \mid \exists T,\ \langle t, T\rangle \in d[\Gamma \cap \mathbb{P}_u]\}\\ &= \bigcup\{v^\frown s \mid \exists T, \langle u^\frown s, T\rangle \in \Gamma \cap \mathbb{P}_u\}\\ &= \bigcup\{v^\frown s \mid u^\frown s \in b\}\\ &= c\,.\end{aligned}$$

So Ω witnesses that c is a PR^*_J-sequence over W.

The other direction can be shown similarly. □

3.2. **Generalization of Solovay's Theorem.**

Solovay's theorem can be generalized for both PR^* and PR^+. In this subsection, we show this.

All through this subsection, in V, let κ be a regular uncountable cardinal and I be a normal precipitous ideal on κ. Moreover, for each $m \leq n \in \omega$, let $\mathbb{P}_n := \mathbb{P}_{I^n}$ and let $\sigma_{m,n} : \mathcal{P}({}^m\kappa) \to \mathcal{P}({}^n\kappa)$ and $\pi_{n,m} : \mathcal{P}({}^n\kappa) \to \mathcal{P}({}^m\kappa)$ be the natural complete embedding and the natural projection associated with I.

First we generalize Solovay's theorem for PR^*.

Theorem 3.3. *Let $\mathbb{P}_\omega$ be the direct limit of $\langle \mathbb{P}_n, \sigma_{m,n} \mid m \leq n \in \omega\rangle$. Let G_ω be a $(V, \mathbb{P}_\omega)$-generic filter and, for each $n \in \omega$, let G_n be the $(V, \mathbb{P}_n)$-generic filter naturally obtained from G_ω. In $V[G_\omega]$, let $\langle M_n, H^m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$ be the iteration of generic ultrapowers of V by I associated with $\langle G_n \mid n \in \omega\rangle$. Then $\langle j_{0,n}(\kappa) \mid n \in \omega\rangle$ is a $PR^*_{j_{0,\omega}(I)}$-sequence over M_ω.*

To prove the above theorem, we need some preparation. Until we complete the proof of the theorem, let $\mathbb{P}_\omega$, G_ω, $\langle G_n \mid n \in \omega\rangle$ and $\langle M_n, H^m, j_{m,n} \mid m \leq n \leq \omega,\ m < \omega\rangle$ be as in the theorem, and, in $V[G_\omega]$, let j_m, κ_m, I_m, $\mathbb{P}^m_k$, d^m_k, G^m_k, $\sigma^m_{k,l}$, $\pi^m_{l,k}$ be as in Section 2.2 for each $m, k, l \in \omega$ with $k \leq l$. Let $I_\omega := j_{0,\omega}(I)$. Note that $H^m = G^m_1$ for each $m \in \omega$.

$\mathbb{P}_\omega$ is the p.o. defined as follows: First let $\sim_\sigma$ be the equivalence relation on $\bigcup_{n\in\omega} |\mathbb{P}_n|$ such that for each $A, B \in \bigcup_{n\in\omega} |\mathbb{P}_n|$, say $A \in \mathbb{P}_m$ and $B \in \mathbb{P}_n$, $A \sim_\sigma B$ iff $\sigma_{m,l}(A) = \sigma_{n,l}(A)$, where $l = max(m, n)$. Let $[A]_\sigma$ denote the equivalence class represented by A. Then

- $|\mathbb{P}_\omega| = \bigcup_{n\in\omega} |\mathbb{P}_n| / \sim_\sigma$,
- if $A \in \mathbb{P}_m$ and $B \in \mathbb{P}_n$ then, letting $l = max(m, n)$, $[A]_\sigma \leq [B]_\sigma$ iff $\sigma_{m,l}(A) \leq \sigma_{n,l}(B)$ in $\mathbb{P}_l$.

For each $m \in \omega$, let $\sigma_{m,\omega} : \mathbb{P}_m \to \mathbb{P}_\omega$ be the complete embedding associated with the direct limit, i.e. the function such that $\sigma_{m,\omega}(A) = [A]_\sigma$ for each $A \in \mathbb{P}_m$. Then $G_m = \sigma_{m,\omega}^{-1}[G_\omega] = \{A \in \mathbb{P}_m \mid [A]_\sigma \in G_\omega\}$.

To prove the theorem, we need the factor lemma for $\mathbb{P}_\omega$ and $\langle M_n, H_m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$. To see this we define $\mathbb{P}_\omega^m$, $\sigma_{k,\omega}^m$, d_ω^m and G_ω^m. Let $m \in \omega$.

Let

- $\mathbb{P}_\omega^m \;:=\; j_m(\mathbb{P}_\omega)$,
- $\sigma_{k,\omega}^m \;:=\; j_m(\sigma_{k,\omega})$, for each $k \in \omega$.

Recall that $\mathbb{P}_k^m = j_m(\mathbb{P}_k)$ and $\sigma_{k,l}^m = j_m(\sigma_{k,l})$ for each $k \leq l \in \omega$. So, in M_m, $\mathbb{P}_\omega^m$ is the direct limit of $\langle \mathbb{P}_k^m, \sigma_{k,l}^m \mid k \leq l \in \omega\rangle$ and $\sigma_{k,\omega}^m : \mathbb{P}_k^m \to \mathbb{P}_\omega^m$ is the induced complete embedding. For each $A \in \bigcup_{k\in\omega} |\mathbb{P}_k^m|$, let $[A]_{\sigma^m}$ denote the equivalence class represented by A. Then $\sigma_{k,\omega}^m(A) = [A]_{\sigma^m}$.

$d_\omega^m : \mathbb{P}_\omega/G_m \to \mathbb{P}_\omega^m$ is defined as follows. Note that $\mathbb{P}_\omega/G_m$ is the p.o. in $V[G_m]$ which is obtained from restricting $\mathbb{P}_\omega$ to $\{[A]_\sigma \mid \exists n \geq m,\, A \in \mathbb{P}_n/G_m\}$. (If $n \geq m$ and $A \in \mathbb{P}_n$ then $[A]_\sigma \in \mathbb{P}_\omega/G_m \Leftrightarrow \forall B \in G_m$, $[A]_\sigma$ and $[B]_\sigma$ are compatible in $\mathbb{P}_\omega \Leftrightarrow \forall B \in G_m$, A and $\sigma_{m,n}(B)$ are compatible in $\mathbb{P}_n \Leftrightarrow A \in \mathbb{P}_n/G_m$.) Then let $d_\omega^m : \mathbb{P}_\omega/G_m \to \mathbb{P}_\omega^m$ be the function such that

- $d_\omega^m([A]_\sigma) \;=\; [d_k^m(A)]_{\sigma^m}$,

for each $A \in \mathbb{P}_{m+k}/G_m$. By Lemma 2.10, d_ω^m is well-defined. Clearly $d_\omega^m \in V[G_m]$. We show that d_ω^m is a dense embedding.

Lemma 3.4. *$d_\omega^m : \mathbb{P}_\omega/G_m \to \mathbb{P}_\omega^m$ is a surjective dense embedding.*

Proof. Because $d_k^m : \mathbb{P}_{m+k}/G_m \to \mathbb{P}_k^m$ is surjective for each $k \in \omega$, d_ω^m is also surjective.

To see that d_ω^m is order preserving, assume $[A]_\sigma \leq [B]_\sigma$ in $\mathbb{P}_\omega/G_m$. Assume A, B is in $\mathbb{P}_{m+k}/G_m$, $\mathbb{P}_{m+l}/G_m$ respectively. Let $i := max(k,l)$. Then $\sigma_{m+k,m+i}(A) \leq \sigma_{m+l,m+i}(B)$ in $\mathbb{P}_{m+i}/G_m$. Because d_i^m is order preserving, $d_i^m(\sigma_{m+k,m+i}(A)) \leq d_i^m(\sigma_{m+l,m+i}(B))$ in $\mathbb{P}_i^m$. Then, by lemma 2.10, $\sigma_{k,i}^m(d_k^m(A)) \leq \sigma_{l,i}^m(d_l^m(B))$. This means that $d_\omega^m([A]_\sigma) \leq d_\omega^m([B]_\sigma)$.

By replacing "$\leq$" by "$\perp$" in the above argument, we can see that d_ω^m preserves incompatibility. □

Let $G_\omega^m \;:=\; d_\omega^m[G_\omega]$. Then G_ω^m is a $(V[G_m], \mathbb{P}_\omega^m)$-generic filter and so is $(M_m, \mathbb{P}_\omega^m)$-generic. We want to show that $G_k^m = (\sigma_{k,\omega}^m)^{-1}(G_\omega^m)$ for each $k \in \omega$, i.e. G_k^m is the $(V[G_m], \mathbb{P}_k^m)$-generic filter naturally obtained from G_ω^m. Assume $k \in \omega$ and $A \in \mathbb{P}_k^m$. Let $B \in \mathbb{P}_{m+k}/G_m$ be such that $A = d_k^m(B)$. (Recall that d_k^m is surjective.) Then $d_\omega^m([B]_\sigma) = [A]_{\sigma^m}$. Then

$$[A]_{\sigma^m} \in G_\omega^m \Leftrightarrow [B]_\sigma \in G_\omega \Leftrightarrow B \in G_{m+k} \Leftrightarrow A \in G_k^m\,.$$

Thus $G_k^m = (\sigma_{k,\omega}^m)^{-1}[G_\omega^m]$.

Note that, by Lemma 2.5, $\langle M_{m+l}, H^{m+k}, j_{m+k,m+l} \mid k \leq l \leq \omega, k < \omega\rangle$ is the iteration of generic ultrapowers of M_m by I_m naturally obtained from G_ω^m.

Now we can start to prove the theorem.

Proof of Theorem 3.3.

In $V[G_\omega]$, let $\vec{\kappa} := \langle \kappa_n \mid n \in \omega\rangle$ and let

$$\Gamma \;:=\; \{\langle t, T\rangle \in (PR_{I_\omega}^*)^{M_\omega} \mid \forall n \in \omega, \vec{\kappa}{\restriction}n \in t^\frown T\}\,.$$

We show that Γ is a $(M_\omega, (PR^*_{I_\omega})^{M_\omega})$-generic filter. For simplicity of notation, we write $PR^*_{I_m}$ for $(PR^*_{I_m})^{M_m}$ for each $m \leq \omega$.

First we show the genericity of Γ. Let $D \in M_\omega$ be a dense subset of $PR^*_{I_\omega}$. We show $\Gamma \cap D \neq \emptyset$. Let $m \in \omega$ and $\bar{D} \in M_m$ be such that $D = j_{m,\omega}(\bar{D})$. $\bar{D}$ is a dense subset of $PR^*_{I_m}$. In M_m, define $E \subseteq \mathbb{P}^m_\omega$ as

$$E := \{[A]_{\sigma^m} \in \mathbb{P}^m_\omega \mid \forall t \in A\, \exists T,\ \langle \vec{\kappa} \restriction m^\frown t,\, T\rangle \in \bar{D}\}\,.$$

Working in M_m, we show that E is dense in $\mathbb{P}^m_\omega$.

Claim . *Assume $k \in \omega$ and $s \in {}^k\kappa_m$. Then there is an $l \in \omega$ such that*

$$B^s_l := \{t \in {}^l\kappa_m \mid \exists T,\ \langle \vec{\kappa} \restriction m^\frown s^\frown t,\, T\rangle \in \bar{D}\} \in ((I_m)^l)^+\,.$$

Proof of Claim. Let $k \in \omega$ and $s \in {}^k\kappa_m$. Assume $B^s_l \in (I_m)^l$ for every $l \in \omega$. Then, by Lemma 2.3, there is an $X_l \in (I_m)^*$ such that $[X_l]^l \cap B^s_l = \emptyset$ for each $l \in \omega$. Let $X := \bigcup_{l\in\omega} X_l$. Then $X \in (I_m)^*$ and if $t \in [X]^l$ then $t \notin B^s_l$. Then $[X]^{<\omega}$ is an $(I_m)^*$-tree and so $\langle \vec{\kappa} \restriction m^\frown s,\, [X]^{<\omega}\rangle \in PR^*_{I_m}$. But, by the construction of X, there is no element of $\bar{D}$ which extends $\langle \vec{\kappa} \restriction m^\frown s,\, [X]^{<\omega}\rangle$. This contradicts $\bar{D}$ is dense in $PR^*_{I_m}$. □.*Claim*

Claim . *E is dense in $\mathbb{P}^m_\omega$.*

Proof of Claim. Let $k \in \omega$ and $A \in \mathbb{P}^m_k$. We find an element of E which extends $[A]_{\sigma^m}$. By the previous claim, for each $s \in A$, there is an $l_s \in \omega$ such that $B^s_{l_s}$ is $(I_m)^{l_s}$-positive. Because $(I_m)^k$ is κ_m-complete, there is an $A' \subseteq A$ and $l \in \omega$ such that A' is $(I_m)^k$-positive and $l_s = l$ for every $s \in A'$. Then let $B := \{s^\frown t \mid s \in A' \wedge t \in B^s_l\}$. Because B^s_l is $(I_m)^l$-positive for each $s \in A'$, B is $(I_m)^{k+l}$-positive, i.e. $B \in \mathbb{P}^m_{k+l}$. Then clearly $\sigma^m_{k,k+l}(A) \geq B$ in $\mathbb{P}^m_{k+l}$ and so $[A]_{\sigma^m} \geq [B]_{\sigma^m}$. On the other hand, if $u \in B$ then there is a T such that $\langle \vec{\kappa} \restriction m^\frown u,\, T\rangle \in \bar{D}$. So $[B]_{\sigma^m} \in E$. □.*Claim*

Return to $V[G_\omega]$.

Because G^m_ω is $(M_m, \mathbb{P}^m_\omega)$-generic, $G^m_\omega \cap E \neq \emptyset$. Let A be such that $[A]_{\sigma^m} \in G^m_\omega \cap E$ and A witnesses that $[A]_{\sigma^m} \in E$, i.e. $\forall t \in A\, \exists T,\ \langle \vec{\kappa} \restriction m^\frown t,\, T\rangle \in \bar{D}$. Assume $A \in \mathbb{P}^m_k$. Then $A \in G^m_k$ and so, by Lemma 2.9, $\vec{\kappa} \restriction [m, m+k) \in j_{m,m+k}(A)$. On the other hand, because $j_{m,m+k}$ is an elementary embedding and does not move $\vec{\kappa} \restriction m$,

$$M_{m+k} \models \forall t \in j_{m,m+k}(A)\, \exists T,\ \langle \vec{\kappa} \restriction m^\frown t,\, T\rangle \in j_{m,m+k}(\bar{D})\,.$$

So, in M_{m+k}, there exists an $(I_{m+k})^*$-tree $\bar{T}$ such that $\langle \vec{\kappa} \restriction m+k,\, \bar{T}\rangle \in j_{m,m+k}(\bar{D})$. Let $T := j_{m+k,\omega}(\bar{T})$. Then

$$\langle \vec{\kappa} \restriction m+k,\, T\rangle = j_{m+k,\omega}(\langle \vec{\kappa} \restriction m+k, \bar{T}\rangle) \in j_{m+k,\omega}(j_{m,m+k}(\bar{D})) = D\,.$$

Thus it suffices to show that $\langle \vec{\kappa} \restriction m+k,\, T\rangle \in \Gamma$. To see this it suffices to show that, for every $l > 0$, $\langle \kappa_{m+k}, \ldots, \kappa_{m+k+l-1}\rangle \in T$. Assume $l > 0$. Let $n := m+k$. Because $\bar{T}$ is an $(I_n)^*$-tree the l-th level of $\bar{T}$, $\bar{T}_{(l)}$, is in $((I_n)^l)^*$. So $\bar{T}_{(l)} \in G^n_l$. Then, by Lemma 2.9, $\langle \kappa_n, \ldots, \kappa_{n+l-1}\rangle \in j_{n,n+l}(\bar{T}_{(l)})$. Then,

$$\langle \kappa_n, \ldots, \kappa_{n+l-1}\rangle = j_{n+l,\omega}(\langle \kappa_n, \ldots, \kappa_{n+l-1}\rangle) \in j_{n+l,\omega}(j_{n,n+l}(\bar{T}_{(l)})) = T_{(l)}\,.$$

This completes the proof of the genericity.

Next we show that Γ is a filter. Clearly Γ is closed upwards. We show that if $\langle t_1, T_1\rangle$ and $\langle t_2, T_2\rangle$ are in Γ then they are compatible in $PR^*_{I_\omega}$. (Because of the genericity of Γ, this suffices.) Assume $\langle t_1, T_1\rangle$, $\langle t_2, T_2\rangle$ are in Γ. Let $n \in \omega$ be such that $t_1, t_2 \subseteq \vec{\kappa}\restriction n$. Then let

$$S_i := (t_i{}^\frown T_i) / (\vec{\kappa}\restriction n)$$

for $i = 0, 1$. Because $\vec{\kappa}\restriction n \in t_i{}^\frown T_i$, S_i is an $(I_\omega)^*$-tree. Then $\langle \vec{\kappa}\restriction n, S_1 \cap S_2\rangle$ is in $PR^*_{I_\omega}$ and is a common extension of $\langle t_1, T_1\rangle$ and $\langle t_2, T_2\rangle$.

This completes the proof of theorem. □.*Theorem*

Next we generalize Solovay's theorem for PR^+.

Theorem 3.5. *Let $\mathbb{P}_\omega$ be the inverse limit of $\langle \mathbb{P}_n, \pi_{n,m} \mid m \leq n \in \omega\rangle$. Let G_ω be a $(V, \mathbb{P}_\omega)$-generic filter and, for each $n \in \omega$, let G_n be a $(V, \mathbb{P}_n)$-generic filter naturally obtained from G_ω. In $V[G_\omega]$, let $\langle M_n, H^m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$ be the iteration of generic ultrapowers of V by I associated with $\langle G_n \mid n \in \omega\rangle$. Then $\langle j_{0,n}(\kappa) \mid n \in \omega\rangle$ is a $PR^+_{j_{0,\omega}(I)}$-sequence over M_ω.*

To prove the theorem we need some preparations. Until we complete the proof of the above theorem, let $\mathbb{P}_\omega$, G_ω, $\langle G_n \mid n \in \omega\rangle$, $\langle M_n, H^m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$ be as in the theorem. In $V[G_\omega]$, let j_m, κ_m, I_m, $\mathbb{P}^m_k$, d^m_k, G^m_k, $\sigma^m_{k,l}$, $\pi^m_{l,k}$ be as in section 2.2 for each $m, k, l \in \omega$ with $k \leq l$. Let $I_\omega = j_{0,\omega}(I)$. Note that $H^m = G^m_1$.

$\mathbb{P}_\omega$ is the p.o. such that

- $|\mathbb{P}_\omega|$ is the set of all sequence $\langle A_n \mid n \in \omega\rangle$ such that $\pi_{n,m}(A_n) = A_m$ for each $m \leq n \in \omega$,
- $\langle A_n \mid n \in \omega\rangle \leq \langle B_n \mid n \in \omega\rangle$ iff $A_n \leq B_n$ in $\mathbb{P}_n$ for every $n \in \omega$.

First we modify $\mathbb{P}_\omega$. In V, let $\mathbb{P}$ be the p.o. of all I^+-trees ordered by inclusion. We see that $\mathbb{P}_\omega$ and $\mathbb{P}$ are equivalent. Note that if T is an I^+-tree then the sequence of levels of T, $\langle T_{(n)} \mid n \in \omega\rangle$, is in $\mathbb{P}_\omega$. Let $e : \mathbb{P} \to \mathbb{P}_\omega$ be the function defined by $e(T) := \langle T_{(n)} \mid n \in \omega\rangle$.

Lemma 3.6. *e is a dense embedding.*

Proof. Clearly e is order preserving. Moreover, if $e(T_1) \leq e(T_2)$ in $\mathbb{P}_\omega$ then $T_1 \leq T_2$ in $\mathbb{P}$. So it suffices to show that $e[\mathbb{P}]$ is dense in $\mathbb{P}_\omega$.

Take an arbitrary $\langle B_n \mid n \in \omega\rangle \in \mathbb{P}_\omega$. By induction on $n \in \omega$, define $A_n \subseteq B_n$ as follows. Let $A_0 := B_0 = \{\langle\rangle\}$. Assuming $A_n \subseteq B_n$ is defined, let $A_{n+1} := \{s \in B_{n+1} \mid s\restriction n \in A_n\}$. Then $T := \bigcup_{n\in\omega} A_n$ is a tree. Moreover, because $A_n \subseteq B_n = \pi_{n+1,n}(B_{n+1})$, $\{\xi \in \kappa \mid s^\frown\langle\xi\rangle \in A_{n+1}\} \in I^+$ for each $s \in A_n$. Thus T is an I^+-tree. Hence $e(T) = \langle A_n \mid n \in \omega\rangle \leq \langle B_n \mid n \in \omega\rangle$. □

We argue using $\mathbb{P}$ instead of $\mathbb{P}_\omega$. Let $G := e^{-1}[G_\omega]$. Then G is $(V, \mathbb{P})$-generic. In V, let $\pi_n : \mathbb{P} \to \mathbb{P}_n$ be the function defined by $\pi_n(T) := T_{(n)}$ for each $T \in \mathbb{P}$. Then π_n is the composition of e and the natural projection from $\mathbb{P}_\omega$ to $\mathbb{P}_n$. Thus G_n is the filter generated by $\pi_n[G] = \{T_{(n)} \mid T \in G\}$.

As is Theorem 3.3, we need the factor lemma for $\mathbb{P}$ and $\langle M_n, H^m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$. We define $\mathbb{P}^m$, π^m_k, d^m and G^m. Let $m \in \omega$.

Let

- $\mathbb{P}^m := j_m(\mathbb{P})$,
- $\pi^m_k := j_m(\pi_k)$.

In M_m, $\mathbb{P}^m$ is the p.o. of all $(I_m)^+$-trees ordered by inclusion and π^m_k is the function defined by $\pi^m_k(T) := T_{(k)}$.

$d^m : \mathbb{P}/G_m \to \mathbb{P}^m$ is defined similarly to d^m_k. For each $T \in \mathbb{P}$, let $f^T_m \in V$ be the function on ${}^m\kappa$ such that $f^T_m(t) = T/t$ for each $t \in {}^m\kappa$. Note that $f^T_m(t)$ is an I^+-tree for each $t \in T_{(m)}$. So if $T \in \mathbb{P}/G_m$, i.e. $T_{(m)} \in G_m$ then $[f^T_m]_{G_m} \in \mathbb{P}^m$. In $V[G_m]$, define $d^m : \mathbb{P}/G_m \to \mathbb{P}^m$ by $d^m(T) := [f^T_m]_{G_m}$ for each $T \in \mathbb{P}/G_m$.

Lemma 3.7. *d^m is a surjective dense embedding.*

Proof. This can be shown in the same way as Lemma 2.4. We show only that d^m preserves incompatibility.

Assume that $T_1, T_2 \in \mathbb{P}/G_m$ and $d^m(T_1), d^m(T_2)$ are compatible in $\mathbb{P}^m$. We show that T_1, T_2 are compatible in $\mathbb{P}/G_m$. Let $g \in V$ be such that $[g]_{G_m}$ is a common extension of $d^m(T_1)$ and $d^m(T_2)$. We may assume $g(t)$ is an I^+-tree for each $t \in {}^m\kappa$. Because $[g]_{G_m}$ is a common extension, $B := \{t \in {}^m\kappa \mid g(t) \subseteq T_1/t\,,\, T_2/t\} \cap T_{1(m)} \cap T_{2(m)} \in G_m$. Because $\pi_m[\mathbb{P}]$ is dense in $\mathbb{P}_m$, there is an $A \subseteq B$ such that $A \in G_m$ and $A \in \pi_m[\mathbb{P}]$. Then A is the m-th level of some I^+-tree and $g(t)$ is an I^+-tree for each $t \in A$. So $T := \bigcup\{t^\frown g(t) \mid t \in A\}$ is an I^+-tree. Moreover $T \subseteq T_1, T_2$ and $T \in \mathbb{P}/G_m$. Thus T_1 and T_2 are compatible in $\mathbb{P}/G_m$. □

Let $G^m := d^m[G]$. Then G^m is $(V[G_m], \mathbb{P}^m)$-generic and thus $(M_m, \mathbb{P}^m)$-generic. We show that each G^m_k is the $\mathbb{P}^m_k$-generic filter naturally obtained from G^m.

Lemma 3.8. *Assume $k \in \omega$. Then G^m_k is the filter generated by $\pi^m_k[G^m] = \{T_{(k)} \mid T \in G^m\}$.*

Proof. G^m_k is a $(M_m, \mathbb{P}^m_k)$-generic filter and $\pi^m_k[G^m]$ generates a $(M_m, \mathbb{P}^m_k)$-generic filter. So it suffices to show that $\pi^m_k[G^m] \subseteq G^m_k$. For each $A \in \mathbb{P}_{m+k}$, let $f^A_m \in V$ be as in Lemma 2.4. Recall that $d^m_k(A) = [f^A_m]_{G_m}$ for each $A \in \mathbb{P}_{m+k}/G_m$ and that $G^m_k = d^m_k[G_{m+k}]$.

Take an arbitrary $B \in \pi^m_k[G^m]$. Then there is an $S \in G^m$ such that $S_{(k)} = B$. Let $T \in G$ be such that $d^m(T) = S$ and $A := T_{(m+k)}$. Note that $A \in G_{m+k}$. Then, for each $t \in T_{(m)}$, $f^A_m(t)$ is the k-th level of $f^T_m(t)$. So, in M_m, $[f^A_m]_{G_m}$ is the k-th level of $[f^T_m(t)]_{G_m}$. Thus $d^m_k(A) = S_{(k)} = B$. Because $A \in G_{m+k}$, $B \in G^m_k$. □

Note that $\langle M_{m+l}, H^{m+k}, j_{m+k,m+l} \mid k \leq l \leq \omega, k < \omega\rangle$ is the iteration of generic ultrapowers of M_m by I_m naturally obtained from G^m.

Now we can start to prove the theorem.

Proof of Theorem 3.5.

In $V[G]$, let $\vec{\kappa} := \langle \kappa_n \mid n \in \omega\rangle$ and let

$$\Gamma := \{\langle t, T\rangle \in (PR^+_{I_\omega})^{M_\omega} \mid \forall n \in \omega, \vec{\kappa}\restriction n \in t^\frown T\}.$$

We show that Γ is $(M_\omega, (PR^+_{I_\omega})^{M_\omega})$-generic. For simplicity of notation, we write $PR^+_{I_n}$ for $(PR^+_{I_n})^{M_n}$ for each $n \leq \omega$.

First we show the genericity. Let $D \in M_\omega$ be a dense subset of $PR^+_{I_\omega}$. We show that $\Gamma \cap D \neq \emptyset$. There is an $m \in \omega$ and $\bar{D} \in M_m$ such that $j_{m,\omega}(\bar{D}) = D$. Then $\bar{D}$ is dense in $PR^+_{I_m}$. In M_m, let $E \subseteq \mathbb{P}^m$ be defined by

$$E = \{T \in \mathbb{P}^m \mid \exists k \in \omega \forall t \in T_{(k)},\ \langle \vec{\kappa}\restriction m^\frown t,\, T/t\rangle \in \bar{D}\}.$$

Working in M_m, we show that E is dense in $\mathbb{P}^m$. Take an arbitrary $S \in \mathbb{P}^m$. We find an $T \in E$ such that $T \leq S$.

Claim . *For some $k \in \omega$,*

$$B_k := \{s \in S_{(k)} \mid \exists S', \langle \vec{\kappa}\restriction m, S\rangle \geq \langle \vec{\kappa}\restriction m^\frown s, S'\rangle \in \bar{D}\} \in ((I_m)^k)^+.$$

Proof of Claim. Assume not. Then, for each $k \in \omega$, there is an $X_k \in (I_m)^*$ such that $[X_k]^k \cap B_k = \emptyset$. Let $X := \bigcap_{k\in\omega} X_k$. Then $X \in (I_m)^*$ and so $S \cap [X]^{<\omega}$ is an $(I_m)^+$-tree. Thus $\langle \vec{\kappa}\restriction m, S \cap [X]^{<\omega}\rangle \in PR^+_{I_m}$. But if $s \in S \cap [X]^{<\omega}$ then $s \notin B_k$. Hence there is no element of $\bar{D}$ which extends $\langle \vec{\kappa}\restriction m, S \cap [X]^{<\omega}\rangle$. This contradicts $\bar{D}$ is dense in $PR^+_{I_m}$. □.*Claim*

Let $k \in \omega$ be such that B_k is $(I_m)^k$-positive. For each $s \in B_k$, let S_s be an $(I_m)^+$-tree witnessing $t \in B_k$. Note that $s^\frown S_s \subseteq S$. Because $\pi^m_k[\mathbb{P}^m]$ is dense in $\mathbb{P}^m_k$, there is an $A \subseteq B$ such that A is the k-th level of some $(I_m)^+$-tree. Then

$$T := \bigcup\{s^\frown S_s \mid s \in A\}$$

is an $(I_m)^+$-tree. Moreover $T \subseteq S$ and k witnesses that $T \in E$. This shows that E is dense.

Return to $V[G]$.

Let $\bar{T}$ be in $G^m \cap E$ and let k be the element of ω witnessing that $\bar{T} \in E$. Let $T := j_{m,\omega}(\bar{T})$.

Claim . *For each $l \in \omega$, $\vec{\kappa}\restriction[m, l) \in T$.*

Proof of Claim. Let $l \in \omega$. Because $\bar{T} \in G^m$, $\bar{T}_{(l)} \in G^m_l$ by Lemma 3.8. Thus, by Lemma 2.9, $\vec{\kappa}\restriction[m, m+l) \in j_{m,m+l}(\bar{T}_{(l)})$. Then, because $j_{m+l,\omega}$ does not move κ_{m+i} for each $i < l$, $\vec{\kappa}\restriction[m, m+l) \in j_{m+l,\omega}(j_{m,m+l}(T_{(l)})) = T_{(l)}$.

□.*Claim*

Because $\bar{T} \in E$ and k witnesses this,

$$M_m \models \text{“}\forall t \in \bar{T}_{(k)},\ \langle \vec{\kappa}\restriction m^\frown t,\ \bar{T}/t\rangle \in \bar{D}\text{”}.$$

Thus, because $j_{m,\omega}$ is elementary and $\vec{\kappa}\restriction[m, m+k) \in T$,

$$\langle \vec{\kappa}\restriction m^\frown \vec{\kappa}\restriction[m, m+k),\ T/\vec{\kappa}\restriction[m, m+k)\rangle = \langle \vec{\kappa}\restriction m+k,\ T/\vec{\kappa}\restriction[m, m+k)\rangle \in D.$$

On the other hand, by the previous claim,

$$\langle \vec{\kappa}\restriction m+k,\ T/\vec{\kappa}\restriction[m, m+k)\rangle \in \Gamma.$$

So $\Gamma \cap D \neq \emptyset$.

Next we show that Γ is a filter. Clearly Γ is closed upwards. So, because of the genericity, it suffices to show that if $\langle t_1, T_1\rangle, \langle t_2, T_2\rangle \in \Gamma$ then $\langle t_1, T_1\rangle$ and $\langle t_2, T_2\rangle$ are compatible in $PR^+_{I_\omega}$.

Assume $\langle t_1, T_1\rangle, \langle t_2, T_2\rangle \in \Gamma$ and $\langle t_1, T_1\rangle \perp \langle t_2, T_2\rangle$ in $PR^+_{I_\omega}$. Let

$$D := \{\langle t, T\rangle \in PR^+_{I_\omega} \mid t \notin t_1{}^\frown T_1 \vee t \notin t_2{}^\frown T_2\}.$$

Then D is dense in $PR^+_{I_\omega}$. Let $\langle t, T\rangle \in \Gamma \cap D$. Without loss of generality, we may assume $t \notin t_1{}^\frown T_1$. Then, because $\langle t, T\rangle \in \Gamma$, t is an initial segment of $\vec{\kappa}$. Then, because $t \notin t_1{}^\frown T_1$, $\vec{\kappa} \notin [t_1{}^\frown T_1]$. This contradicts to that $\langle t_1, T_1\rangle \in \Gamma$.

This completes the proof of theorem. □.*Theorem*

3.3. Observations about PR^* and PR^+ from Theorem 3.3 and 3.5.

In this subsection, we observe well-known facts about PR^* and PR^+ from the point of view of Theorem 3.3 and Theorem 3.5. If U is a κ-complete filter on κ then Prikry Forcing associated with U does not add any bounded subset of κ and so does not affect V_κ. It is known that this can be generalized to PR^* and PR^+: if I has the strong saturation property then PR^*_I does not affect V_κ and if I is strategically closed then PR^+_I does not affect V_κ. We show this using Theorem 3.3 and Theorem 3.5.

First we define strategically closedness of ideals.

For each p.o. $\mathbb{Q}$ and $\delta \in On$, let $\Omega_\delta(\mathbb{Q})$ be the following two players game. In $\Omega_\delta(\mathbb{Q})$, Player I and II build a descending sequence $\langle q_\xi \mid \xi \in \delta - \{0\}\rangle$ in $\mathbb{Q}$, where Player I plays odd stages and Player II plays even and limit stages. Player II wins iff the game can be continued for δ stages.

We say that $\mathbb{Q}$ is *δ-strategically closed* if Player II has a winning strategy in the game $\Omega_\delta(\mathbb{Q})$. Here, a winning strategy for Player II is a function τ from the set of all descending sequences $\langle q_\eta \mid \eta \in \xi - \{0\}\rangle$ with $\xi < \delta$ to $\mathbb{Q}$ such that if Player II plays $\tau(\langle q_\eta \mid \eta \in \xi - \{0\}\rangle)$ in each ξ-th stage then Player II wins. An ideal I is called *δ-strategically closed* if $\mathbb{P}_I$ is δ-strategically closed.

Lemma 3.9. *Let κ be a regular uncountable cardinal and I be a normal ideal on κ. For each $m \leq n \in \omega$, let $\sigma_{m,n} : \mathbb{P}_{I^m} \to \mathbb{P}_{I^n}$ and $\pi_{n,m} : \mathbb{P}_{I^n} \to \mathbb{P}_{I^m}$ be the natural complete embedding and projection associated with the Fubini power of I, respectively.*

1. *Assume $\delta < \kappa$ and I is δ-saturated. Then the direct limit of $\langle \mathbb{P}_{I^n}, \sigma_{m,n} \mid m \leq n \in \omega\rangle$ has the δ-c.c..*
2. *Assume $\delta > \omega$ and I is δ-strategically closed. Then the inverse limit of $\langle \mathbb{P}_{I^n}, \pi_{n,m} \mid m \leq n \in \omega\rangle$ is δ-strategically closed.*

Proof. For each $n \in \omega$, let $\mathbb{P}_n := \mathbb{P}_{I^n}$ and $\dot{G}_n$ be the canonical name of a $\mathbb{P}_n$-generic filter.

(1). It suffices to show that $\Vdash_{\mathbb{P}_n}$ "$\mathbb{P}_{n+1}/\dot{G}_n$ has the δ-c.c.". We show this by induction on $n \in \omega$. Note that if $n = 1$ then this is true because I is δ-saturated. Assume $n \in \omega$ and that this is true for each $m \leq n$. We show this for $n + 1$.

Let G_n be a $(V, \mathbb{P}_n)$-generic filter and let $j_n : V \to M_n \cong Ult(V, G_n)$ be the generic elementary embedding. Because j_n is elementary and j_n does not move δ, $j_n(\mathbb{P}_I)$ has δ-c.c. in M_n. By the induction hypothesis, in V, $\mathbb{P}_n$ has the δ-c.c. and so I^n is a κ-complete δ-saturated ideal. Thus ${}^\kappa M_n \cap V[G_n] \subseteq M_n$. So $j_n(\mathbb{P}_I)$ has the δ-c.c. in $V[G_n]$. By Lemma 2.4, $\mathbb{P}_{n+1}/G_n$ and $j_n(\mathbb{P}_I)$ are equivalent in $V[G_n]$. Thus $\mathbb{P}_{n+1}/G_n$ has the δ-c.c. in $V[G_n]$.

(2). (2) can be shown in the same way as (1). But we need a slightly long argument for treating the inverse limit of p.o.'s. Instead we directly prove that $\mathbb{P}$ in the proof of Theorem 3.5 is δ-strategically closed. Recall that $\mathbb{P}$ is the p.o. of all I^+-trees ordered by inclusion. Let τ be a winning strategy for Player II in the game $\Omega_\delta(\mathbb{P}_I)$. Using τ, we give a winning strategy $\bar{\tau}$ for Player II in $\Omega_\delta(\mathbb{P})$.

Let ξ be in $\delta - \{0\}$ and $\langle T_\eta \mid \eta \in \delta - \{0\}\rangle$ be a descending sequence in $\mathbb{P}$. Let $S := \bigcap\{T_\eta \mid \eta \in \xi - \{0\}\}$. If

$$T := \{t \in S \mid \forall k \in |t|,\ t(k) \in \tau(\langle Suc_{T_\eta}(t) \mid \eta \in \xi - \{0\}\rangle)\} \in \mathbb{P}$$

then let $\bar{\tau}(\langle T_\eta \mid \eta \in \xi - \{0\}\rangle)$ be the above T. Otherwise let $\bar{\tau}(\langle T_\eta \mid \eta \in \xi - \{0\}\rangle)$ be an arbitrary element of $\mathbb{P}$.

Let ξ be even and $\langle T_\eta \mid \eta \in \xi - \{0\}\rangle$ be a descending chain in $\mathbb{P}$. Let S, T be as above. Note that if $t \in T$ then $Suc_T(t) = Suc_S(t) \cap \tau(\langle Suc_{T_\eta} \mid \eta \in \xi - \{0\}\rangle)$. Now assume that, for each $t \in S$, $\langle Suc_{T_\eta} \mid \eta \in \xi - \{0\}\rangle$ is a play in $\Omega_\delta(\mathbb{P}_I)$ in which Player II plays according to τ. Then, because τ is a winning strategy, $Suc_T(t) = \tau(\langle Suc_{T_\eta}(t) \mid \eta \in \xi - \{0\}\rangle) \in I^+$ for each $t \in T$. Note that $\langle\rangle \in T \neq \emptyset$. So $T \in \mathbb{P}$, i.e. $\bar{\tau}(\langle T_\eta \mid \eta \in \xi - \{0\}\rangle) = T$. Moreover $T \subseteq S$, i.e. T is below each T_η. By induction on ξ, we can see:

> Assume ξ is even and $\langle T_\eta \mid \eta \in \xi - \{0\}\rangle$ is a play in which Player II plays according to $\bar{\tau}$. Let S, T be as in the definition of $\bar{\tau}$. Then:
> i). For each $t \in S$, $\langle Suc_{T_\eta}(t) \mid \eta \in \xi - \{0\}\rangle$ is a play in $\Omega_\delta(\mathbb{P}_I)$ in which Player II plays according to τ.
> ii). $T = \bar{\tau}(\langle T_\eta \mid \eta \in \xi - \{0\}\rangle)$ and T can be played at the ξ-th stage.

In particular, ii) implies that $\bar{\tau}$ is a winning strategy for Player II in $\Omega_\delta(\mathbb{P})$. □

Theorem 3.10. *Let κ be a regular uncountable cardinal and let I be a normal ideal on κ.*

(1) *Assume that $\delta < \kappa$ and I is δ-saturated. Let Γ be (V, PR_I^*)-generic. Then for each $\alpha, \beta < \kappa$ and $f \in {}^\alpha\beta \cap V[\Gamma]$, there is a function $F \in V$ such that for every $\xi \in \alpha$, $|F(\xi)|^V < \delta$ and $f(\xi) \in F(\xi)$. (We say that F $<\delta$-covers f.)*

(2) *Assume that $\delta > \omega$ and I is δ-strategically closed. Then $PR_I^\dagger$ does not add any bounded subset of δ.*

Proof. (1). Assume not. Let α, β, $\dot{f}$ and $\langle t, T\rangle$ be such that $\alpha, \beta < \kappa$ and $\langle t, T\rangle \in PR_I^*$ forces $\dot{f} \in {}^\alpha\beta$ and there is no $F \in V$ which $<\delta$-covers $\dot{f}$. Let $\mathbb{P}_\omega$ be the direct limit of $\langle \mathbb{P}_{I^n} \mid n \in \omega\rangle$ as for the natural complete embeddings and let G_ω be $(V, \mathbb{P}_\omega)$-generic. Let $\langle M_n, j_{m,n} \mid m \leq n \leq \omega\rangle$ be as in Theorem 3.3. Let $M = M_\omega$, $j := j_{0,\omega}$ and $\vec{\kappa} = \langle j_{0,n}(\kappa) \mid n \in \omega\rangle$.

We work in $V[G_\omega]$. By Theorem 3.3 and Lemma 3.2, $t^\frown\vec{\kappa}$ is a $PR_{j(I)}^*$-sequence over M. Let Γ_t be the $(M, PR_{j(I)}^*)$-generic filter generated by $t^\frown\vec{\kappa}$ and let f be the interpretation of $j(\dot{f})$ by Γ_t. Note that $j(\langle t, T\rangle) \in \Gamma_t$ because $j(t) = t$ and $\vec{\kappa}\restriction n \in j(T)$ for each $n \in \omega$. So, because j is elementary and does not move α, β and δ, $f \in {}^\alpha\beta$ and there is no $F \in M$ which δ-covers f. On the other hand, because $\mathbb{P}_\omega$ has the δ-c.c. and $f \in V[G_\omega]$, there is an $F \in V$ which $<\delta$-covers f. We may assume $dom(F) = \alpha$ and $F(\xi) \subseteq \beta$ for each $\xi < \alpha$. Then $F = j(F) \in M$. This is a contradiction.

(2). We show (2) almost the same argument. Assume the contrary. Then there is an $\alpha < \delta$, a $PR_I^\dagger$-name $\dot{x}$ and $\langle t, T\rangle \in PR_I^\dagger$ such that $\langle t, T\rangle$ forces $\dot{x} \subseteq \alpha$ and $\dot{x} \notin V$. Let $\mathbb{P}_\omega$ be the inverse limit of $\langle \mathbb{P}_{I^n} \mid n \in \omega\rangle$ as for the natural projections and let G_ω be a $(V, \mathbb{P}_\omega)$-generic filter such that $\langle T_{(n)} \mid n \in \omega\rangle \in G_\omega$. Let $\langle M_n, j_{m,n} \mid m \leq n \leq \omega\rangle$ be as in Theorem 3.5. Let $M = M_\omega$, $j := j_{0,\omega}$ and $\vec{\kappa} = \langle j_{0,n}(\kappa) \mid n \in \omega\rangle$.

Work in $V[G_\omega]$. Let Γ_t be a $(M, PR_{j(I)}^\dagger)$-generic filter generated by $t^\frown\vec{\kappa}$. Let x be the interpretation of $j(\dot{x})$ by Γ_t. Then because $\langle T_{(n)} \mid n \in \omega\rangle \in G_\omega$, $\vec{\kappa}\restriction n \in j(T)$ for each $n \in \omega$ and so $j(\langle t, T\rangle) \in \Gamma_t$. Then, by the elementarity of j, $x \subseteq \alpha$ and $x \notin M$. On the other hand, because $\mathbb{P}_\omega$ is δ-strategically closed, $x \in V$. Because $x \subseteq \alpha < \kappa$, $x = j(x) \in M$. This is a contradiction. □

Next we discuss $\aleph_1$-semiproperness of PR^* and PR^+. First we review $\aleph_1$-semiproperness of partial orderings.

A p.o. $\mathbb{P}$ is called $\aleph_1$-*semiproper* if there is a cardinal $\lambda > 2^{2^{|tcl(\mathbb{P})|}}$ and a club $C \subseteq [\mathcal{H}_\lambda]^\omega$ such that for every $N \in C$ and $p \in \mathbb{P} \cap N$, there is a $p^* \leq p$ which forces "$N[\dot{G}] \cap \omega_1^V = N \cap \omega_1^V$". Here $\dot{G}$ is the canonical name of a $\mathbb{P}$-generic filter and, for each $(V, \mathbb{P})$-generic filter G, $N[G] := \{\dot{x}_G \mid \dot{x}$ is a $\mathbb{P}$-name $\wedge\ \dot{x} \in N\}$. We call the above p^* a semi master condition for N.

Theorem 3.11. *Let κ be a regular uncountable cardinal and I be a normal precipitous ideal on κ. Let $\sigma_{m,n} : \mathbb{P}_{I^m} \longrightarrow \mathbb{P}_{I^n}$ be the natural complete embedding and $\pi_{n,m} : \mathbb{P}_{I^n} \longrightarrow \mathbb{P}_{I^m}$ be the natural projection for $m \leq n \in \omega$.*

1. *If the direct limit of $\langle \mathbb{P}_{I^n}, \sigma_{m,n} \mid m \leq n \in \omega\rangle$ is $\aleph_1$-semiproper then PR_I^* is $\aleph_1$-semiproper.*
2. *If the inverse limit of $\langle \mathbb{P}_{I^n}, \pi n, m \mid m \leq n \in \omega\rangle$ is $\aleph_1$-semiproper then PR_I^+ is $\aleph_1$-semiproper.*

Proof. (2) can be shown in the same way as (1). So we show only (1). Let $\mathbb{P}_\omega$ be the direct limit of $\langle \mathbb{P}_{I^n}, \sigma_{m,n} \mid m \leq n \in \omega\rangle$.

In $V^{\mathbb{P}_\omega}$, let $\langle M_n, j_{m,n} \mid m \leq n \leq \omega\rangle$ be as in Theorem 3.5 and let $M = M_\omega$, $j = j_{0,\omega}$. For each $t \in {}^{<\omega}\kappa$, let $\dot{\Gamma}_t$ be a $\mathbb{P}_\omega$-name for the $(M, PR^*_{j(I)})$-generic filter generated by $t^\frown\langle j_{0,n}(\kappa) \mid n \in \omega\rangle$. Note that if $\dot{x} \in V$ is a PR_I^*-name then $j(\dot{x}) \in M$ is a $PR^*_{j(I)}$-name. So there is a $\mathbb{P}_\omega$-name $\dot{a} \in V$ such that

$$V^{\mathbb{P}_\omega} \models \text{“}\dot{a} = j(\dot{x})_{\dot{\Gamma}_t} := \text{the interpretation of } j(\dot{x}) \text{ by } \dot{\Gamma}_t\text{”}$$

In V, let λ be a cardinal such that

- $\kappa, I, \mathbb{P}_\omega, PR_I^* \in \mathcal{H}_\lambda$,
- If $\dot{x} \in \mathcal{H}_\lambda$ is a PR_I^*-name and $t \in {}^{<\omega}\kappa$ then there is a $\mathbb{P}_\omega$-name $\dot{a} \in \mathcal{H}_\lambda$ such that $V^{\mathbb{P}_\omega} \models$ "$\dot{a} = j(\dot{x})_{\dot{\Gamma}_t}$" .

Let $F : \mathcal{H}_\lambda \longrightarrow \mathcal{H}_\lambda$ be a function witnessing the second condition above, i.e. for each PR_I^*-name $\dot{x} \in \mathcal{H}_\lambda$ and $t \in {}^{<\omega}\kappa$, $F(\dot{x}, t)$ is a $\mathbb{P}_\omega$-name such that $V^{\mathbb{P}_\omega} \models$ "$F(\dot{x}, t) = j(\dot{x})_{\dot{\Gamma}_t}$".

We show that if N is a countable elementary submodel of $\langle \mathcal{H}_\lambda, \in, F, \kappa, I, ...\rangle$ and $\langle t, T\rangle \in N \cap PR_I^*$ then there is a semi master condition for N below $\langle t, T\rangle$. Let N and $\langle t, T\rangle$ be as above. Because $\mathbb{P}_\omega$ is $\aleph_1$-semiproper, there is a semi master condition $p \in \mathbb{P}_\omega$ for N. Let G_ω be a $(V, \mathbb{P}_\omega)$-generic filter containing p. In $V[G_\omega]$, let j and M be as above. Note that ω_1 is absolute among V, M and $V[G_\omega]$.

We work in $V[G_\omega]$. Let Γ_t be the interpretation of $\dot{\Gamma}_t$ by G_ω. Because N is countable in V, $j(N) = j[N]$. Then, because $t \in N$ and N is closed under F,

$$\begin{aligned} j(N)[\Gamma_t] &= \{\dot{y}_{\Gamma_t} \mid \dot{y} \in j(N) \wedge \dot{y} \text{ is a } PR^*_{j(I)}\text{-name}\} \\ &= \{j(\dot{x})_{\Gamma_t} \mid \dot{x} \in N \wedge \dot{x} \text{ is a } PR^*_I\text{-name}\} \\ &= \{F(\dot{x}, t)_{G_\omega} \mid \dot{x} \in N \wedge \dot{x} \text{ is a } PR^*_I\text{-name}\} \\ &\subseteq N[G_\omega] \,. \end{aligned}$$

So, because $j \restriction \omega_1 = id$ and G_ω contains p,

$$j(N)[\Gamma_t] \cap \omega_1 \subseteq N[G_\omega] \cap \omega_1 = N \cap \omega_1 \subseteq j(N) \cap \omega_1$$

and thus $j(N)[\Gamma_t] \cap \omega_1 = j(N) \cap \omega_1$. This implies that there is a semi master condition for $j(N)$ in Γ_t. Note that $j(\langle t, T\rangle) \in \Gamma_t$. Thus $M_\omega \models$ "there is a semi master condition for $j(N)$ below $j(\langle t,T\rangle)$". So, by elementarity of j, $V\models$"there is a semi master condition for N below $\langle t,T\rangle$". $\square$

Corollary 3.12. *Let κ be a regular uncountable cardinal and I be a normal ideal on κ.*

(1) *If I is ω_1-saturated then PR_I^* is $\aleph_1$-semiproper.*
(2) *If I is $\omega+1$-strategically closed then PR_I^+ is $\aleph_1$-semiproper.*

References

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