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There is an independent splitting family

JÖRG BRENDLE[1]

The Graduate School of Science and Technology, Kobe University
Rokko-dai 1-1, Nada-ku, Kobe 657-8501, Japan
神戸市灘区六甲台 1-1 神戸大学大学院自然科学研究科
brendle@kurt.scitec.kobe-u.ac.jp

ABSTRACT

Let $\mathfrak{s}$ be the splitting number, that is, the size of the least splitting family. We show there is an independent splitting family of size $\mathfrak{s}$.

Introduction

This is an old note written back in 1996 and originally not intended for publication, the reason being that the question it addressed had been solved 20 years earlier. However, recent work of Hrušák and Steprāns [HS] and others indicated there was some interest in the size of the smallest independent splitting family, and since this note showed this was actually equal to the splitting number $\mathfrak{s}$ we decided to publish it after all. We tried to keep the original text, but inserted a few references. We apologize that the contents of this note is quite disjoint from our talk at the conference, but the latter paper has been accepted already for publication elsewhere.[2]

We call $\mathcal{I} \subseteq [\omega]^\omega$ an *independent family* iff every Boolean combination of elements of $\mathcal{I}$ is infinite (i.e., iff for all finite partial functions $\tau : \mathcal{I} \to \{1, -1\}$, the set $A_\tau = \bigcap_{A \in dom(\tau)} A^{\tau(A)}$ is infinite where $A^1 = A$ and $A^{-1} = \omega \setminus A$). Given $A, B \in [\omega]^\omega$ we say A *splits* B iff both

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2 J. Brendle, L. Halbeisen, and B. Löwe, *Silver measurability and its relation to other regularity properties*, Mathematical Proceedings of the Cambridge Philosophical Society, to appear.

$A \cap B$ and $B \setminus A$ are infinite. $\mathcal{S} \subseteq [\omega]^\omega$ is a *splitting family* iff for all $B \in [\omega]^\omega$ there is $A \in \mathcal{S}$ which splits B. $\mathcal{I}$ is an *independent splitting family* iff it's both independent and splitting. We answer a question of K. Kunen [Mi, Problem 4.6] by showing

THEOREM 1. (ZFC) *There is an independent splitting family.*

It turns out this has been proved for the first time 20 years ago by P. Simon [Si]. It has been reproved independently by S. Shelah (unpublished) and the present author. — We also develop some related combinatorics.

1. Proof of Theorem 1

Before starting out with the proof, we need to introduce several of the classical cardinal invariants of the continuum. Let $\mathfrak{s}$ be the size of the smallest splitting family (the *splitting number*); $\mathfrak{i}$ stands for the cardinality of the least maximal independent family (the *independence number*). Given $A, B \in [\omega]^\omega$, write $A \subseteq^* B$ iff $A \setminus B$ is finite; similarly, we define $\subset^*$. Given functions f and g in the Baire space ω^ω, say that f *eventually dominates* g ($g <^* f$, in symbols) iff $\{n \in \omega;\ g(n) \geq f(n)\}$ is finite; similarly, we define $\leq^*$. The *dominating number* $\mathfrak{d}$ is defined to be the size of the smallest $\mathcal{D} \subseteq \omega^\omega$ such that every member of ω^ω is eventually dominated by a member of $\mathcal{D}$ (such families are called *dominating families*). It is well–known that $\omega_1 \leq \mathfrak{s} \leq \mathfrak{d} \leq \mathfrak{i} \leq \mathfrak{c}$ holds in ZFC (where $\mathfrak{c}$ stands for the cardinality of the continuum). The inequality $\mathfrak{s} \leq \mathfrak{d}$ is due to P. Nyikos [vD]; and $\mathfrak{d} \leq \mathfrak{i}$ was proved by S. Shelah [Va]. We shall use the main idea of the first proof, as well as the second result for our argument.

We call a sequence $P = \langle I_n;\ n \in \omega \rangle$ a *partition* iff it is a partition of ω into finite adjacent intervals (i.e. $0 = \min(I_0) < \ldots < \max(I_n)+1 = \min(I_{n+1}) < \ldots$). Given a strictly increasing function $f \in \omega^\omega$ (with $f(0) \geq 1$), let P_f, the *partition associated with* f, be defined by $I_0 = [0, f(0)), \ldots, I_n = [f^n(0), f^{n+1}(0)), \ldots$ where we put $f^{n+1}(0) = f(f^n(0))$. Given $\mathcal{F} \subseteq \omega^\omega$, let $\mathcal{P}_\mathcal{F} = \{P_f;\ f \in \mathcal{F}\}$ be the family of associated partitions. We say $A \in [\omega]^\omega$ *splits* a partition $P = \langle I_n;\ n \in \omega \rangle$ iff there are infinitely many $n \in \omega$ with $I_n \subseteq A$ and infinitely many $m \in \omega$ with $I_m \cap A = \emptyset$. We shall prove

THEOREM 2. *Let $\mathcal{D} \subseteq \omega^\omega$ be a dominating family of size $\mathfrak{d}$. Then there is an independent family $\mathcal{I} \subseteq [\omega]^\omega$ (also of size $\mathfrak{d}$) such that every partition from $\mathcal{P}_\mathcal{D}$ is split by a member from $\mathcal{I}$.*

Before proving Theorem 2, let us see how to deduce Theorem 1 from it. Note that the argument is a straightforward reformulation of the proof of $\mathfrak{s} \leq \mathfrak{d}$.

Proof of Theorem 1 from Theorem 2. We claim that the $\mathcal{I}$ provided by Theorem 2 is splitting. Given $B \in [\omega]^\omega$, define $g_B \in \omega^\omega$ such that $g_B(n)$ is the least $k \in B$ larger than n. Choose $f \in \mathcal{D}$ eventually dominating g_B, and take $A \in \mathcal{I}$ splitting the partition P_f. We claim that A splits B.

To see this, simply note that if $P_f = \langle I_n;\ n \in \omega\rangle$, then almost all I_n have non-trivial intersection with B (because $g_B(f^n(0)) < f^{n+1}(0)$ for almost all n). Since A avoids infinitely many of the I_n's and contains infinitely many I_m's, it splits B. □

Proof of Theorem 2. Let $\mathcal{D} = \{f_\alpha;\ \alpha < \mathfrak{d}\}$ be an enumeration of $\mathcal{D}$. We shall recursively construct sets A_α for $\alpha < \mathfrak{d}$ such that

(i) A_α is independent from $\mathcal{I}_\alpha = \{A_\beta;\ \beta < \alpha\}$;

(ii) A_α splits the partiton P_{f_α}.

So suppose $\mathcal{I}_\alpha$ has been produced. Let B be any set independent from $\mathcal{I}_\alpha$. Such a set exists by $|\mathcal{I}_\alpha| < \mathfrak{d} \leq \mathfrak{i}$. We describe how to modify B so that it splits P_{f_α} and remains independent.

Fix a finite partial function $\tau : \alpha \to \{1, -1\}$ and look at $A_\tau = \bigcap_{\beta \in dom(\tau)} A_\beta^{\tau(\beta)}$. Also let $P_{f_\alpha} = \langle I_n;\ n \in \omega\rangle$. Since A_τ has infinite intersection with both B and $\omega \setminus B$, we can define $g_\tau \in \omega^\omega$ such that $g_\tau(n)$ is the least $k > n$ such that both $A_\tau \cap B \cap (\bigcup_{i=n}^{k-1} I_i)$ and $A_\tau \cap (\omega \setminus B) \cap (\bigcup_{i=n}^{k-1} I_i)$ are non-empty. Let $\mathcal{G}$ be the closure of the family of the g_τ's under taking finite maxima. (That is, if $g_0, ..., g_n \in \mathcal{G}$, then $g \in \mathcal{G}$ where $g(k) = \max\{g_0(k), ..., g_n(k)\}$ for all $k \in \omega$.) Since $|\mathcal{G}| < \mathfrak{d}$, we can find $f \in \omega^\omega$ which is not dominated by any member of $\mathcal{G}$. Without loss, f is strictly increasing.

We partition ω into the four sets

$$C_m = \bigcup_{k\in\omega} \Big[f^{4k+m}(0), f^{4k+m+1}(0)\Big), \qquad m \in 4.$$

Notice that there is $m \in 4$ such that for all τ there are infinitely many $n \in C_m$ with $g_\tau(n) < f(n)$. (If there were no such m we could find a τ_m witnessing the failure for each m; then the maximum of the g_{τ_m} would eventually dominate f, a contradiction.) Without loss $m = 0$. Now put

$$D_m = \bigcup_{k\in\omega} \bigcup_{n\in J_{4k+m}} I_n, \qquad m \in 4,$$

where $J_\ell = \big[f^\ell(0), f^{\ell+1}(0)\big)$. We next define A_α such that $A_\alpha \cap (D_0 \cup D_1) = B \cap (D_0 \cup D_1)$ and $A_\alpha \cap (D_2 \cup D_3) = D_2$. It is immediate from the second clause of this definition that A_α splits P_{f_α}.

We still have to check A_α is independent of $\mathcal{I}_\alpha$. For this take $\tau : \alpha \to \{1, -1\}$ a finite partial function. Choose $n \in C_0$ with $g_\tau(n) < f(n)$. Let k be such that $n \in J_{4k}$. Then $g_\tau(n) < f^{4k+2}(0)$, hence both $A_\tau \cap B$ and $A_\tau \cap (\omega \setminus B)$ intersect $I = \bigcup_{i \in [f^{4k}(0), f^{4k+2}(0))} I_i$ non-trivially. Since $B \cap I = A_\alpha \cap I$, this is still true for B replaced by A_α. Hence both intersections $A_\tau \cap A_\alpha$ and $A_\tau \cap (\omega \setminus A_\alpha)$ are infinite, and we're done. $\square$

2. The partition–splitting number

We now try to shed some more light on a phenomenon which was crucial in the above proof. Call a family $\mathcal{S} \subseteq [\omega]^\omega$ *partition–splitting* iff every partition is split by some member of $\mathcal{S}$. It is immediate from the way Theorem 1 was proved from Theorem 2 that every partition–splitting family is a splitting family as well. Let $\mathfrak{ps}$ denote the size of the smallest partition–splitting family. The *unbounding number* $\mathfrak{b}$ is the cardinality of the least family $\mathcal{F} \subseteq \omega^\omega$ such that no $g \in \omega^\omega$ eventually dominates all members of $\mathcal{F}$ (such families are called *unbounded families*). Clearly $\mathfrak{b} \leq \mathfrak{d}$. Then we have

THEOREM 3. (Kamburelis–Węglorz [KW]) $\mathfrak{ps} = \max\{\mathfrak{b}, \mathfrak{s}\}$.

Proof. $\mathfrak{ps} \geq \mathfrak{s}$ follows from the remark in the preceding paragraph.

Next, given $A \in [\omega]^\omega$ co–infinite, define $g_A \in \omega^\omega$ by $g_A(n) =$ the least $k > n$ such that the interval $[n, k)$ intersects both A and $\omega \setminus A$. We see immediately that if $f \geq^* g_A$ then P_f is not split by A. $\mathfrak{ps} \geq \mathfrak{b}$ follows.

Finally, we show that $\mathfrak{ps} \leq \max\{\mathfrak{b}, \mathfrak{s}\}$. Modifications of the argument shall be used several times later on. Given $B \in [\omega]^\omega$ and $f \in \omega^\omega$, define $C(B, f) = \bigcup_{m \in B} I^f_m$ where $P_f = \langle I^f_m;\ m \in \omega \rangle$. We shall prove that $\mathcal{C} = \{C(B, f);\ B \in \mathcal{S} \text{ and } f \in \mathcal{F}\}$ is partition–splitting if $\mathcal{S} \subseteq [\omega]^\omega$ is splitting and $\mathcal{F} \subseteq \omega^\omega$ is unbounded.

To see this, let $P = \langle J_\ell;\ \ell \in \omega \rangle$ be any partition. Define $g_P \in \omega^\omega$ such that $g_P(n) =$ the least $k > n$ such that at least two of the intervals J_ℓ are contained in the interval $[n, k)$. We claim that if $f \not\leq^* g_P$, then there are infinitely many m such that I^f_m contains some J_ℓ.

For this, take n such that $f(n) > g_P(n)$. Find m such that $n \in I^f_m$. Note that $n < f^{m+1}(0)$, hence $g_P(n) < f^{m+2}(0)$. This means at least two intervals J_ℓ are contained in $I^f_m \cup I^f_{m+1}$. Hence either some J_ℓ is contained in I^f_m, or some J_ℓ is contained in I^f_{m+1}.

Let $A = A(P, f)$ be the set of all m such that I^f_m contains some J_ℓ. If B splits A, then $C(B, f)$ splits the partition P, and we're done. $\square$

COROLLARY 4. $\mathfrak{ps} \leq \mathfrak{d}$. $\square$

(Of course, this also follows from Theorem 2; for there, we produced an independent partition–splitting family of size $\mathfrak{d}$.)

We briefly mention duality (see Blass for a detailed account). To many cardinal invariants, we can associate a *dual cardinal* which is gotten essentially by negating the basic statement in the definition of the given cardinal and by replacing a quantifier of the form $\exists^\infty n$ ("there are infinitely many n") by one of the form $\forall^\infty n$ ("for almost all n") or vice–versa. So $\mathfrak{b}$ and $\mathfrak{d}$ are dual to each other. The dual of $\mathfrak{s}$ is the *reaping* (or: *refinement*) number $\mathfrak{r}$ which is defined as the size of the smallest $\mathcal{R} \subseteq [\omega]^\omega$ such that no $A \in [\omega]^\omega$ splits all elements of $\mathcal{R}$ (or, equivalently, given $A \in [\omega]^\omega$ there is $R \in \mathcal{R}$ with either $R \subseteq^* A$ or $R \subseteq^* \omega \setminus A$). The proof that $\mathfrak{s} \leq \mathfrak{d}$ dualizes to $\mathfrak{b} \leq \mathfrak{r}$. A maximal independent family is easily seen to be reaping and, hence, we see $\mathfrak{r} \leq \mathfrak{i}$. Similarly, we say a family $\mathcal{P}$ of partitions is *partition–reaping* iff there is no $A \in [\omega]^\omega$ splitting all members of $\mathcal{P}$. $\mathfrak{pr}$, the *partition–reaping number*, is the size of the smallest partition–reaping family. We now get

PROPOSITION 5. [Br] $\mathfrak{pr} = \min\{\mathfrak{r}, \mathfrak{d}\}$.

Proof. In the proof of Theorem 1 (from Theorem 2) we saw that, given $B \in [\omega]^\omega$, if $A \in [\omega]^\omega$ splits the partition P_{g_B}, then it also splits B. Hence, if $\mathcal{R} \subseteq [\omega]^\omega$ is such that no $A \in [\omega]^\omega$ splits all members of $\mathcal{R}$, then no $A \in [\omega]^\omega$ can split all members of $\mathcal{P} = \{P_{g_B};\ B \in \mathcal{R}\}$; and $\mathfrak{pr} \leq \mathfrak{r}$ follows.

By the second paragraph of the previous proof, we conclude that if $\mathcal{F} \subseteq \omega^\omega$ is dominating, then $\{P_f;\ f \in \mathcal{F}\}$ is not split by a single $A \in [\omega]^\omega$ — and hence $\mathfrak{pr} \leq \mathfrak{d}$.

From the last part of the previous proof, we see that if $\mathcal{P}$ is a family of partitions of size less than $\min\{\mathfrak{r}, \mathfrak{d}\}$, then all elements of $\mathcal{P}$ are split by $C(B, f)$ where $f \not\leq^* g_P$ for $P \in \mathcal{P}$ and B splits all $A(P, f)$. $\square$

COROLLARY 6. $\mathfrak{pr} \geq \mathfrak{b}$. $\square$

We digress a little further to comment on a problem addressed by J. Steprāns [St]. The $\aleph_0$*-splitting number* $\mathfrak{s}(\omega)$ is the size of the smallest $\mathcal{S} \subseteq [\omega]^\omega$ such that given any countable $\{A_j;\ j \in \omega\} \subseteq [\omega]^\omega$ there is $S \in \mathcal{S}$ splitting all A_j. Similarly we may define the $\aleph_0$*-partition splitting number* $\mathfrak{ps}(\omega)$ to be the size of the smallest $\mathcal{S} \subseteq [\omega]^\omega$ such that all members of any countable family of partitions are split by a single $S \in \mathcal{S}$. Clearly, $\mathfrak{s}(\omega) \geq \mathfrak{s}$, $\mathfrak{ps}(\omega) \geq \mathfrak{ps}$ and $\mathfrak{ps}(\omega) \geq \mathfrak{s}(\omega)$. Steprāns asked whether $\mathfrak{s} = \mathfrak{s}(\omega)$. A modification in the proof of Theorem 3 gives

PROPOSITION 7. $\mathfrak{ps}(\omega) = \mathfrak{ps}$.

Proof. It suffices to show that $\mathfrak{ps}(\omega) \leq \max\{\mathfrak{b}, \mathfrak{s}\}$. For this, we show the family $\mathcal{C}$ defined in the proof of Theorem 3 is actually $\aleph_0$–partition–splitting.

Given a set $P = \{P_j = \langle J_{j\ell}; \ell \in \omega\rangle;\ j \in \omega\}$ of partitions, define $g_P \in \omega^\omega$ such that $g_P(n) =$ the least $k > n$ such that there is i between n and k such that for each $j < n$ at least one interval $J_{j\ell}$ is contained in each of $[n, i)$ and $[i, k)$. The rest of the argument goes through as before. □

COROLLARY 8. $\mathfrak{s} \geq \mathfrak{b}$ *implies* $\mathfrak{s} = \mathfrak{s}(\omega)$. □

On the other hand, A. Kamburelis [KW] proved that $\mathfrak{s} <$ cov (meager) implies $\mathfrak{s} = \mathfrak{s}(\omega)$, where cov (meager) is the size of the smallest covering of the real line by meager sets. Hence, if $\mathfrak{s} < \mathfrak{s}(\omega)$ is at all consistent, we must have cov (meager) $\leq \mathfrak{s} < \mathfrak{s}(\omega) \leq \mathfrak{b}$.

3. Independent splitting families of different cardinalities

Equipped with the ideas from the last section, we investigate independent splitting families in somewhat more detail. It is relatively easy to modify the argument in the proof of Theorems 1 and 2 to get an independent partition–splitting family of size $\mathfrak{c}$. Concerning smaller cardinalities we have

THEOREM 9. *There is an independent partition–splitting family of size* $\mathfrak{ps}$.

Proof. We construct such a family $\mathcal{I}$ of size $\max\{\mathfrak{b}, \mathfrak{s}\}$ by modifying the argument for $\mathfrak{ps} \leq \max\{\mathfrak{b}, \mathfrak{s}\}$ in the proof of Theorem 3. By Theorem 2, we can assume $\mathfrak{b} < \mathfrak{d}$.

Let $\{f_\alpha;\ \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be an unbounded family of strictly increasing functions which is well–ordered by $<^*$ (i.e. $\alpha < \beta$ implies $f_\alpha <^* f_\beta$). Also choose $\{B_\gamma;\ \gamma < \mathfrak{s}\} \subseteq [\omega]^\omega$ a splitting family; and let $\{D_{\alpha\gamma};\ \langle\alpha, \gamma\rangle \in \mathfrak{b} \times \mathfrak{s}\}$ be an independent family of size $\mathfrak{ps}$. Finally fix a partition $\langle E_k;\ k \in \omega\rangle$ of ω into countably many countable sets. Since $\mathfrak{b} < \mathfrak{d}$ we find $f \in \omega^\omega$ which is not eventually dominated by any f_α on any E_k (that is, $\{n \in E_k;\ f(n) > f_\alpha(n)\}$ is infinite for all k and all α).

We're ready to define the sets $C_{\alpha\gamma}$, where $\langle\alpha, \gamma\rangle \in \mathfrak{b} \times \mathfrak{s}$, as follows. Let $K_\alpha = \{n;\ f_\alpha(n) \geq f(n)\}$. Put $C_{\alpha\gamma} \cap K_\alpha = K_\alpha \cap (\bigcup_{m \in B_\gamma} I_m^\alpha)$ where $P_{f_\alpha} = \langle I_m^\alpha;\ m \in \omega\rangle$ is the partition associated with f_α; and let $C_{\alpha\gamma} \cap (\omega \setminus K_\alpha) = (\omega \setminus K_\alpha) \cap (\bigcup_{k \in D_{\alpha\gamma}} E_k)$. We claim that $\mathcal{I} = \{C_{\alpha\gamma};\ \langle\alpha, \gamma\rangle \in \mathfrak{b} \times \mathfrak{s}\}$ is the family we are seeking.

We first check $\mathcal{I}$ is independent. Let $\tau : \mathfrak{b} \times \mathfrak{s} \to \{1, -1\}$ be a finite partial function. Fix α maximal in the first coordinate of the domain of τ. Note that if β is in the first coordinate of the domain of τ, then $(\omega \setminus K_\alpha) \subseteq^* (\omega \setminus K_\beta)$. By choice of the $D_{\beta\gamma}$ and by definition of the $C_{\beta\gamma}$, we now see they're independent on the set $\omega \setminus K_\alpha$.

The proof $\mathcal{I}$ is partition–splitting is a minor variation on the proof of Theorem 3, and therefore we confine ourselves to a brief sketch. Given a partition $P = \langle J_\ell;\ \ell \in \omega\rangle$, define g_P as before. Find $\alpha < \mathfrak{b}$ such that $f_\alpha \not\leq^* f \circ g_P$. If n is such that $f_\alpha(n) > f(g_P(n))$, and $n \in I^\alpha_m$, then some interval J_ℓ will belong to either I^α_m or to I^α_{m+1} as before; furthermore, we will have that f_α dominates f on all of $[n, g_P(n))$, and, a fortiori, on J_ℓ; hence $J_\ell \subseteq K_\alpha$. This allows us to conclude as in Theorem 3. □

Let κ be a cardinal. A collection $\{T_\alpha;\ \alpha < \kappa\}$ of subsets of ω is called a *tower* iff $\alpha < \beta$ implies $T_\beta \subset^* T_\alpha$, and there is no $T \in [\omega]^\omega$ such that $T \subseteq^* T_\alpha$ for all $\alpha < \kappa$. Let $\mathfrak{t}$, the *tower number*, be the size of the smallest tower. It's well–known that $\mathfrak{t} \leq \mathfrak{b}$ and $\mathfrak{t} \leq \mathfrak{s}$. We're ready to prove

THEOREM 10. *There is an independent splitting family of size* $\mathfrak{s}$.

Proof. By the previous result we can assume $\mathfrak{s} < \mathfrak{b}$. The construction will be quite similar to the one in the preceding theorem.

Let $\{T_\alpha;\ \alpha < \mathfrak{t}\}$ be a tower; fix $\{B_\gamma;\ \gamma < \mathfrak{s}\}$ a splitting family and $\{D_{\alpha\gamma};\ \langle\alpha,\gamma\rangle \in \mathfrak{t}\times\mathfrak{s}\}$ an independent family as before. Using $\mathfrak{t} < \mathfrak{b}$, we easily find a partition $\langle E_k;\ k \in \omega\rangle$ of ω into countably many countable sets such that $E_k \cap T_\alpha$ is infinite for all k and all α. Define $C_{\alpha\gamma}$, for $\langle\alpha,\gamma\rangle \in \mathfrak{t}\times\mathfrak{s}$, by $C_{\alpha\gamma} \cap (\omega \setminus T_\alpha) = B_\gamma \cap (\omega \setminus T_\alpha)$ and $C_{\alpha\gamma} \cap T_\alpha = T_\alpha \cap (\bigcup_{k \in D_{\alpha\gamma}} E_k)$. As in the proof of the previous theorem, we see $\mathcal{I} = \{C_{\alpha\gamma};\ \langle\alpha,\gamma\rangle \in \mathfrak{t}\times\mathfrak{s}\}$ is independent. To see it's splitting fix $A \in [\omega]^\omega$; then find $\alpha < \mathfrak{t}$ such that $A \setminus T_\alpha$ is infinite; next find $\gamma < \mathfrak{s}$ such that $A \setminus T_\alpha$ is split by B_γ. Then $C_{\alpha\gamma}$ also splits $A \setminus T_\alpha$, and, a fortiori, A. □

Note added in December 2003. Michael Hrušák remarked that the assumption $\mathfrak{t} < \mathfrak{b}$ is unnecessary for the proof of Theorem 10. Instead one may replace a given tower $\{T_\alpha;\ \alpha < \mathfrak{t}\}$ by $\{T'_\alpha;\ \alpha < \mathfrak{t}\}$ where $T'_\alpha = \{(m,n);\ m \in T_\alpha \text{ and } n < m\}$. Then we can easily find the required partition $\langle E_k;\ k \in \omega\rangle$.

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